Addenda:

(1) As usual in SUSY, to close SUSY algebra introduce an auxiliary non-dynamical field B_μ:

$$S = -\frac{T}{2}\int d\tau d\sigma \left[\partial_a X\cdot\partial^a X - \bar\psi\cdot i\partial\!\!\!/\psi - B\cdot B\right]$$

$$\begin{cases} \delta X^\mu = \bar\varepsilon\psi^\mu \\ \delta\psi^\mu = -\varepsilon\, i\partial\!\!\!/ X^\mu + \varepsilon B^\mu \\ \delta B^\mu = -\bar\varepsilon\, i\partial\!\!\!/\psi^\mu \end{cases}$$

$$\delta\bar\psi^\mu = \bar\varepsilon\, i\partial\!\!\!/ X^\mu + \bar\varepsilon B^\mu$$

This also explains the dof count:

matter sector	X^μ, B^μ	2d bosonic dofs
	ψ^μ	2d fermionic dofs
2d gravity sector:	$e_\alpha{}^a$, A	2 bosonic dofs
	χ_α	2 fermionic "

Superspace:

– 4d $\quad x^\mu\ \theta^a\ \theta^{\dot a} \qquad a,\dot a = 1,2 \quad \theta^a = \varepsilon^{ab}\theta_b$

left chiral superfield:

$$\varphi(x^\mu,\theta^a) = \varphi(x) + \theta^a\psi_a + \theta^a\theta_a F$$

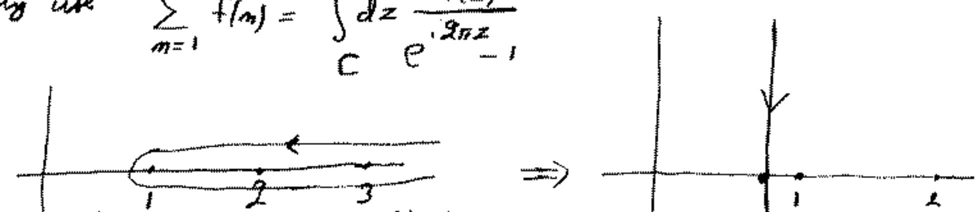
– 2d: σ, θ^a

$$Y^\mu(\sigma,\theta) = X^\mu(\sigma) + \bar\theta\psi^\mu + \tfrac{1}{2}\bar\theta\theta B^\mu$$

(2) To find $\sum_0^\infty (m+\frac{1}{2}) \equiv \zeta(-1,\frac{1}{2}) \qquad \zeta(z,\alpha) \equiv \sum_0^\infty \frac{1}{(m+\alpha)^z}$

one must study the analytic continuation

Probably use $\displaystyle\sum_{m=1}^\infty f(m) = \int_C dz \frac{f(z)}{e^{i2\pi z}-1}$



Or more simply from (see Magnus-Oberhettinger-Soni)

$$\zeta(z,\alpha) = \frac{1}{\Gamma(z)}\int_0^\infty dt\, \frac{t^{z-1}}{e^{t}-1}\, e^{-(\alpha-1)t} \qquad \text{(expand \& integrate!)}$$

$$\zeta(z,1) = \zeta(z) = \frac{1}{2^z-1}\,\zeta(z,\tfrac{1}{2}) \Rightarrow \boxed{\zeta(-1,\tfrac{1}{2}) = (\tfrac{1}{2}-1)\zeta(-1) = \frac{1}{24}}$$

Proof:

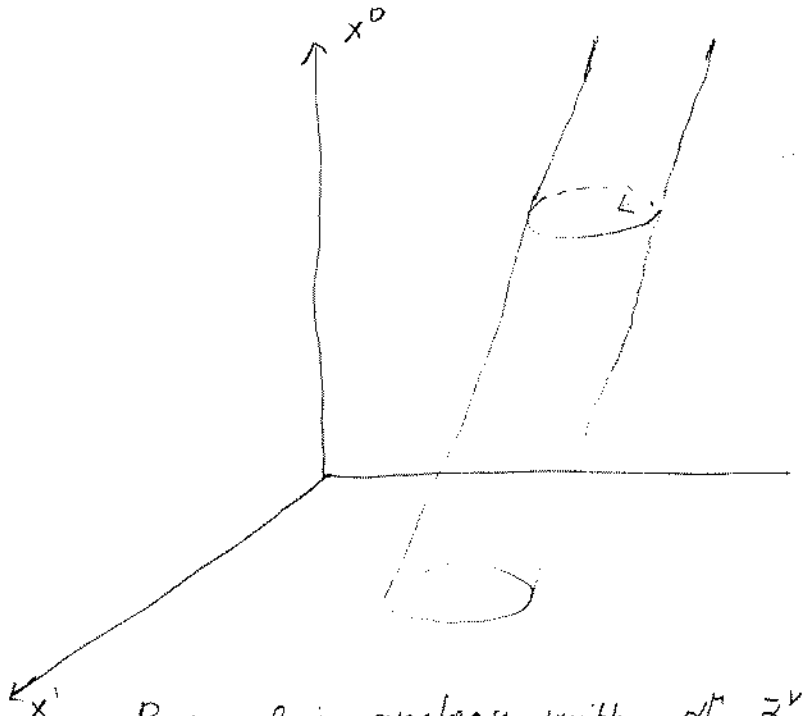
$$\zeta(z,\tfrac{1}{2}) = \frac{1}{\Gamma(z)}\int_0^\infty dt\, t^{z-1}\left[\frac{e^{\frac{1}{2}t}}{e^t-1} = \frac{e^{\frac{1}{2}t}(e^{\frac{1}{2}t}-1)}{(e^t-1)(e^{\frac{1}{2}t}-1)} = \frac{e^t-e^{\frac{1}{2}t}}{(e^t-1)(e^{\frac{1}{2}t}-1)} = \frac{1}{e^{\frac{1}{2}t}-1} - \frac{1}{e^t-1}\right]$$

$$= 2^z\zeta(z) - \zeta(z)$$

$$\zeta(-1,a) = \sum_0^\infty (m+a) = -\frac{1}{12}(1-6a+6a^2) \quad \text{or} \quad \sum_1^\infty (m-\theta) = \frac{1}{24} - \frac{1}{8}(2\theta-1)^2 \quad \text{Polchinski (2.9.19)}$$

note

6. Superstrings in space-time, supergravities



... find massless states; interactions of which are described by supergravities

string → point, $\alpha' \sim \frac{1}{T} \to 0$

$\underbrace{x^{2,3,4,\dots,9}}$ what to do with these?

$\eta_{\mu\nu} \to G_{\mu\nu}(x)$!!

Proceed in analogy with $\alpha^{\mu}_{-1}\,\tilde{\alpha}^{\nu}_{-1}|0\,p\rangle$ (p. 46; 50) and separate

Closed:

$$M^2 = \frac{4}{l_s^2}\Big[\sum_1^{\infty}(\underset{\tau-\sigma}{\overset{R}{\alpha_{-m}\cdot\alpha_m}} + \underset{\tau+\sigma}{\overset{L}{\tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m}}) + \sum_{r=\frac{1}{2}}^{\infty}(r\,b_{-r}\cdot b_r + r\,\tilde{b}_{-r}\cdot\tilde{b}_r) - \underset{-\frac{1}{2}}{a_{NS}} - \underset{-\frac{1}{2}}{\tilde{a}_{NS}}\Big] \qquad \boxed{a_{NS} = \tfrac{1}{2}\,!}$$

- $|0,p\rangle^R_{NS} \otimes |0,p\rangle^L_{NS}$ $\quad M^2 = -\frac{4}{l_s^2}$ out!
- $b^{\mu}_{-\frac{1}{2}}|0,p\rangle^R_{NS} \otimes b^{\nu}_{-\frac{1}{2}}|0,p\rangle^L_{NS}$ $\quad M^2 = 0 \Rightarrow \underset{35}{G_{\mu\nu}}, \underset{28}{B_{\mu\nu}}, \underset{1}{\phi}$

  ($b^i_{-\frac{1}{2}}\,\tilde{b}^j_{-\frac{1}{2}}|0,p^+p\rangle_{NS}$ in light cone gauge)
- $\alpha^{\mu}_{-1}\,\tilde{\alpha}^{\nu}_{-1}|0,p\rangle_{NS}$ $\quad M^2 = \frac{4}{l_s^2} = \frac{2}{\alpha'}$ out!

⋮

bosonic graviton is now massive!

One finds that consistently states with even number of b_{-r}'s (or $\tilde{b}_{-r}$'s) can be eliminated (meaning that if they are thrown out they do not reappear) = GSO projection

R ⟳ R

$$M^2 = \frac{4}{l_s^2}\Big[\sum_1^\infty (\underbrace{\alpha_{-m}\cdot\alpha_m}_{m a_m^\dagger \cdot a_m} + \bar\alpha_{-m}\cdot\bar\alpha_m) + \sum_1^\infty (m d_{-m}\cdot d_m + m\bar d_{-m}\cdot \bar d_m) - 0\Big]$$

$\boxed{a_R = \bar a_R = 0\ !}$

spinor index in 10 (or 8) dims

- $|A,p\rangle_R^R \otimes |B,p\rangle_R^L$ $\quad M^2 = 0$

bosonic states, "bispinors"

$C_\mu \ C_{\mu\nu\lambda}$ $\quad 8 + 56 = 64$ Type IIA

$C_0 \ C_{\mu\nu} \ C_{\mu\nu\lambda\kappa}$ $\quad 1 + 28 + 35 = 64$ Type IIB

R ⟳ NS (or NS ⟳ R)

$$M^2 = \frac{4}{l_s^2}\Big[N_R^\alpha + N_L^{\bar\alpha} + \sum_1^\infty m d_{-m}\cdot d_m + \sum_{\frac12}^\infty r\bar b_{-r}\cdot \bar b_r - \frac12\Big]$$

- $|A\,p\rangle_R^R \otimes |0\,p\rangle_{NS}^L$ $\quad M^2 = -\frac{2}{l_s^2}$ $\quad \Leftarrow$ tachyon again thrown out by GSO projection

- $\underbrace{|A\,p\rangle_R}_{\text{spin } \frac12} \otimes \underbrace{b^\mu_{-\frac12}}_{\text{spin } 1}|0\,p\rangle_{NS}^L$ $\quad M^2 = 0$

$$\tfrac12 \times 1 = \tfrac12 + \tfrac32 = \underset{8}{\chi^a} + \underset{56}{\psi^{\mu a}}$$

Here one has to discuss the chiralities of $|A\rangle_R \otimes b^\mu_{-\frac12}|0\rangle_{NS}$

$\bar b^\mu_{-\frac12}|0\rangle_{NS} \otimes |A\rangle_R$

Opposite: type IIA

Same: type IIB

Ramond vacuum $|A\,0\rangle_R$ $\qquad$ NS vacuum has no spinor index!

$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu} \Rightarrow \{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$. $\quad$ # of dofs of a spinor in d:

$(\gamma^\mu = i\sqrt2\, d_0^\mu)$

zero mode,

$\psi(\sigma) = d_0 + \sum_{m\neq 0} d_m$

like $X = x + l_s^2 p\tau + \sum_{m\neq0}$

$$\begin{cases} 2^{d/2} & \text{Dirac spinor} \quad \gamma^\mu_{AB} \\ 2^{d/2-1} & \text{Majorana or Weyl} \\ 2^{d/2-2} & \text{" and "} \end{cases}$$

Spinor rep of $SO(1,d-1)$: $J_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ should satisfy p.52

$p_\mu \gamma^\mu_{AB} |B\,p\rangle_R = 0$

↑ one of the constraint eqs.

Now that we have massless states of the string how do they interact?

$\Rightarrow$ Strings in Background Fields

$\nearrow$ string coupl. const

2d Einstein-Hilbert $\int d^2\sigma \sqrt{-h}\, R_h$ p. 74

Add two new terms:

dilaton

$$S = -\frac{T}{2}\int d^2\sigma \sqrt{-h}\left\{ h^{ab} \underbrace{G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu} + \varepsilon^{ab} B_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + \frac{\alpha'}{2} R_h(X)\phi(X)\right\}$$

let X_0 = some const BG

$\dim T X^2 = 1$

$\dim X = \frac{1}{\sqrt{T}} \sim \sqrt{\alpha'}$

$$= \left[G_{\mu\nu}(X_0) + (X-X_0)^\alpha \partial_\alpha G_{\mu\nu}(X_0) + \frac{1}{2!}(\ )^\alpha(\ )^\beta \partial_\alpha\partial_\beta G_{\mu\nu} + \cdots\right]\partial_a X^\mu \partial_b X^\nu$$

$$\approx \left[\ 1 \ + \ \underline{X\cdot\partial} \ + \ X^2\partial^2 \ + \cdots \ \right](\partial X)^2$$

$$= \# \frac{1}{\sqrt{T}}\cdot\frac{1}{\text{Radius}} = \# \frac{\alpha'}{\text{Radius}} \quad \text{perturbative for small } \alpha'$$

$\Rightarrow$ Interactions! $\Rightarrow$ "Non-linear σ model"

or

$$= \left[\eta_{\mu\nu} + \varepsilon_{\mu\nu}(k) e^{ik\cdot X} + \cdots\right]\partial_a X^\mu \partial_b X^\nu$$

$$= \left[\eta_{\mu\nu} + \varepsilon_{\mu\nu}\left(1 + ik\cdot X - \frac{1}{2}k_\alpha k_\beta X^\mu X^\nu + \cdots\right)\right]\partial_a X^\mu \partial_b X^\nu$$

$\Rightarrow$ graviton vertex operator : $\varepsilon_{\mu\nu}(k)\, \partial_a X^\mu \partial^a X^\nu e^{ik\cdot X}$:

The above action is classically Lorentz-reparametrisation-Weyl invariant but quantisation destroys these unless $d=26$ even if $G_{\mu\nu}(X) = \eta_{\mu\nu}$, $B_{\mu\nu} = \phi = 0$. When $G_{\mu\nu} = G_{\mu\nu}(X)$ this theory is a 2d interacting field theory, all the couplings start to "run":

$$\beta^G_{\mu\nu} = \frac{\partial G_{\mu\nu}}{\partial \log\Lambda} \leftarrow \text{regulator}$$

and invariance is true only if $\beta^G_{\mu\nu} = 0 = \beta^B_{\mu\nu} = \beta^\phi_{\mu\nu}$

These eqs generalise "$d=26$" to non-constant $G_{\mu\nu}$.

Parenthesis: String coupling g_s

$$e^{-S} = e^{-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}\left\{\cdots\cdots + \alpha' R\phi\right\}} \sim \left[e^{-\langle\phi\rangle}\right]^{2-2\tilde{g}} \sim (g_s^2)^{\tilde{g}-1}$$

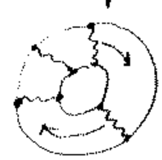
$$\chi = \frac{1}{4\pi}\int d^2\sigma\sqrt{-h}\,R = \text{Euler number} = 2 - 2\tilde{g}$$ ← genus

$\exists$ a potential for dilaton $\Rightarrow \exists\langle\phi\rangle$. Then $g_s = e^{\langle\phi\rangle}$

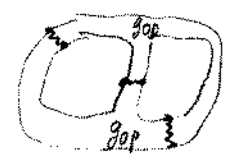
Why is this "string coupling"?

$$\left[\begin{array}{l}\phi \to \phi - c \\ \Rightarrow g_s \to e^c g_s\end{array}\right.$$

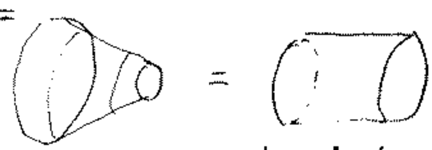
open string loop



2loop open string

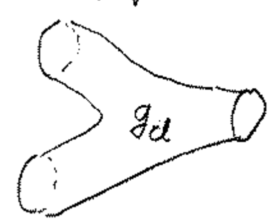


(pull out internal circle



closed string propag

⇓ pull out



closed string vertex

$$\Rightarrow g_{open}^2 = g_{cl}$$

Adding one handle costs g_s^2 (since $\tilde{g} \to \hat{g}+1$) or g_{cl}^2 :

$$\Rightarrow \boxed{g_{open}^2 \cong g_{cl} \cong g_s \cong e^{\langle\phi\rangle}}$$

$$\left\{\begin{array}{lll} g_s^2 & \text{is} & \text{handle exp parameter} \\ \dfrac{1}{N_c^2} & & \text{— " —}\end{array}\right.$$

constants appear!

$$\left\{\begin{array}{l} g_{open}^2 = \dfrac{2\pi^{D/2}}{2^{(10-D)/4}}\, g_{cl} \\ g_{cl} = (2\pi)^2 g_s \end{array}\right.$$

The equations $\beta^G_{\mu\nu}=0$, $\beta^B_{\mu\nu}=0$, $\beta^\phi_{\mu\nu}=0$ are the eqs. of motion following from the field theory action

strings appear as point particles

$$(\alpha')^{d-2} S = \int d^dx\, \underbrace{\sqrt{G}\, e^{-\phi}\left[R\right.}_{\Downarrow} + (\nabla\phi)^2 - \frac{1}{12}H^2 + \frac{D-26}{3}\Big] + O(\alpha')$$

$$\sqrt{G}\; R_E \quad \text{by } G^E_{\mu\nu} = e^{-\frac{2\phi}{d-2}} G_{\mu\nu}$$

$$H_{\mu\nu\alpha} = \partial_\mu B_{\nu\alpha} + \partial_\nu B_{\alpha\mu} + \partial_\alpha B_{\mu\nu}\,, \quad H = dB$$

$$\begin{cases} \frac{1}{\alpha'}\beta^B_{\mu\nu} = \nabla^\alpha\left[e^{-\phi} H_{\mu\nu\alpha}\right] = 0 & \text{"Maxwell"} \\ 3\frac{1}{\alpha'}\beta^\Phi_{\mu\nu} = \frac{d-26}{3\alpha'} + (\nabla\phi)^2 - 2\Box\phi - R + \frac{1}{12}H^2 = 0 & \text{"scalar"} \\ \frac{1}{\alpha'}\beta^G_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} + \nabla_\mu\nabla_\nu\phi = 0 & \text{"gravity"} \end{cases}$$

Conformal transformation to Einstein frame:

$$\boxed{\tilde{G}_{\mu\nu} = e^{2\Omega}G_{\mu\nu} \;\Rightarrow\; \tilde{R} = e^{-2\Omega}\left[R - 2(d-1)\nabla^2\Omega - (d-2)(d-1)\,\partial_\mu\Omega\,\partial^\mu\Omega\right]}$$

Try $\tilde{G}_{\mu\nu} = e^{\sigma\phi}G_{\mu\nu}$

$$\Rightarrow \sqrt{G}\,e^{-\phi}R = \sqrt{\tilde{G}}\, e^{-\underbrace{[\sigma(d-2)+2]}_{\text{require}=0}\frac{\phi}{2}}\left\{\tilde{R} + \sigma(d-1)\frac{1}{\sqrt{\tilde{g}}}\partial_\mu\left(\sqrt{\tilde{g}}\,\partial^\mu\phi\right) - \frac{1}{4}\sigma^2(d-1)(d-2)\,\partial_\mu\phi\,\partial^\mu\phi\right\}$$

$$\sigma = -\frac{2}{d-2}$$

Gives the "Einstein frame": $\sqrt{G}\,e^{-\phi} = \sqrt{\tilde{G}}\, e^{\frac{2}{d-2}\phi}$

$$(\alpha')^{d-2}S = \int d^dx\sqrt{G}\left[R - \frac{1}{d-2}(\nabla\phi)^2 - e^{-\frac{4\phi}{d-2}}\frac{1}{12}H^2 + e^{\frac{2}{d-2}\phi}\,\frac{d-26}{3}\right]$$

Cf <u>Brans - Dicke</u>

$$\int d^4x\,\sqrt{g}\left[f(\phi)R + \frac{1}{2}\nabla^\mu\phi\,\nabla_\mu\phi - V(\phi)\right]$$

or <u>Reissner- Nordström</u>:

$$\int d^4x\,\sqrt{-g}\left\{+\frac{1}{16\pi G}(R+2\Lambda) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right\}$$

Type IIA sugra: (non-chiral) $\qquad F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu \equiv F_2$

$$S = \underbrace{\frac{1}{2\kappa^2}}_{\equiv \frac{1}{16\pi G_{10}}} \int d^{10}x \sqrt{-G} \left\{ R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{12} e^{-\phi} H_3^2 - \frac{1}{4} e^{\frac{3}{2}\phi} F_2^2 - \frac{1}{48} e^{\frac{1}{2}\phi} F_4^2 \right.$$

$$\left. - \frac{1}{2304} \frac{1}{\sqrt{-G}} \varepsilon^{\mu_0 \cdots \mu_9} B_{\mu_0\mu_1} \underbrace{F^4_{\mu_2 \cdots \mu_5}} F^4_{\mu_6 \cdots \mu_9} \right\} + \text{fermions}$$

field strength constructed from the R-R 3-form $C_{\mu\nu\lambda} \equiv F_4$

$B_2 \wedge F_4 \wedge F_4$

Fields

{R: periodic BC for ψ
{NS: anti " " " ψ

			# of physical dofs	
	NS-NS	ϕ	1	
		$B_{\mu\nu}$	28	← couples to a string
		$G_{\mu\nu}$	35	
bispinors	R-R	C_μ	8	← couples to point
		$C_{\mu\nu\lambda}$	56	← " " 2d surface
	NS-R	χ_a $s=\frac{1}{2}$	8	dilatino
		ψ^μ_a $s=\frac{3}{2}$	56	gravitino
	R-NS	χ'_a	8	
		$\psi^{\mu\,\prime}_a$	56	

$128_b + 128_f$ dofs

$8^2 = 36 - 1 + 28 + 1 \qquad d = 10 \to 8$

$\quad = 3 - 1 + 1 + 1 \qquad d = 4 \to 2$

= D branes carry RR charges $\binom{8}{3} = 56$

dim $= p$, p even for IIA

Type II B sugra (chiral, more physical)

$$S_{IIB} = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-G} \Big\{ e^{-2\phi} \left[R + 4(\nabla\phi)^2 - \frac{1}{2} H^2 \right]$$

$$- \frac{1}{2}\left(F_1^2 + \tilde{F}_3^2 + \tilde{F}_5^2 \right) \Big\} - \frac{1}{32\pi G_{10}} \int C_4 \wedge H_3 \wedge F_3 + \text{ferm's}$$

RR 1- 3- 5- forms $\quad (C_4 \wedge H_3)_{\mu_1 \cdots \mu_7} = (C_4)_{[\mu_1 \cdots \mu_4} (H_3)_{\mu_5 \mu_6 \mu_7]}$

States;

NS-NS	ϕ	1
	$B_{\mu\nu}$	28
	$R_{\mu\nu}$	35
R-R	C_0	1
	$C_2 \to C_{\mu\nu}$	28
	$C_4 \to C_{\mu\nu\lambda\kappa}$	35

+ fermion states as for IIA

$H \equiv H_3 = dB$

$F_1 = dC_0$

$F_3 = dC_2 \qquad \tilde{F}_3 = F_3 - C_0 \wedge H_3$

$F_5 = dC_4 \qquad \tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$

self-dual!
$* \tilde{F}_5 = \tilde{F}_5$

A general prototype action could be:

$$S = \frac{1}{2\kappa_d^2} \int d^d x \sqrt{-g} \left\{ R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2} \sum_{m=1} \frac{1}{m!} e^{a_m \phi} \underbrace{F_m^2}_{F_{\mu_1 \cdots \mu_m} F^{\mu_1 \cdots \mu_m}} + \cdots \right\}$$

$a_m = -\frac{m-5}{2}$

(Lyng-Petersen hep-th) 9909131 ch. 3.1)

$$\begin{cases} d = 10 \\ d = 11 \quad a_m = 0, \ \phi = 0 \quad \text{M-theory!} \end{cases}$$

p-brane: source of charge for the p+1 form RR gauge field $C \Rightarrow m = p+2$ form F

3-brane: source of $C_{\mu\nu\lambda\kappa} \Rightarrow F_{\mu\nu\lambda\kappa\rho} \cong \partial_{[\mu} C_{\nu\lambda\kappa\rho]}$

$$d = 1 + p + \underbrace{d-(p+1)}_{\text{dims transverse to the p-brane}}$$

$$z^\mu = (t, x^i, y^a) = (z^0, x^1, \ldots, x^p, \underbrace{y^1, \ldots y^{d-(p+1)}}_{\text{transverse}})$$

$p=3$: $\quad 10 = 4 + 6$