The density of states can be computed for various string theories (heterotic: Bowick-Wijewardhane, PRL 54, 2485 (85))

$$\frac{dN}{dM} = \rho(M) = \frac{1}{\sqrt{\alpha'}}(\sqrt{\alpha'}M)^{-a} e^{bM} \qquad b \equiv \frac{1}{T_H} \sim \sqrt{\alpha'}$$

$M \sim \frac{1}{\sqrt{\alpha'}}$

bosonic	$a = \frac{25}{2}$	$b = 4\pi\sqrt{\alpha'}$
heterotic	$= 10$	$b = (2+\sqrt{2})\pi\sqrt{\alpha'}$
open SuStr	$= \frac{9}{2}$	$= \pi\sqrt{8\alpha'}$
closed "	$= 10$	$= \pi\sqrt{8\alpha'}$

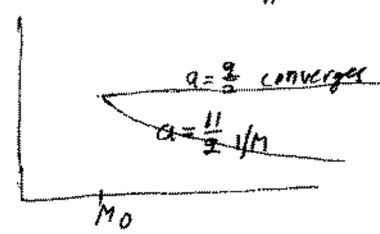
$$\log Z \cong \int_0^{M_0} dM + \int_{M_0}^{\infty} dM\, \rho(M)\, \underbrace{V_9 \int \frac{d^9k}{(2\pi)^9} e^{-\beta E}}_{V_9\left(\frac{TM}{2\pi}\right)^{\frac{9}{2}} e^{-\beta M}} \qquad \left(\frac{1}{2}\log\frac{1+e^{-\beta E}}{1-e^{-\beta E}} \approx e^{-\beta E}\right)$$

Want $T \gg M_0$

$\alpha' M^2 = N \gg 1$

$$= \frac{1}{T_H} V_9 T^9 \int_{M_0}^{\infty} dM \left(\frac{M}{T}\right)^{\frac{9}{2}} \left(\frac{M}{T_H}\right)^{-a} e^{\frac{M}{T_H} - \frac{M}{T}}$$

$$\sim T_H^{a-1} \cdot V_9\, T^{\frac{9}{2}} \int_{M_0}^{\infty} dM\, M^{\frac{9}{2}-a}\, e^{-(\beta-b)M} \qquad (\beta-b)M = t \qquad T = T_H$$



$$\sim T_H^{a-1} V_9\, T^{\frac{9}{2}} \left(\frac{1}{\beta-b}\right)^{1+\frac{9}{2}-a} \underbrace{\int_{(\beta-b)M_0}^{\infty} dt\, t^{\frac{11}{2}-a-1} e^{-t}}_{\equiv \Gamma\left((\beta-b)M_0,\, \frac{11}{2}-a\right)}$$

$$\boxed{\log Z \sim V_9 T_H^9 \left(\frac{T_H}{T_H - T}\right)^{\frac{11}{2}-a} \int_{\frac{T_H-T}{T_H T}M_0}^{\infty} dt\, t^{\frac{9}{2}-a} e^{-t}}$$

Depends sensitively on a: $\sim (T_H - T)^{a-\frac{11}{2}}$

Entropy favours joining strings

$$\frac{M}{2} \;\; \frac{M}{2} \Longrightarrow \underbrace{M}_{\text{many more states available}}$$

5. Superstrings (superficially)

$h_{ab}\,,\ X^\mu(\sigma^a) \quad \Longrightarrow \quad$ what in SuSy ? $\quad$ bosonic $\to$ fermionic

$$ds^2 = G_{\mu\nu}\, dx^\mu dx^\nu = G_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} d\sigma^a d\sigma^b \equiv h_{ab}\, d\sigma^a d\sigma^b$$

$$h_{ab} = \begin{pmatrix} \dot X^2 & \dot X \cdot X' \\ \dot X \cdot X' & X'^2 \end{pmatrix}$$

Answer:

Need 2d Clifford:

$$\{\rho^a, \rho^b\} = -2\eta^{ab} \qquad i\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad i\rho^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \underbrace{\rho^0\rho^1}_{\text{the "}\gamma_5\text{" in 2d}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \sigma^3$$

$$i\rho^a \partial_a = \begin{pmatrix} 0 & \partial_\tau - \partial_\sigma \\ -\partial_\tau - \partial_\sigma & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & \partial_- \\ -\partial_+ & 0 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}^{R}_{L} \qquad i\partial\!\!\!/ \psi = \underbrace{2\begin{pmatrix} 0 & \partial_- \\ -\partial_+ & 0 \end{pmatrix}}_{\text{real}} \underbrace{\begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}}_{\Rightarrow \text{ real fermionic}} = 0 \Rightarrow \begin{cases} \partial_- \psi_+ = 0 & \psi_+ = \psi(\sigma^+) \\ \partial_+ \psi_- = 0 & \psi_- = \psi(\sigma^-) \equiv \psi_R(\sigma^-) \end{cases}$$

$$\bar\psi = \psi^\dagger \rho^0 = \psi^T \rho^0$$

Then: $\qquad = (\psi_- \ \ \psi_+)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i(\psi_+ \ \ -\psi_-)$

consult here a discussion of spinor reps of Lorentz(4)
$\lambda_a,\ \lambda^a = \varepsilon^{ab}\lambda_b\,,\quad \lambda_a \to M_a{}^b \lambda_b$
$\lambda_{\dot a} \to M^*{}_{\dot a}{}^b \lambda_b$

$$\begin{cases} \delta X^\mu = \bar\xi \psi^\mu & \left(i\underbrace{(\xi_+ \ \ -\xi_-)}_{\text{spinor parameter}}\begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} = i\left(\xi_+\psi^\mu_- - \xi_-\psi^\mu_+\right) \right. \\ & \qquad\qquad = -\psi^\mu_+ \xi_- \ ! \\ \delta\psi^\mu = \xi\left(-i\partial\!\!\!/ X^\mu - i\rho^a \bar\psi^\mu \chi_a\right) & \text{but } \bar\psi \xi = \bar\xi \psi \ ! \\ \delta\chi_a = \partial_a \xi & \\ \delta e^a_b = -2i\,\bar\xi \rho^a \chi_a & h_{ab} = e_a^c e_b^d \eta_{cd} \quad e \sim \sqrt{h} \ ! \text{ zweibein} \end{cases}$$

one also needs a "gravitino"

simple if ξ = const, global SuSy

$$\underbrace{\underbrace{h_{ab} \quad X^\mu}_{3\ +\ d}}_{\to\ d-2 \text{ physical dofs, bosonic}} \qquad \underbrace{\psi^\mu \quad \chi_a}_{\text{one must finally get same no. of physical fermionic dofs}}$$

The action will again have the symmetries

- d-dim Poincaré
- reparametrisation invariance
- local susy $\xi(\sigma^a)$
- Weyl $\delta h_{ab} = \Lambda(\sigma^a) h_{ab}$

$$\delta \chi_a = \frac{1}{2}\Lambda(\sigma^a)\chi_a$$

$$\delta \psi^\mu = -\frac{1}{2}\Lambda(\sigma^a)\psi^\mu$$

$\Rightarrow$ superconformal gauge $h_{ab} = \eta_{ab}$, $\chi_a = 0$:

$$S = -\frac{T}{2}\int \underbrace{d\tau d\sigma}_{\frac{1}{2}d\sigma^+ d\sigma^-}\left[\partial_a X^\mu \partial^a X_\mu - \bar\psi^\mu i\rho^a\partial_a \psi_\mu\right]$$

$$\left[-4\partial_+ X\cdot\partial_- X - i(\psi_+, -\psi_-)\begin{pmatrix} 2\partial_-\psi_+ \\ -2\partial_+\psi_- \end{pmatrix}\right]$$

$$-2i(\psi_+\partial_-\psi_+ + \psi_-\partial_+\psi_-)$$

$$\boxed{S = T\int d\sigma^+ d\sigma^-\left[\partial_+ X\cdot\partial_- X + \frac{i}{2}(\psi_+\cdot\partial_-\psi_+ + \psi_-\cdot\partial_+\psi_-)\right]}$$

EOM $\quad \partial_+\partial_- X = 0 \quad$ + BC (see later)

$$\begin{cases} \partial_-\psi_+ = 0 & \psi_+ = \psi_+(\sigma^+) \\ \partial_+\psi_- = 0 & \psi_- = \psi_-(\sigma^-) \end{cases}$$

Global susy $\begin{cases} X \equiv X^\mu \\ \psi \equiv \psi^\mu \end{cases}$

$$\delta\psi = -i\,\partial\!\!\!/ X\,\xi = \begin{pmatrix} 0 & -2\partial_- X \\ 2\partial_+ X & 0 \end{pmatrix}\begin{pmatrix} \xi_- \\ \xi_+ \end{pmatrix}$$

$$= \begin{pmatrix} -2\partial_- X\,\xi_+ \\ 2\partial_+ X\,\xi_- \end{pmatrix} \equiv \begin{pmatrix} \delta\psi_- \\ \delta\psi_+ \end{pmatrix} \qquad \delta\bar\psi = +i\,\bar\xi\,\partial\!\!\!/ X$$

$$\delta X = \bar\xi\psi = i(\xi_+\psi_- - \xi_-\psi_+)$$

Taking $\xi_+ = 0$:

$$\delta\mathcal{L} = i\xi_- \left[-\partial_+\psi_+ \cdot \partial_- X + \partial_+ X \cdot \partial_- \psi_+ - \psi_+ \cdot \partial_+\partial_- X \right]$$

vanishes "on shell", using partial int & EOM

The constraints $T_{ab} = 0$ now contain fermionic terms:

$$T_{ab} = \partial_a X \cdot \partial_b X + \frac{i}{4}\bar{\Psi}\cdot(\rho_a\partial_b + \rho_b\partial_a)\psi - \frac{1}{2}\eta_{ab}\left[\partial_c X \cdot \partial^c X + \frac{i}{2}\bar{\Psi}\cdot\rho^c\partial_c\psi\right] = 0$$

(makes $T^a{}_a = 0$) (EOM of h_{ab})

$$J^a = \frac{1}{2}\rho^b\rho^a\psi\cdot\partial_b X = 0 \qquad \text{(Gravitino } \chi_a \text{ EOM)}$$

BC:

$$\delta S_F \sim \int d^2\sigma \left[\delta\psi_+ \cdot \partial_-\psi_+ + \underbrace{\psi_+ \partial_- \delta\psi_+}_{\partial_-(\psi_+\delta\psi_+) - \partial_-\psi_+\cdot\delta\psi_+} + \cdots\right]$$

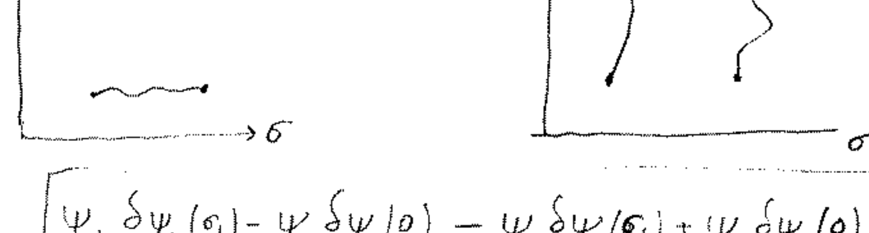
$$= \int d^2\sigma \left[2\delta\psi_+ \cdot \underbrace{\partial_-\psi_+}_{=0 \text{ EOM}} + \partial_-(\psi_+\delta\psi_+) + \cdots + \partial_+(\psi_-\delta\psi_-)\right]$$

$$\Rightarrow \text{need} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[(\partial_\tau - \partial_\sigma)(\psi_+\delta\psi_+) + \underbrace{(\partial_\tau + \partial_\sigma)(\psi_-\delta\psi_-)}_{F(\tau+\sigma)}\right] = 0$$

$\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

$$\int d\tau \int d\sigma \left[\partial_\tau f(\tau,\sigma) - \partial_\sigma f(\tau,\sigma)\right]$$

$$= \int_0^{\sigma_1} d\sigma\left[f(\tau_f\,\sigma) - f(\tau_i\,\sigma)\right] - \int_{\tau_i}^{\tau_f} d\tau\left[f(\tau\,\sigma_1) - f(\tau\,\sigma{=}0)\right]$$



This matters!
Choose
$f(\tau\,\sigma_1) = f(\tau\,0)$

$$\Rightarrow \boxed{\psi_+\delta\psi_+(\sigma_1) - \psi_+\delta\psi_+(0) - \psi_-\delta\psi_-(\sigma_1) + \psi_-\delta\psi_-(0) = 0}$$

Closed

0

2π or π

$(\sigma_1 = 2\pi)$

$$\psi_+ \delta\psi_+ - \psi_- \delta\psi_- (\sigma) = \psi_+ \delta\psi_+ - \psi_- \delta\psi_- (\sigma + 2\pi)$$

or $\sigma_1 = \pi$

ψ_+, ψ_- independently (anti)periodic

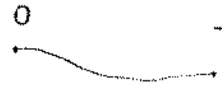
$$\psi_+ (\sigma + 2\pi) = \eta_4 \psi_+ (\sigma) \qquad \delta\psi_+ (\sigma + 2\pi) = \mp \delta\psi_+ (\sigma)$$

$$\psi_- (\sigma + 2\pi) = \eta_3 \psi_- (\sigma) \qquad \delta\psi_- (\sigma + 2\pi) = \mp \delta\psi_- (\sigma)$$

$$\left.\begin{array}{ll} \eta_3 = \eta_4 = 1 & RR \\ \eta_3 = \eta_4 = -1 & NS-NS \end{array}\right\} \Rightarrow \text{bosons after quantisation}$$

$$\begin{array}{ll} \eta_3 = -\eta_4 = 1 & R-NS \\ \eta_3 = -\eta_4 = -1 & NS-R \end{array} \Rightarrow \text{fermions}$$

Open



0 π ψ_+, ψ_- not independent

ψ_+ related to ψ_- by (anti)periodicity

$$\psi_- (\tau 0) = \eta_1 \psi_+ (\tau 0)$$

$$\psi_- (\tau \pi) = \eta_2 \psi_+ (\tau \pi)$$

$$\delta L \sim \psi_+ (\tau\pi) \delta\psi_+ (\tau\pi) - \psi_- (\tau\pi) \delta\psi_- (\tau\pi) - \left[\psi_+ (\tau 0) \delta\psi_+ (\tau 0) - \psi_- (\tau 0) \delta\psi_- (\tau 0)\right]$$

$$\psi_+ (\tau\pi) \left[\delta\psi_+ (\tau, \pi) - \eta_2 \delta\psi_- (\tau\pi)\right] - \psi_+ (\tau 0) \left[\delta\psi_+ (\tau, 0) - \eta_1 \delta\psi_- (\tau, 0)\right]$$

$$= 0$$

it seems that from $\psi_- (\tau, 0) = \eta_1 \psi_+ (\tau, 0)$ one can conclude $\delta\psi_- (\tau, 0) = \eta_1 \delta\psi_+ (\tau, 0)$ so that $[\ ] = 0$

What is the relevant Grassman algebra? Are its basis elements $(n=2)$

$0 \quad \psi_+ \quad \psi_- \quad \psi_+\psi_- = -\psi_-\psi_+$?

Then $\delta\psi_+$ and $\delta\psi_-$ must also be one of these. But what?

$$\left(\psi_+\psi_+ = 0 \Rightarrow \delta\psi_+ \psi_+ + \psi_+ \delta\psi_+ = 0\right)$$

Expansions:

Closed: $\psi_- = \sum \psi_t \, e^{-2it\sigma^-}$ (Right)

$t \in \mathbb{Z}$ R $\to e^{+2im(\sigma+\pi)}$

$t \in \mathbb{Z}+\frac{1}{2}$ NS $\to e^{+2i(m+\frac{1}{2})(\sigma+\pi)}$

$\psi_+ = \sum \tilde{\psi}_t \, e^{-2it\sigma^+}$

$(0 < \sigma < 2\pi)$

$X = x + \frac{1}{2} l_s^2 p \tau + i l_s \sum \frac{1}{m} [\alpha_m e^{-im\sigma^-} + \ldots$

or:

$\psi_- = \sum_{m \in \mathbb{Z}} d_m e^{-2im\sigma^-}$ R

$\sum_{r \in \mathbb{Z}+\frac{1}{2}} b_r e^{-2ir\sigma^-}$ NS

similarly $\bar{d}_m$ $\bar{b}_r$

Open: $\psi_\pm = \sum \psi_t \, e^{-it\sigma^\pm}$

$t \in \mathbb{Z}$ R $\quad e^{-in}$

$t \in \mathbb{Z}+\frac{1}{2}$ NS

$\psi_\pm = \frac{1}{\sqrt{2}} \sum d_m e^{-im\sigma^\pm}$ R

$= \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z}+\frac{1}{2}} b_r e^{-ir\sigma^\pm}$ NS

$$\psi_+(\sigma) = \sum \psi_t \, e^{-i(m+\frac{1}{2})\sigma} \cdot e^{-it\tau}$$

$$\psi_-(\sigma) = \sum \psi_t \, e^{+i(m+\frac{1}{2})\sigma} \, e^{-it\tau}$$

$$\psi_+(\pi) = \sum \ \cdots \ e^{-i\frac{\pi}{2}} \qquad \psi_-(\pi) = \sum \cdot \ e^{i\frac{\pi}{2}}$$

$$= e^{i\pi} \cdot \psi_+(\pi)$$

Hamiltonian:

p.63: $S = \frac{T}{2} \int d\tau \, d\sigma \left[\dot{X}^2 - X'^2 + i \, \psi_+ (\partial_\tau \psi_+ - \partial_\sigma \psi_+) + i \, \psi_- (\partial_\tau \psi_- + \partial_\sigma \psi_-) \right] = \int d\tau L$

p.39 $\quad H = \frac{\partial L}{\partial \dot{q}} \dot{q} - L \qquad \Pi = T \dot{X} \qquad \Pi_{\psi_+} = \frac{\partial L}{\partial \partial_\tau \psi_+} = i \frac{T}{2} \psi_+$

$$\boxed{\Rightarrow H = \frac{T}{2} \int d\sigma \left[\dot{X}^2 + X'^2 + i \, \psi_+ \cdot \partial_\sigma \psi_+ - i \, \psi_- \cdot \partial_\sigma \psi_- \right]}$$

Quantize fermions: $\left\{ \psi_+^\mu(\tau,\sigma), \, i \frac{T}{2} \psi_+^\nu(\tau,\sigma') \right\} = i\hbar \, \delta(\sigma - \sigma') \, \eta^{\mu\nu}$

$\Rightarrow$ Fermion anticommutators will be of type

$$\{ \psi_t^\mu, \psi_r^\nu \} = \eta^{\mu\nu} \delta_{t+r,0}$$

$aa^\dagger + a^\dagger a = 1 \qquad aa = a^\dagger a^\dagger = 0$

$a|0\rangle = 0$ If $N = a^\dagger a$ then $N a^\dagger |0\rangle = a^\dagger \underbrace{a a^\dagger}_{1-a^\dagger a} |0\rangle = a^\dagger|0\rangle$

$\{\psi^i_t, \psi^j_{-t}\} = \delta_{ij}$ choose ψ_{-t}, $t>0$, as creation op +

H

T_{++} T_{--} $J_\pm$

We had: p. 40, 36 $\quad H = L_0 + \bar{L}_0 - 2a \quad$ closed

$$L_0 = \underbrace{\frac{1}{2}\alpha_0\cdot\alpha_0}_{\frac{1}{8} l_s^2 p^2} + \underbrace{\sum_{m=1}^{\infty} \alpha_{-m}\cdot\alpha_{+m}}_{\sum_1^\infty m\, a_m^\dagger \cdot a_m \equiv N} \qquad \left(L_k = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{k-n}\cdot\alpha_n\right)$$

$$H = \frac{1}{4} l_s^2 p^2 + N + \bar{N} - 2a$$

p. 51
$$H = L_0 - a = \frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_1^\infty \alpha_{-m}\cdot\alpha_m - a = \frac{1}{2} l_s^2 p^2 + N - a \qquad \text{open}$$

Now add fermionic terms:

p. 37, 64 without T $\quad T_{++} = \partial_+ X\cdot\partial_+ X + \frac{i}{2}\psi_+\cdot\partial_+\psi \qquad T_{--} = \partial_- X\cdot\partial_- X + \frac{i}{2}\psi_-\cdot\partial_-\psi_-$

$J_- = \psi_-\cdot\partial_- X \qquad J_+ = \psi_+\cdot\partial_+ X \qquad$ supercurrent

Fourier of $T_{++}(\sigma^+) \to \bar{L}_m \qquad T_{--}(\sigma^-) \to L_m$

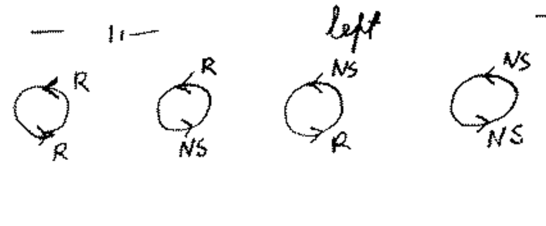
$J_-(\sigma^-) \to G_m \qquad J_+(\sigma^+) \to \bar{G}_m$

+ $m>0$

Open $0<\sigma<\pi$:
d_{-m}: create a periodic (R) fermionic excitation $\sim \cos m\sigma$
b_{-m}: " an antiperiodic (NS) — " — $\sim \cos(m+\frac{1}{2})\sigma$

Closed: $0<\sigma<2\pi$:
d_{-m}: create a periodic (R) right-moving fermionic excitation $\sim e^{-2im\sigma^-}$
$\bar{d}_{-m}$: — " — left — — " — $\sim e^{-2im\sigma^+}$
b_{-m}: — " — an antiperiodic (NS) right — — " — $\sim e^{-2im\sigma^-}$
$\bar{b}_{-m}$: — " — left — " —

Sectors:



$$\psi_t \to \overset{t\in\mathbb{Z}}{d_n} \text{ or } \overset{t\in\mathbb{Z}+\frac{1}{2}}{b_n}$$

$$\begin{cases} L_m = \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{-p}\cdot\alpha_{m+p} + \frac{1}{2}\sum_t \left(\frac{m}{2}+t\right)\psi_{-t}\cdot\psi_{m+t} & m>0 \\ \bar{L}_m = \cdots \end{cases}$$

closed: $L_0 = \frac{l_s^2}{8}p^2 + \sum_{m=1}^{\infty}\alpha_{-m}\cdot\alpha_m + \sum_{t>0} t\,\psi_{-t}\cdot\psi_t \qquad \bar{L}_0 = \frac{l_s^2}{8}p^2 + \sum\tilde{\alpha}_{-m}\cdot\alpha_m + \sum t\,\bar{\psi}_{-t}\cdot\bar{\psi}_t$

cf p. 45: $H = L_0 + \bar{L}_0 - 2a \quad \Rightarrow \frac{1}{2}\alpha' M^2 = \sum_1^{\infty}\alpha_{-p}\cdot\alpha_p + \sum_{t>0} t\,\psi_{-t}\cdot\psi_t - a + (\alpha \to \tilde{\alpha}, \dots)$

open $L_0 = \frac{1}{2} l_s^2 p^2 + \quad$ —— " ——

$H = L_0 - a \qquad \Rightarrow \alpha' M^2 = \sum_1^{\infty}\alpha_{-p}\cdot\alpha_p + \sum_{t>0} t\,\psi_{-t}\cdot\psi_t - a$

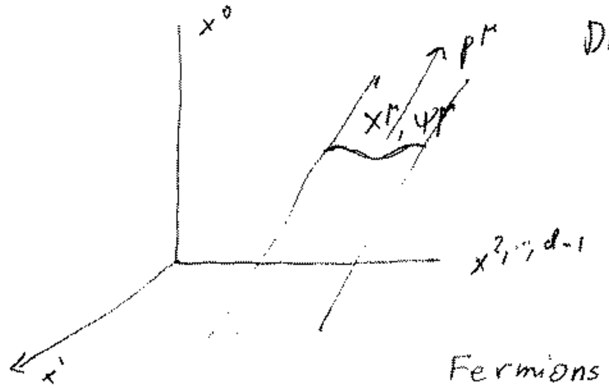
$$G_t = \sum_{p\in\mathbb{Z}} \alpha_{-p}\cdot\psi_{t+p} \qquad \bar{G}_t = \cdots$$

One gets a graded Lie algebra

$$\begin{cases} [L_m, L_n] = (m-n)L_{m+n} + \frac{d}{8}m(m^2-2a)\delta_{m+n,0} \\ [L_m, G_t] = \left(\frac{m}{2}-t\right)G_{m+t} & \begin{cases} a_R = 0 \\ a_{NS} = \frac{1}{2}\end{cases} \quad d=10 \\ \{G_t, G_s\} = 2L_{t+s} + \frac{d}{2}\left(t^2 - \frac{a}{2}\right)\delta_{t,-s} \end{cases}$$

The really interesting part of the bosonic string was getting the graviton, $M=0$ spin 2 particle $\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0p\rangle$.

How do we get fermions in d?



Dirac: $\not{p}\, u(p) = p_\mu \gamma^\mu u(p) = p_\mu \gamma^\mu_{\alpha\beta} u^\beta(p) = 0$

Dirac spinor
$$\psi = \begin{pmatrix} \psi_a \\ \bar{\chi}^{\dot{a}} \end{pmatrix} \sim \begin{matrix} D^{(\frac{1}{2},0)}(\Lambda) \\ D^{(0,\frac{1}{2})}(\Lambda) \end{matrix}$$

For Majorana $\psi = \chi$, real 2 dofs

Weyl: definite chirality, spin projection

Fermions in d ~ spinor reps. of $SO(1, d-1)$

Naive derivation of a:

$$L_0 = \frac{1}{2}\sum_{n\in Z}\alpha_{-n}\cdot\alpha_n + \frac{1}{2}\sum_t t\,\psi_{-t}\cdot\psi_t$$

$$[\alpha_m^\mu, \alpha_{-m}^\nu] = m\,\eta^{\mu\nu} \qquad \underbrace{\text{include only tr. dims}} \qquad \psi_t^\mu\psi_{-t}^\nu + \psi_{-t}^\nu\psi_t^\mu = \eta^{\mu\nu}$$

$$\alpha_m\cdot\alpha_{-m} - \alpha_{-m}\cdot\alpha_m = m(d-2) \qquad \psi_t\cdot\psi_{-t} + \psi_{-t}\cdot\psi_t = d-2$$

$$= \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{m>0}\alpha_{-m}\cdot\alpha_m + \frac{1}{2}\underbrace{\sum_{m'=-m>0}\alpha_{m'}\cdot\alpha_{-m'}}_{\alpha_{-m'}\cdot\alpha_{m'}+m(d-2)} + \frac{1}{2}\sum_{t>0} + \frac{1}{2}\sum_{t'=-t>0}(-t')\underbrace{\psi_{t'}\cdot\psi_{-t'}}_{-\psi_{-t'}\psi_{t'}+d-2}$$

$$= \frac{1}{2}\alpha_0^2 + \sum_1^\infty \alpha_{-m}\cdot\alpha_m + \sum_{t>0}^\infty \psi_{-t}\cdot\psi_t + \overbrace{\frac{d-2}{2}\sum_1^\infty m - \frac{d-2}{2}\sum_{t>0} t}$$

$$= \begin{cases} 0 & \text{for } R,\ t\in Z \\ \frac{d-2}{2}\left[\zeta(-1) - \sum_{t=\frac{1}{2}}^\infty t\right] & \text{for } NS,\ t\in Z+\frac{1}{2} \end{cases}$$

You might try

$$\sum_{t=m+\frac{1}{2}}^\infty t = \sum_{m=0}^\infty (m+\tfrac{1}{2}) = \zeta(-1) + \frac{1}{2}\sum_0^\infty 1 = \zeta(-1) + \frac{1}{2} + \frac{1}{2}\sum_1^\infty 1$$

$$= \zeta(-1) + \frac{1}{2} + \frac{1}{2}\underbrace{\zeta(0)}_{-\frac{1}{2}} = -\frac{1}{12} + \frac{1}{2} - \frac{1}{4} = \frac{1}{6}$$

but correct analytic continuation gives (Lüst-Theisen, p. 155)

$$\sum_{m\geq 0}(m+a) = -\frac{1}{12}(6a^2-6a+1) = \zeta(-1, a)$$

$$\underset{a=\frac{1}{2}}{=} -\frac{1}{12}\left(\frac{3}{2} - 3 + 1\right) = \frac{1}{24}$$

(naive $= \zeta(-1) + a + a\,\zeta(0) = -\frac{1}{12} + a - \frac{a}{2} = -\frac{1}{12} + \frac{1}{2}a = -\frac{1}{12}(1-6a)$)

↑ this naive continuation misses the $-\frac{1}{2}a^2$ term!!

$$\Rightarrow -\frac{d-2}{2}\left(\frac{1}{12} + \frac{1}{24}\right) = \frac{d-2}{16} \underbrace{=\frac{1}{2}}_{a_{NS}=\frac{1}{2}} \Rightarrow d = 10$$