4.3 open string

L & R are connected by BC!

$\Rightarrow$ one set α^μ_m ; L_m

$$X^\mu = x^\mu + l_s^2 p^\mu \tau + \frac{i l_s}{2} \sum_{m \neq 0} \frac{1}{m} \alpha^\mu_m \left[e^{-im(\tau-\sigma)} + e^{-im(\tau+\sigma)} \right]$$

$$+ \, i l_s \sum_{m \neq 0} \frac{1}{m} \alpha^\mu_m e^{-im\tau} \cos m\sigma$$

$$\dot{X}^\mu = \underbrace{l_s^2 p^\mu}_{\downarrow} + l_s \sum_{m \neq 0}^{\infty} \alpha^\mu_m \cos m\sigma = l_s \sum_{m \in \mathbb{Z}} \alpha^\mu_m \cos m\sigma \qquad \alpha^\mu_0 = l_s p^\mu$$

$$\int_0^\pi d\sigma \, T l_s^2 p^\mu = p^\mu$$

$$\begin{cases} L_k = \int_0^\pi d\sigma \left[e^{ik(\tau+\sigma)} T_{++} + e^{ik(\tau-\sigma)} T_{--} \right] = \frac{1}{2} \sum \alpha_{k-m} \cdot \alpha_m & k \neq 0 \\ L_0 = \frac{1}{2} \alpha_0 \cdot \alpha_0 + \sum_1^\infty \alpha_{-m} \cdot \alpha_m & H = L_0 - a \end{cases}$$

Require again $L_{k>0}|phys\rangle = 0 \quad (L_0 - a)|phys\rangle = 0$

$$(L_0 - a)|ph\rangle = \Big[\frac{1}{2} l_s^2 \underbrace{p^2}_{-M^2} + \underbrace{\sum_1^\infty \alpha_{-m} \cdot \alpha_m}_{\sum_1^\infty m a^\dagger_m a_m} - a \Big] |phys\rangle = 0$$

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{\pi l_s^2}$$

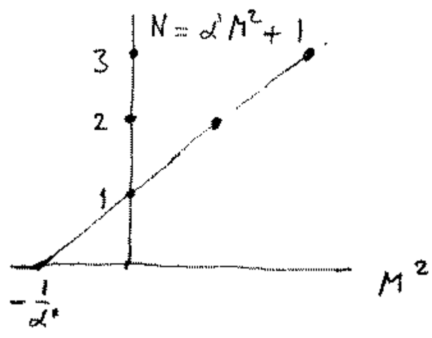
$$\frac{1}{2} l_s^2 M^2 = \sum_1^\infty m a^\dagger_m a_m - a \qquad 2\pi T = \frac{1}{\alpha'} = \frac{2}{l_s^2}$$

$$\boxed{M^2_{open} = \frac{1}{\alpha'} \Big[\sum_1^\infty m a^\dagger_m a_m - a \Big] \quad a=1 \quad M^2_{closed} = \frac{2}{\alpha'} \Big[\sum_1^\infty (m a^\dagger_m a_m + m \bar{a}^\dagger_m \bar{a}_m) - 2a \Big]}$$

$\hookrightarrow e^{-im\sigma_+}$

(1) $|0, p^\mu\rangle \quad M^2 = -\frac{1}{\alpha'}$ tachyon

(2) $a^{\mu\dagger}_1 |0, p^\mu\rangle = \alpha^\mu_{-1} |0, p^\mu\rangle \quad M^2 = 0$



$$\boxed{\begin{array}{l} x^+ = 0 \qquad p^- = \frac{1}{l_s}\alpha_0^- \\ [x^-, p^+] = -i, [x^i, p^j] = i\delta_{ij} \\ [\alpha^i_m, \alpha^j_m] = m\delta_{ij}\delta_{m,-m} \\ \alpha_m^- = \frac{1}{l_s p^+}\frac{1}{2}\sum_{p\in\mathbb{Z}} \underset{\sim}{\alpha}_{m-p}\cdot\underset{\sim}{\alpha}_p \end{array}}$$

4.4 Proving $d=26$, $a=1$:

Take light cone gauge quantisation

prove Lorentz algebra

$$[J_{\mu\nu}, J_{\alpha\beta}] = i\left(+\eta_{\mu\alpha}J_{\nu\beta} - \eta_{\nu\alpha}J_{\mu\beta} - \eta_{\mu\beta}J_{\nu\alpha} + \eta_{\nu\beta}J_{\mu\alpha}\right)$$

In particular $[J_{-i}, J_{-j}] = 0$ since $\eta_{-i} = 0$, $J_{--} = 0$

Mnemonic:

$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \qquad x_\mu p_\nu - p_\nu x_\mu = i\eta_{\mu\nu}$

$[x_\mu p_\nu - x_\nu p_\mu, x_\alpha p_\beta - x_\beta p_\alpha] =$

$$x_\mu \underbrace{p_\nu x_\alpha}_{\hat{=}-i\eta_{\nu\alpha}} p_\beta - x_\alpha \underbrace{p_\beta x_\mu}_{-i\eta_{\beta\mu}} p_\nu + x_\nu \underbrace{p_\mu x_\beta}_{-i\eta_{\mu\beta}} p_\alpha - x_\beta \underbrace{p_\alpha x_\nu}_{-i\eta_{\alpha\nu}} p_\mu - \left\{ x_\mu \underbrace{p_\nu x_\beta}_{-i\eta_{\nu\beta}} p_\alpha - x_\beta \underbrace{p_\alpha x_\mu}_{-i\eta_{\mu\alpha}} p_\nu + x_\nu \underbrace{p_\mu x_\alpha}_{-i\eta_{\mu\alpha}} p_\beta - x_\alpha \underbrace{p_\beta x_\nu}_{-i\eta_{\nu\beta}} p_\mu \right.$$

$$= -i\eta_{\nu\alpha}J_{\mu\beta} - i\eta_{\mu\beta}J_{\nu\alpha} + i\eta_{\mu\alpha}J_{\nu\beta} + i\eta_{\nu\beta}J_{\mu\alpha}$$

We had the conserved Noether currents associated with $\delta X^\mu = \omega^\mu{}_\nu X^\nu + \varepsilon^\mu$

$$P_a{}^\mu = T\partial_a X^\mu \qquad J_a^{\mu\nu} = T(X^\mu\partial_a X^\nu - X^\nu\partial_a X^\mu) \leftarrow = -J^{\nu\mu} \text{ since } \omega_{\mu\nu} = -\omega_{\nu\mu}$$

$$p^\mu = \int d\sigma\, P_\tau{}^\mu(\tau,\sigma) = \int_0^{\pi} d\sigma\, P_\tau{}^\mu(\tau,\sigma)$$

$$J^{\mu\nu} = \int_0^{\pi \text{ or } 2\pi} d\sigma\left(X^\mu P^\nu - X^\nu P^\mu\right) = T\int_0^{\pi \text{ or } 2\pi} d\sigma\left(X^\mu\dot{X}^\nu - X^\nu\dot{X}^\mu\right)$$

$\alpha = 0$ terms: $\quad X^\mu = x^\mu + \frac{1}{2}l_s^2 p^\mu\tau \quad \dot{X}^\mu = \frac{1}{2}l_s^2 p^\mu \quad \pi T = \frac{1}{l_s^2}$

$[0, 2\pi]$

$$p^\mu = T\int_0^{\sigma_1} d\sigma\, \dot{X}^\mu = T\frac{1}{2}l_s^2\sigma_1 p^\mu = p^\mu \quad \text{if } \sigma_1 = 2\pi$$

$$J^{\mu\nu} = T\int_0^{\sigma_1 = 2\pi} d\sigma\left[\left(x^\mu + \underbrace{\tfrac{1}{2}l_s^2 p^\mu\tau}_{\text{cancels}}\right)\tfrac{1}{2}l_s^2 p^\nu - (\mu\to\nu)\right]$$

$$= \underbrace{T\frac{1}{2}l_s^2\cdot 2\pi}_{=1}\left(x^\mu p^\nu - x^\nu p^\mu\right)$$

Altogether

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_1^\infty \frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right) \qquad \text{(open)}$$

$$\Rightarrow J^{-i} = x^- p^i - x^i p^- - i\sum_1^\infty \frac{1}{n}\left(\alpha^-_{-n}\alpha^i_n - \alpha^i_{-n}\alpha^-_n\right)$$

Recapitulate what has been done

$$\begin{cases} X^+(\tau,\sigma) = \quad l_s^2 p^+ \tau & \text{defines LCG; } x^+ = 0,\ \alpha^+_n = 0 \\[1em] X^-(\tau,\sigma) = \underset{\substack{\text{constant of}\\ \text{integration class'lly}\\ \Rightarrow \text{new quantum}\\ \text{variable}}}{x^-} + \underbrace{l_s^2 p^-}_{\equiv l_s\alpha_0^-} \tau + i l_s \sum_{n\neq 0} \frac{1}{n} \underbrace{\alpha^-_n}_{=\frac{1}{l_s p^+}\frac{1}{2}\sum \alpha_{n-p}\cdot\alpha_p = \frac{1}{l_s p^+} L_n} e^{-in\tau}\cos n\sigma & \\[1em] \qquad = \frac{1}{p^+}(L_0 - a) & a:\ \text{normal ordering const} \quad L_0 = \frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_1^\infty \alpha_{-p}\cdot\alpha_p \\[1em] X^i(\tau,\sigma) = x^i + \underbrace{l_s^2 p^i}_{\equiv l_s\alpha_0^i}\tau + i l_s \sum_{n\neq 0}\frac{1}{n}\alpha^i_n e^{-in\tau}\cos n\sigma & \end{cases}$$

So the physical operators are

$$p^+ \quad x^- \qquad [x^-, p^+] = -i \ \ (\eta^{+-} = -1) \qquad \Rightarrow \qquad \left[x^-, \frac{1}{p^+}\right] = i\frac{1}{(p^+)^2}$$

$$[x^i, p^j] = i\delta_{ij} \qquad [\alpha^i_n, \alpha^j_m] = n\delta_{ij}\delta_{n,-m} \qquad [x^i, \alpha^j_n] = i l_s \delta_{ij}\delta_{n0} \quad (\alpha_0^j = l_s p^j)$$

$$\Rightarrow \left[L_n, \alpha^i_m\right] = -m\alpha^i_{n+m} \qquad \left[L_n, x^i\right] = -i l_s \alpha^i_n$$

$$\left[L_n, L_m\right] = (n-m)L_{n+m} + \frac{d-2}{12} n(n^2-1)\delta_{n,-m}$$

$$\left[\alpha^-_n, \alpha^-_m\right] = \frac{1}{l_s p^+}(n-m)\alpha^-_{n+m} + \left[\frac{d-2}{12(l_s p^+)^2} n(n^2-1) - \frac{2n}{l_s p^+}\alpha_0^-\right]\delta_{n,-m}$$

!!

Zwiebach p. 230

Claim is that after careful evaluation Green-Schw-Witten I p. 99

1st done by Goddard-Rebbi -Thorn Nuovo Cimento 19A, 425 (1970) old stuff!

$$[J^{-i}, J^{-j}] = \frac{2}{l_s^2 (p^+)^2} \sum \Delta_m \left(\alpha_{-m}^i \cdot \alpha_m^j - \alpha_{-m}^j \alpha_m^i \right)$$

$$\Delta_m = m\left[1 - \frac{d-2}{24}\right] + \frac{1}{m}\left[\frac{d-2}{24} - a\right]$$

$\Rightarrow$ $d = 26$, $a = 1$ for Lorentz invariance