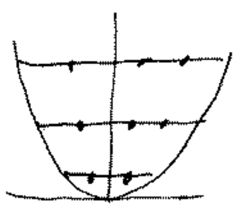
4.5 Density of states:

$$\alpha' M^2 = \underbrace{\sum_{n=1}^{\infty} \sum_{\mu=1}^{d-2}}_{\text{summed only over transverse dofs}} n\, a^{\dagger}_{n\mu} a_{n\mu} - 1 \qquad \begin{cases} a^{\mu}_{n} = \frac{1}{\sqrt{n}} \alpha^{\mu}_{n} \\ a^{\dagger\mu}_{n} = \frac{1}{\sqrt{n}} \alpha^{\mu +}_{n} = \frac{1}{\sqrt{n}} \alpha^{\mu}_{-n} \end{cases} \quad n > 0$$

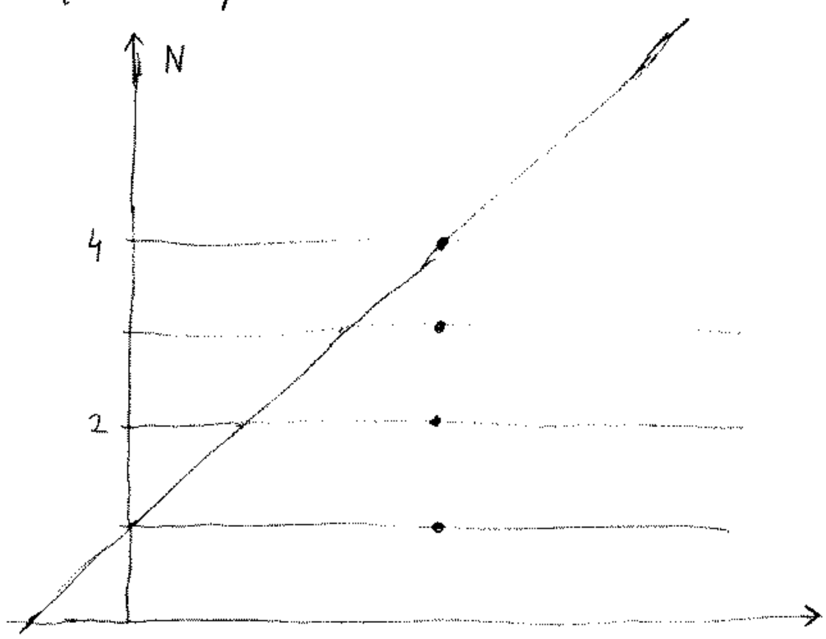
$$\sum_{1}^{\infty} n N_n = N_1 + 2N_2 + \underset{\substack{\uparrow \\ \text{\# of particles on level 3} \\ \text{each contributing 3 units to } \alpha' M^2}}{3N_3} + 4N_4 + \cdots \equiv \alpha' M^2 + 1 \equiv N$$

Compare harmonic oscillator: $N = n_1 + n_2 + \cdots + n_{max}$ particles so that n_i go to level i: $E_i = \hbar\omega\left(i + \frac{1}{2}\right)$



$$E = \hbar\omega\left[n_1 + 2n_2 + 3n_3 + \cdots + \frac{N}{2}\right] \qquad N \leq \frac{E}{\frac{1}{2}\hbar\omega}$$

Suppose $N = \alpha' M^2 + 1 \gg 1$. How many different combinations of N_i are possible?



$N = 4$ obtained by

$(N_1\ N_2\ N_3\ N_4) =$

$(0\ 0\ 0\ 1)$	$a_4^{\dagger}$
$(1\ 0\ 1\ 0)$	$a_3^{\dagger} a_1^{\dagger}$
$(0\ 2\ 0\ 0)$	$(a_2^{\dagger})^2$
$(4\ 0\ 0\ 0)$	$(a_1^{\dagger})^4$
$(2, 1, 0, 0)$	$(a_1^{\dagger})^2 a_2^{\dagger}$

5 different ways of partitioning 4 in integers so that their sum is 4

$p(N) = p(4) = 5$

$p(5) = 7$

$N = 5$: $\quad 5 \quad 4+1 \quad 3+1+1 \quad 2+1+1+1 \quad 1+1+1+1+1$

$\qquad\qquad 3+2 \quad 2+2+1$

$p(10) = 42 \qquad p(100) = 190\,569\,292$

What is $p(N)$ analytically (at large N)?

Claim: the generating function of $p(N)$ is

$$Z(x) = \prod_{N=1}^{\infty} \frac{1}{1-x^N} = \sum_{1}^{\infty} p(N)\, x^N$$

$$= \frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1}{1-x^3}\cdots$$

$$= (1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^3+x^6+\cdots)(1+x^4+\cdots)(1+x^5+\cdots)$$

Compare the grand partition function $\mathcal{Z}(\mu) = \sum (e^{\beta\mu})^N Z(N)$ of ideal B-E gas at $\mu = 0$

$$\mathcal{Z}(\beta, \mu=0) = \prod_{m} \frac{1}{1-e^{-\beta E_m}} = \prod_{1}^{\infty} \frac{1}{1-x^m} \qquad \text{if } x = e^{-\beta E_0}, \quad E_m = m E_0$$

$$= \int_0^{\infty} dE\, \underbrace{g(E)}_{\frac{dN}{dE}}\, e^{-\beta E}$$

$\frac{dN}{dE}$ = density of states at fixed total E

(derive this: $Z = \sum_{\text{states}} e^{-\beta E_{\text{state}}} = \sum_{m_i=0}^{\infty} e^{-\beta \underbrace{E_0 (m_1 + 2m_2 + 3m_3 + \cdots)}_{=E}}$

$$= \sum_{m_1} (e^{-\beta E_0})^{m_1} \sum_{m_2=0}^{\infty} (e^{-2\beta E_0})^{m_2} \cdots$$

$$= \sum_0^{\infty} x^{m_1} \cdot \sum_0^{\infty} (x^2)^{m_2} \cdot \sum_0^{\infty} (x^3)^{m_3} \cdots = \prod_{N=1}^{\infty} \frac{1}{1-x^N}$$

Since the upper limits are ∞, this really is $\mathcal{Z}(\mu=0) = \sum_{N=0}^{\infty} Z(N)$)

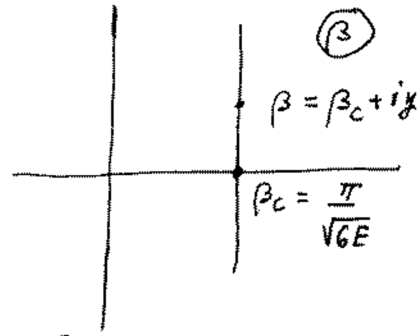
Invert the Laplace transform:

Dedekind eta function (put $x = e^{2\pi i \tau}$

$$\eta(\tau) = e^{i\pi \frac{1}{12}\tau} \prod_{1}^{\infty} \left(1 - e^{2\pi i \tau \cdot m}\right) = (-i\tau)^{-1/2}\, \eta\!\left(-\frac{1}{\tau}\right)$$

$\beta = \beta_c + iy$

$$\rho(E) = \int \frac{d\beta}{2\pi i}\, e^{\beta E + \log Z(\beta)}$$



Now do steepest descent:

$$e^{f(\beta)} = e^{f(\beta_c) + \underbrace{f'(\beta_c)}_{=0}(\beta-\beta_c) + f''(\beta_c)(\beta-\beta_c)^2 + \cdots}$$

$$f'(\beta) = E + \frac{d}{d\beta}\log Z(\beta)$$

put $E_0 = 1$

here $$\log Z(\beta) = -\sum_{N=1}^{\infty} \log(1 - e^{-\beta N})$$

$$\approx -\int_1^\infty dN\, \log(1-e^{-\beta N}) = -\frac{1}{\beta}\overbrace{\int_\beta^\infty dx\, \log(1-e^{-x})}^{= -L_2(e^{-\beta})}$$

$$\approx \frac{1}{\beta}\frac{\pi^2}{6} + \frac{1}{2}\log\frac{\beta}{2\pi} + O(\beta)$$

to get the pre-exp. factor you need the log!

$$\Rightarrow f'(\beta) = E - \frac{\pi^2}{6}\frac{1}{\beta^2} = 0 \quad \Rightarrow \quad \beta_c = \sqrt{\frac{\pi^2}{6E}}$$

$$f(\beta_c) = \sqrt{\frac{\pi^2}{6}}\sqrt{E} + \frac{\pi^2}{6}\frac{1}{\sqrt{\frac{\pi^2}{6E}}} = 2\sqrt{\frac{\pi^2}{6}E} + \frac{1}{2}\log\frac{1}{2\pi}\sqrt{\frac{\pi^2}{6E}}$$

$$f''(\beta_c) = 2\frac{\pi^2}{6\beta_c^3} = 2\frac{1}{\sqrt{\frac{\pi^2}{6}}}E^{3/2}$$

$$\Rightarrow \rho(E) = e^{f(\beta_c)}\int_{-\infty}^{\infty}\frac{dy}{2\pi}\, e^{-\frac{1}{2}f''(\beta_c)y^2} = \frac{e^{f(\beta_c)}}{\sqrt{2\pi f''(\beta_c)}}$$

$$= \frac{e^{2\sqrt{\frac{\pi^2}{6}E} + \log\frac{1}{\sqrt{2\pi}}\left(\frac{\pi^2}{6E}\right)^{\frac{1}{4}}}}{\sqrt{2\pi\, 2\left(\frac{\pi^2}{6}\right)^{-\frac{1}{2}}E^{3/2}}} = \frac{1}{4\sqrt{3}\,E}\, e^{2\sqrt{\frac{\pi^2}{6}E}}$$

with $E = N$: $$\boxed{p(N) = \frac{1}{4\sqrt{3}N}\, e^{2\pi\sqrt{\frac{N}{6}}}}$$

3d Particles in a box:

$$\rho(E,N) \sim \frac{1}{E}(VE^3)^N$$

$$\rho(E,\mu=0) \sim \frac{1}{E}(VE^3)^{\frac{1}{8}}\, e^{\#(VE^3)^{\frac{1}{4}}}$$

Now actually $\alpha' M^2 = \sum_{m=1}^{\infty} \sum_{\mu=1}^{d-2} m\, a^{+}_{\mu m} a_{\mu m} - 1$

$$\Rightarrow \prod_{N=1}^{\infty} \left[\frac{1}{1-x^N}\right]^{d-2} \qquad \text{simply } Z \Rightarrow Z^{d-2}$$

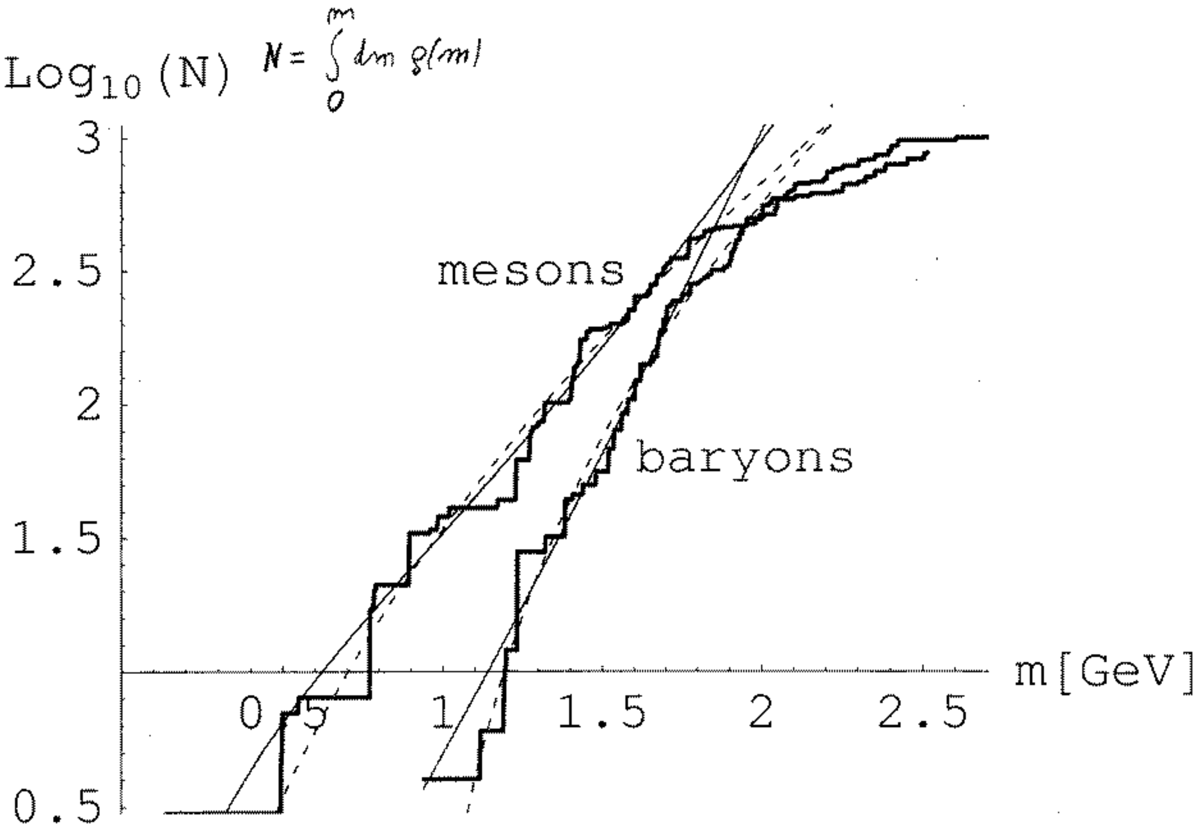
$$P(N,d) = \frac{1}{\sqrt{2}} \left(\frac{d-2}{24}\right)^{\frac{d-1}{4}} \frac{1}{N^{\frac{d+1}{4}}} e^{2\pi\sqrt{\frac{d-2}{6}N}} \underset{d=26}{=} \frac{1}{\sqrt{2}} \frac{1}{N^{\frac{27}{4}}} e^{4\pi\sqrt{N}}$$

$$\underset{N\sim\alpha' M^2}{\Rightarrow} \varrho(M) \sim \frac{M}{M^{\frac{27}{2}}} e^{4\pi\sqrt{\alpha'} M}$$ exponential density of states

$\Rightarrow$ Maximum (Hagedorn) temp. $T_H = \frac{1}{4\pi\sqrt{\alpha'}}$

$\frac{dN}{dM} = 2M\frac{dN}{dM^2} \sim$

Hagedorn mass spectrum today:



Ensemble of strings:

(a) One MB particle mass m in a box V_d:

$$Z(TVN{=}1) = \mathrm{Tr}\, e^{-\beta H} = \sum_{\text{states}} e^{-\beta E} = V \int \frac{d^d p}{(2\pi)^d}\, e^{-\beta\sqrt{\bar p^2 + m^2}}$$

$$\bar p = \frac{2\pi}{L}(n_1, \dots, n_d)$$

$$E^2 = \frac{4\pi^2}{L^2}(n_1^2 + \dots + n_d^2) + m^2$$

$$\underset{d=3}{=} \begin{cases} \dfrac{V}{\lambda_{th}^3} = V\left(\dfrac{mT}{2\pi}\right)^{\frac{3}{2}} e^{-\beta m} & T \ll m \\ \dfrac{1}{\pi^2} V T^3 & T \gg m \end{cases}$$

$$Z(N) = \frac{1}{N!} Z^N(1), \quad Z(\mu) = \sum (e^{\beta\mu})^N Z(N)$$

$$Z(1) = \frac{2}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)} V_d \int_0^\infty dp\, p^{d-1} e^{-\sqrt{(\beta p)^2 + (\beta m)^2}} \qquad \beta p = x \quad p = Tx$$

$$= \frac{2}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)} (T^d V_d) \int_0^\infty dx\, x^{d-1} e^{-\sqrt{x^2 + (\beta m)^2}} \cong K_\nu(\beta m)$$

$$e^{-\beta m - \frac{x^2}{2\beta m}}$$

$$e^{-\beta m} \underbrace{\int_0^\infty dx\, x^{d-1} e^{-\frac{x^2}{2\beta m}}}_{\approx \frac{1}{2}\Gamma\left(\frac{d}{2}\right)(2\beta m)^{\frac{d}{2}}}$$

$$Z(1,m) = \underbrace{T^d V_d}_{\text{dimless}} \cdot \overbrace{\left(\frac{\beta m}{2\pi}\right)^{\frac{d}{2}}}^{\text{dimless}} e^{-\beta m} = V_d \left(\frac{Tm}{2\pi}\right)^{\frac{d}{2}} e^{-\beta m}$$

$\uparrow$ # of spatial dims

(b) one string in a box $V_{d=25}$:

$$\alpha' M^2 = N - 1 \approx N = \sum_{m=1}^{\infty} \sum_{\mu=1}^{d-2} m\, a^+_{\mu m} a_{\mu m}$$

both oscillation modes and $\bar{p}$ are excited

$$(M^2 = 2p^+p^- + \bar{p}^2)$$

$$e^{-\beta F} = Z(\beta) = \sum_N p(N, d)\; Z(M(N))$$

$$\approx \int_{N_0} dN \frac{1}{\sqrt{2}N^{27/4}} e^{4\pi\sqrt{N}}\, V_{d-1} \left(\frac{TM}{2\pi}\right)^{\frac{25}{2}} e^{-\beta M} \Bigg|_{N=\alpha' M^2}$$

$$\approx \int_{M_0}^{\infty} dM\, 2\alpha' M \frac{1}{\sqrt{2}(\alpha')^{\frac{27}{4}} M^{\frac{27}{2}}} V_{25} \left(\frac{TM}{2\pi}\right)^{\frac{25}{2}} e^{+(4\pi\sqrt{\alpha'} - \beta)M}$$

$$\boxed{\text{powers of } M \text{ cancel !!}}$$

$$= \sqrt{2}\frac{1}{\alpha'^{\frac{23}{4}}} V_{25} \left(\frac{T}{2\pi}\right)^{\frac{25}{2}} \int_{M_0}^{\infty} dM\, e^{+(4\pi\sqrt{\alpha'} - \beta)M}$$

$$\frac{1}{T_H} - \frac{1}{T} = \frac{T - T_H}{T T_H}$$

$$\approx \frac{2^{11}}{\pi} \underbrace{V_{25} (T T_H)^{\frac{25}{2}}}_{\text{dimless!}} \frac{T}{T_H - T} e^{-4\pi\sqrt{N_0}\frac{T_H - T}{T}}$$

$$\approx \frac{2^{11}}{\pi} V_{25} T_H^{25} \frac{T_H}{T_H - T}$$

$$\boxed{\text{Maximum } T = T_H = \frac{1}{4\pi\sqrt{\alpha'}}}$$

$F = E - TS$ diverges logarithmically: $F = -T \log Z \cong T_H \log(T_H - T)$

$dF = -S\,dT - p\,dV \qquad E = F + TS$

$$S = -\frac{\partial F}{\partial T} = \frac{T_H}{T_H - T} \qquad \cong T_H \log(T_H - T) + \frac{T_H^2}{T_H - T} \Bigg\}\ \text{all diverge} \sim \frac{1}{T_H - T}$$

$$p = -\frac{\partial F}{\partial V}$$

Lots of further points to observe:
- many strings $\Rightarrow Z(N) = (Z(1))^N$, string backgrounds
- compactification of extra dims