Einstein-Hilbert action

$\dim G_d = \frac{1}{m^{d-2}}$; $\dim g = 1$; $\dim R = \partial\partial g \sim m^2$; usual units: $[\Lambda] = \frac{1}{sec^2}$

$\frac{1}{m^d} \cdot 1 \cdot m^2$; $g^{\mu\nu} R_{\mu\nu} = R_{\mu\nu\alpha\beta}$

$$S = -s \frac{1}{16\pi G_d} \int d^d x \sqrt{|g|} \, (R + \Lambda)$$

$s = -$ $(-+++\ldots)$
$+$ $(+++\ldots)$

$g = -\det g_{\mu\nu}$

metric conv!! $\eta_{\mu\nu} = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}$; $\det \eta_{\mu\nu} = -1$

If $x^\mu \to x'^\mu$ then $d^d x' = dx'^1 dx'^2 \cdots dx'^d = \left| \left| \underbrace{\frac{\partial x'}{\partial x}}_{=\det \frac{\partial x'^\mu}{\partial x^\nu}} \right| \right| d^d x$

Reminder: under $x^\mu \to x'^\mu$

$$\begin{cases} v'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} v^\alpha & \text{Ex: } dx'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} dx^\alpha \quad \text{contrav.} \quad \text{vector space} \\ v'_\mu = \frac{\partial x^\beta}{\partial x'^\mu} v_\beta & \text{Ex: } \frac{\partial \varphi(x)}{\partial x'^\mu} = \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial \varphi}{\partial x^\beta} \quad \text{cov.} \quad \text{dual space} \end{cases}$$

$\equiv \partial'_\mu \varphi$

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = J_{\alpha\mu} J_{\beta\nu} g_{\alpha\beta} = J^T_{\mu\alpha} g_{\alpha\beta} J_{\beta\nu}$$

$$\det g' = \det J^T \det g \det J = (\det J)^2 \det g$$

$$g' = (\det J)^2 g = \left(\det \frac{\partial x^\alpha}{\partial x'^\mu}\right)^2 g = \left(\det \frac{\partial x}{\partial x'}\right)^2 \cdot g$$

$$d^d x' = \left(\det \frac{\partial x'^\mu}{\partial x^\alpha}\right) \underbrace{d^d x}_{\det \frac{\partial x^\nu}{\partial x'^\beta} d^d x'}$$

$$\det \frac{\partial x'^\mu}{\partial x^\alpha} \cdot \det \frac{\partial x^\nu}{\partial x'^\beta} = 1$$

$$\det \frac{\partial x'}{\partial x} \cdot \det \frac{\partial x}{\partial x'} = 1$$

$$\sqrt{g}\, d^d x \to \sqrt{g'}\, d^d x' = \det \frac{\partial x}{\partial x'} \sqrt{g} \cdot \det \frac{\partial x'}{\partial x} \, d^d x = \sqrt{g}\, d^d x$$

$\boxed{\delta(\sqrt{g} R)}$

$g = -\det g_{\mu\nu}$

$-g = \det g_{\mu\nu} = e^{\mathrm{Tr} \log(g)}$

$$\delta(\sqrt{g} R_{\mu\nu} g^{\mu\nu}) = \underbrace{\delta(\sqrt{g})}_{\frac{1}{2}\frac{1}{\sqrt{g}}\delta g} g^{\alpha\beta} R_{\alpha\beta} + \sqrt{g} R_{\alpha\beta} \underbrace{\delta g^{\alpha\beta}}_{-g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu}} + \underbrace{\sqrt{g} g^{\mu\nu} \delta R_{\mu\nu}}_{= \partial_\alpha [\quad]}$$

$g = -\det$

$= g \cdot g^{\mu\nu} \delta g_{\mu\nu} = -g \, g_{\mu\nu} \delta g^{\mu\nu}$

$g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$!

Computed explicitly in SW 12.4

surprisingly, this is a surface term: remove!

Use $\boxed{\det A = e^{\mathrm{Tr} \log A}}$

$$\delta(\det A) = \left[e^{\mathrm{Tr}\log(A + \delta A)} - e^{\mathrm{Tr}\log A}\right]$$

$$= \left[e^{\mathrm{Tr}\log A(1 + A^{-1}\delta A)} - e^{\mathrm{Tr}\log A}\right]$$

$$= \left(e^{\mathrm{Tr}\log A + \underbrace{\mathrm{Tr}\log(1 + A^{-1}\delta A)}_{\mathrm{Tr} A^{-1}\delta A}} - e^{\mathrm{Tr}\log A}\right)$$

$$= e^{\mathrm{Tr}\log A}(1 + \mathrm{Tr} A^{-1}\delta A - 1) = \det A \; \mathrm{Tr} A^{-1}\delta A$$

$g = \det g_{\mu\nu}$ and $\varepsilon_{\mu\nu\lambda\sigma}$

$x \to x'$: $\quad \det g' = \left(\det \frac{\partial x^\alpha}{\partial x'^\beta}\right)^2 \det g$

For any $m\times m$ matrix A_{ij}

$$\det A = \sum \varepsilon_{i_1 i_2 \cdots i_m} A_{1 i_1} A_{2 i_2} \cdots A_{m i_m}$$

$$= \sum \varepsilon_{i_1 i_2 \cdots i_m} A_{i_1 1} A_{i_2 2} \cdots A_{i_m m} \qquad v'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu} v_\alpha$$

or

$$= \frac{1}{m!} \sum \varepsilon_{i_1 \cdots i_m} \varepsilon_{j_1 \cdots j_m} A_{i_1 j_1} \cdots A_{i_m j_m} \qquad v_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} v_\mu$$

$$\varepsilon_{\mu_1' \cdots \mu_m'} \det A = \varepsilon_{\mu_1 \cdots \mu_m} A_{\mu_1 \mu_1'} \cdots A_{\mu_m \mu_m'}$$

Taking $A_{\mu\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}$ this looks like the tensor transf. formula but for the $\left(\det \frac{\partial x^\mu}{\partial x^{\mu'}}\right)^{-1} = \det \frac{\partial x^{\mu'}}{\partial x^\mu}$

$$\varepsilon_{\mu_1' \cdots \mu_m'} = \left(\det \frac{\partial x^{\mu'}}{\partial x^\mu}\right) \cdot \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_m}}{\partial x^{\mu_m'}} \varepsilon_{\mu_1 \cdots \mu_m}$$

$\Rightarrow$ define $\quad \varepsilon_{\mu_1 \cdots \mu_m} = \sqrt{|\det g_{\mu\nu}|}\, \underbrace{\tilde{\varepsilon}_{\mu_1 \cdots \mu_m}}$

$\Downarrow g^{\mu\nu} \qquad\qquad \tilde{\varepsilon}_{12\cdots m} = +1$ etc. ; numerical tensor

$\varepsilon^{\mu_1 \cdots \mu_m}$

Then $\int d^2\sigma \sqrt{h}\; \varepsilon^{ab} \partial_a X \partial_b X$ is Weyl inv:

$$\sqrt{h}\, \varepsilon^{ab} = \sqrt{h}\, h^{ac} h^{bd} \sqrt{h}\, \tilde{\varepsilon}_{cd}$$

$$\to \underbrace{\Lambda \;\; \tfrac{1}{\Lambda} \;\; \tfrac{1}{\Lambda} \;\; \Lambda}_{=1} * \text{same}$$

$\det g_{\mu\nu} \cdot \det g^{\mu\nu} = 1$!

$$\varepsilon^{\mu_1 \cdots \mu_m} = \sqrt{|g|}\, g^{\mu_1\nu_1} \cdots g^{\mu_m \nu_m} \tilde{\varepsilon}_{\nu_1 \cdots \nu_m} = \sqrt{|g|}\, \tilde{\varepsilon}^{\mu_1 \cdots \mu_m} \underbrace{\det g^{\mu\nu}}_{\frac{1}{g}} = \frac{\mathrm{sign}(g)}{\sqrt{|g|}} \tilde{\varepsilon}^{\mu_1 \cdots \mu_m}$$

If $\varepsilon_{0123} = 1$ then $\varepsilon^{0123} = -1$!

for $(-+++)$

$$\boxed{\delta(\sqrt{g}R) \hat{=} -\sqrt{g}\,(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R)\delta g_{\mu\nu} = +\sqrt{g}(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}} \qquad \boxed{\delta\sqrt{g} = \tfrac{1}{2}\sqrt{g}\; g^{\mu\nu}\delta g_{\mu\nu} = -\tfrac{1}{2}\sqrt{g}\,g_{\mu\nu}\delta g^{\mu\nu}}$$

$$\delta\left[\sqrt{g}\,(g^{\alpha\beta}R_{\alpha\beta}+\Lambda)\right] = \tfrac{1}{2}\sqrt{g}\; g^{\mu\nu}R\,\delta g_{\mu\nu} - \sqrt{g}\,R^{\mu\nu}\delta g_{\mu\nu} + \Lambda\,\tfrac{1}{2}\sqrt{g}\; g^{\mu\nu}\delta g_{\mu\nu}$$

$$= -\sqrt{g}\;\delta g_{\mu\nu}\left(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R - \tfrac{1}{2}\Lambda g^{\mu\nu}\right)$$

Note:

$$\boxed{\begin{aligned} g^{\mu\nu}\delta g_{\mu\nu} &\overset{!}{=} -g_{\mu\nu}\delta g^{\mu\nu} \\ R^{\mu\nu}\delta g_{\mu\nu} &= -R_{\mu\nu}\underbrace{\delta g^{\mu\nu}}_{-g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}} \end{aligned}}$$

$$\delta S = +\frac{1}{16\pi G_d}\int d^dx\,\sqrt{g}\,\underbrace{\left(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R - \tfrac{1}{2}\Lambda g^{\mu\nu}\right)}_{=0 \text{ at extremum}}\delta g_{\mu\nu}$$

<u>dS & AdS spaces:</u>

better $\Lambda \to -2\Lambda$! (p. 90)

$$R^{\mu\nu} = \frac{\Lambda}{2-d}\,g^{\mu\nu} \quad\Rightarrow\quad R = g^{\mu\nu}R_{\mu\nu} = \frac{\Lambda}{2-d}\,\underbrace{g^{\mu\nu}g_{\mu\nu}}_{\delta^\mu{}_\mu = d}$$

$$R = \Lambda\frac{d}{2-d} \qquad g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu{}_\nu$$

$$R^{\mu\nu} - \tfrac{1}{2}(R+\Lambda)g^{\mu\nu} = \left[\frac{1}{2-d} - \frac{1}{2}\left(\frac{d}{2-d}+1\right)\right]\Lambda g^{\mu\nu}$$

$$2-d-(2-d) = 0$$

Maximally symmetric spaces even have

$$R_{\mu\nu\alpha\beta} = \frac{R}{d(d-1)}\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right)$$

$$R_{\mu\nu\alpha\beta} = \frac{R}{d(d-1)}\Big[\underbrace{g_{\mu\alpha}g^{\mu\alpha}}_{\delta^\mu{}_\mu = d}\underbrace{g_{\nu\beta}g^{\nu\beta}}_{d} - g_{\mu\beta}\underbrace{g_{\nu\alpha}g^{\mu\alpha}}_{\delta^\mu{}_\nu}g^{\nu\beta}\Big]\ldots$$

<u>Reminder:</u>

covariant derivative

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \overset{\text{Christoffel}}{\Gamma^\mu_{\nu\alpha}}\,V^\alpha \xrightarrow{x\to x'} \overset{\text{transforms like tensor}}{\frac{\partial x'^\mu}{\partial x^\alpha}\cdot\frac{\partial x^\beta}{\partial x'^\nu}\cdot\nabla_\beta V^\alpha}$$

$$(D_\mu = \partial_\mu + igA_\mu)$$

$$\nabla_\nu V_\mu = \partial_\nu V_\mu - \Gamma^\alpha_{\mu\nu}V_\alpha$$

Curvature tensor

$$(\nabla_\lambda\nabla_\nu - \nabla_\nu\nabla_\lambda)V_\mu = -V_\alpha R^\alpha{}_{\mu\nu\lambda} \qquad \left([D_\mu, D_\nu] = \frac{i}{g}F_{\mu\nu}\right)$$

$$(\nabla_\lambda\nabla_\nu - \nabla_\nu\nabla_\lambda)V^\mu = V^\alpha R^\mu{}_{\alpha\nu\lambda}$$

$$R_{\mu\nu\lambda\rho} = \tfrac{1}{2}\left(\partial_\rho\partial_\nu g_{\mu\lambda} - \partial_\rho\partial_\mu g_{\nu\lambda} - \partial_\lambda\partial_\nu g_{\mu\rho} + \partial_\lambda\partial_\mu g_{\nu\rho}\right) + g_{\alpha\beta}\left(\Gamma^\alpha_{\lambda\mu}\Gamma^\beta_{\nu\rho} - \Gamma^\alpha_{\rho\mu}\Gamma^\beta_{\nu\lambda}\right)$$

linearised gravity

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{8\pi G}\,h_{\mu\nu} \approx \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{Pl}}$$

$$R_{\mu\nu\lambda\rho} \qquad R_{\nu\rho} = g^{\mu\lambda}R_{\mu\nu\lambda\rho} \qquad R = g^{\nu\rho}R_{\nu\rho} = R_{\mu\nu\lambda\rho}$$

Including matter :

$$S = \int d^dx \left\{ \left(-\frac{1}{16\pi G}\right) \sqrt{g}\, (R + \overset{-2\Lambda}{\Lambda}) + \sqrt{g}\, \mathcal{L}_m(g_{\mu\nu}) \right\}$$

$$\delta S = \int d^dx \left\{ \left(-\frac{1}{16\pi G}\right)(-\sqrt{g}) \left[R^{\mu\nu} - \frac{1}{2} g^{\mu\nu}(R+\Lambda) \right] \delta g_{\mu\nu} + \frac{1}{2}\sqrt{g} \left(g^{\mu\nu}\mathcal{L}_m + 2\frac{\delta \mathcal{L}}{\partial g^{\mu\nu}} \right) \delta g_{\mu\nu} \right.$$

$$= \int d^dx \frac{1}{16\pi G} \sqrt{g} \left\{ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu}(R+\Lambda) + 8\pi G \left(g^{\mu\nu}\mathcal{L}_m + 2\frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}} \right) \right\} \delta g_{\mu\nu}$$

since
$$\left\{ \begin{aligned} \delta \sqrt{g}\, \mathcal{L}_m(g_{\mu\nu}) &= +\frac{1}{2}\sqrt{g}\, g^{\mu\nu}\, \delta g_{\mu\nu}\, \mathcal{L}_m + \sqrt{g}\, \frac{\partial \mathcal{L}_m}{\partial g_{\mu\nu}}\, \delta g_{\mu\nu} \\ &= \frac{1}{2}\sqrt{g} \left(g^{\mu\nu}\mathcal{L}_m + 2\frac{\partial \mathcal{L}_m}{\partial g_{\mu\nu}} \right) \delta g_{\mu\nu} \\ &= \frac{1}{2}\sqrt{g} \left(-g_{\mu\nu}\mathcal{L}_m + 2\frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}} \right) \delta g^{\mu\nu} \end{aligned} \right.$$

$$\left(T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} \right)$$

note sign change when $\delta g_{\mu\nu} \to \delta g^{\mu\nu}$

$$= -\int d^dx \frac{1}{16\pi G} \sqrt{g} \left\{ + R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R+\Lambda) + 8\pi G \left(g_{\mu\nu}\mathcal{L}_m - 2\frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}} \right) \right\} \delta g^{\mu\nu}$$

$$\Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R + \overset{-2\Lambda}{\Lambda}) = 8\pi G \left(2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}_m \right) \equiv 8\pi G\, T_{\mu\nu}}$$

Ex 1: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4} F_{\sigma\varrho}F_{\alpha\beta}\, g^{\sigma\alpha} g^{\varrho\beta}$

only one term !

$$\frac{\partial}{\partial g^{\mu\nu}} (g^{\sigma\alpha} g^{\varrho\beta}) = \frac{\partial g^{\sigma\alpha}}{\partial g^{\mu\nu}} g^{\varrho\beta} + g^{\sigma\alpha} \frac{\partial g^{\varrho\beta}}{\partial g^{\mu\nu}} = (\delta_{\sigma\mu}\delta_{\alpha\nu} + \cancel{\delta_{\sigma\nu}\delta_{\alpha\mu}}) g^{\varrho\beta} + g^{\sigma\alpha}(\delta_{\varrho\mu}\delta_{\beta\nu} + \cancel{\delta_{\varrho\nu}\delta_{\beta\mu}})$$

$$\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = -\frac{1}{4} \left\{ (F_{\mu\varrho} F_{\nu\beta}\, g^{\varrho\beta} + F_{\sigma\mu}F_{\alpha\nu}\, g^{\sigma\alpha} \right\} = -\frac{1}{2} F_{\mu\varrho} F_{\nu}{}^{\varrho} = \frac{1}{2} F_{\mu\varrho} F^{\varrho}{}_{\nu}$$

$$\Rightarrow \boxed{T_{\mu\nu} = -F_{\mu\varrho} F_{\nu}{}^{\varrho} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}} \Rightarrow T_{\mu}{}^{\mu} = -F_{\mu\varrho}F^{\mu\varrho} + F_{\alpha\beta}F^{\alpha\beta} = 0$$

Ex 2 Vacuum energy : I would write $T_{\mu\nu} = \varepsilon_{vac}\, g_{\mu\nu} = -p_{vac}\, g_{\mu\nu}$

$$\boxed{\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\equiv G_{\mu\nu}} + \tilde{\Lambda}\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}}$$

if $T_{\mu\nu} = (\varepsilon + p)u^{\mu}u^{\nu} - p\, g^{\mu\nu}$

$\tilde{\Lambda} = 8\pi G\, \varepsilon_{vac}$

corresponds to $-\frac{1}{2}\Lambda = \tilde{\Lambda}$ above

Next study $g_{\mu\nu}$ for given special $T_{\mu\nu}$, starting from $T_{\mu\nu} = 0$!

Local $\to$ global structure ?