$$\overbrace{\langle phys | L_k | phys\rangle}^{=0} = 0 \qquad k>0 \qquad L_k$$

$$\langle phys | L_{-k} | phys\rangle = \underbrace{\langle phys | L_k^\dagger | phys\rangle}_{=0} \qquad k>0$$

only $\langle ph | L_0 | ph\rangle = a \neq 0$

Summary: For closed $\circlearrowleft$ $\sigma = 0 = 2\pi$

$$X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau + \frac{i}{2} l_s \sum_{\substack{m\neq 0\\ -\infty}}^{\infty} \frac{1}{m}\left[\alpha_m^\mu e^{-i2m(\tau-\sigma)} + \bar\alpha_m^\mu e^{-i2m(\tau+\sigma)}\right]$$

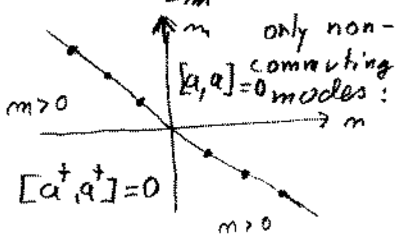
$\boxed{\tfrac{1}{2}}$ $\boxed{1}$

$$\alpha_m^{\mu\dagger} = \alpha_{-m}^\mu$$

(reality, herm. of X^μ)

$$[\alpha_m^\mu, \underbrace{\alpha_m^\nu}_{\alpha_{-m}^{\nu\dagger}}] = \sqrt{-mm}\,\delta_{m,-m}\eta^{\mu\nu} \underset{\substack{\text{choose}\\ m>0}}{\Longrightarrow} \Big[\underbrace{\frac{1}{\sqrt{m}}\alpha_m^\mu, \frac{1}{\sqrt{-m}}\alpha_{-m}^{\nu\dagger}}\Big] = \delta_{-m,m}\eta^{\mu\nu}$$

$$\underset{m>0}{[a_m, a_{-m}^\dagger]} = \delta_{m,-m} \quad \text{(forget } \eta^{\mu\nu})$$



$$=$$

$$N_m = a_m^\dagger a_m = \frac{1}{m}\alpha_{-m}\cdot\alpha_m \qquad m>0$$

$$\underbrace{\alpha_{-m}\cdot\alpha_m}_{\text{normal ordered}} = m\underbrace{N_m}_{\substack{\text{number of excitations}\\ \text{of level } m,\ m>0}}$$

$$= \alpha_m\cdot\alpha_{-m} + md \qquad mN_m \;!$$

$$\begin{cases} \frac{1}{\sqrt{m}}\alpha_{m>0}^\mu & \text{annihilates excit. } m>0,\mu \\ \frac{1}{\sqrt{m}}\alpha_m^{\mu\dagger} = \frac{1}{\sqrt{m}}\alpha_{-m}^\mu & \text{creates excit } m>0,\mu \end{cases}$$

$$\alpha_0^\mu = \frac{1}{2} l_s \hat p^\mu = \bar\alpha_0^\mu \qquad L_0 = \frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_{m=1}^\infty \alpha_{-m}\cdot\alpha_m$$

ground state of oscillators

$$\alpha_1^\mu,\ \frac{1}{\sqrt2}\alpha_2^\mu,\ \frac{1}{\sqrt3}\alpha_3^\mu, \dots \qquad \overbrace{|0, p^\mu\rangle} = 0$$

$$[\alpha_{-1}^\mu]^{m_1}, [\tfrac{1}{\sqrt2}\alpha_{-2}^\mu]^{m_2}, [\tfrac{1}{\sqrt3}\alpha_{-3}^\mu]^{m_3}, \dots |0, p^\mu\rangle = \text{state with}$$

m_1 $m=1$ excitations $\mu_1\cdots\mu_{m_1}$
m_2 $m=2$ -"- $\mu_1\mu_2\cdots\mu_{m_2}$
⋮

moving with p^μ

The states have to satisfy

$$\begin{cases} L_m |state\rangle = 0 & m>0 \\ L_0 |state\rangle = a|state\rangle & (\text{same for } \bar L) \end{cases}$$

Addendum: ζ-function

$$
\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{\Gamma(s)}\int_0^\infty dx \frac{x^{s-1}}{e^x-1} \quad \mathrm{Re}s>1 \\
&= \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s} = \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty dx\frac{x^{s-1}}{e^x+1} \quad \mathrm{Re}s>0 \\
&= 2(2\pi)^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s) = \prod_{p=\mathrm{prime}}\frac{1}{1-\frac{1}{p^s}}. \qquad (1.56)
\end{aligned}
$$

The middle line follows from the indentity

$$
\frac{1}{e^x+1} = \frac{1}{e^x-1} - \frac{2}{e^{2x}-1}. \qquad (1.57)
$$

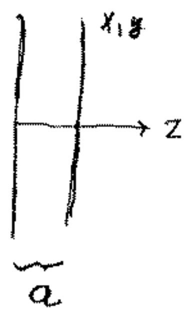
From this analytic continuation we have a beautiful result of the sum of all positive integers:

$$
\zeta(-1) = 1+2+3+4+\ldots = \frac{2}{4\pi^2}\sin(-\frac{\pi}{2})\Gamma(2)\zeta(2) = -\frac{1}{12}. \qquad (1.58)
$$

This may look like something completely unphysical, but the contrary is the case. For example, when you hear that bosonic string theories are consistent only in a world of 26 dimensions, this number 26 is the solution of $1+2+3+4+.. = 2/(2-d)$.

Compare Casimir effect, another (relative) zero-point energy computation

2 infinite plates:



$$
\begin{aligned}
E &= \sum\frac{1}{2}\hbar\omega = \frac{1}{2}\hbar c\sum\sqrt{k_x^2+k_y^2+k_z^2} \\
&= \frac{1}{2}\hbar c\sum_{n=0,\pm1,\ldots}\int\frac{L^2}{(2\pi)^2}d^2k\sqrt{\mathbf{k}_T^2+(\frac{\pi n}{a})^2}, \\
&= \frac{1}{2}\hbar c\sum_{n=0,\pm1,\ldots}\left(-\frac{L^2}{6\pi}\right)\left(\frac{\pi}{a}|n|\right)^3 \\
&= \frac{1}{2}\hbar c\left(-\frac{\pi^2}{6a^3}\right)L^2\sum_{n=0,\pm1,\ldots}|n|^3 \qquad \boxed{\zeta(-3)=\frac{1}{120}} \\
&= -\frac{\pi^2\hbar c}{720a^3}L^2, \qquad (1.72)
\end{aligned}
$$

where the momentum integral was done by setting $s=0$ in

$$
\int\frac{d^2k}{(2\pi)^2}(\mathbf{k}_T^2+m^2)^{1/2-s} = \frac{m^{3-2s}}{4\pi}\frac{1}{s-3/2}
$$

Claim: negative norm states $a_m^{0\dagger}|0\rangle$ are eliminated from the spectrum if $d=26$, $a=1$

($X^0(\tau,\sigma) = \cdots \quad \sum_n \frac{1}{n} a_n^0 \cdots$

this is like simplifying with $X^0(\tau,\sigma)=\tau$; better to choose $X^+ = a + bp^+\tau \Rightarrow$ light cone gauge)

Energy, mass of the states: $\quad H = L_0 + \bar{L}_0 - 2a$

$$|m_1, m_2, m_3, \; ; \underbrace{k^\mu}_{\text{contributes } -k^2 = k_0^2 - \vec{k}^2 \text{ to } M^2} \rangle$$

$$(L_0 - a)|m_1, \; ; k^\mu\rangle = 0 = (\bar{L}_0 - a)|m_1 \cdots ; k^\mu\rangle$$

$$\Big(\underbrace{\tfrac{1}{2}\alpha_0\cdot\alpha_0} + \overset{R}{\sum_1^\infty} \alpha_{-m}\cdot\alpha_m + \underbrace{\tfrac{1}{2}\alpha_0\cdot\alpha_0} + \overset{L}{\sum_1^\infty} \bar{\alpha}_{-m}\cdot\bar{\alpha}_m - 2a\Big)|m_1 \cdots ; k^\mu\rangle = 0$$

$$\Rightarrow \tfrac{1}{4} l_s^2 \underbrace{k^2}_{-M^2} + N_R + N_L - 2a = 0 \qquad T = \frac{1}{2\pi\alpha'} = \frac{1}{\pi l_s^2}$$

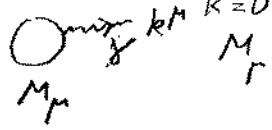
$$\Rightarrow \boxed{M^2 = \frac{4}{l_s^2}(N_R + N_L) - \frac{8}{l_s^2}a} = \boxed{\underbrace{\frac{2}{\alpha'}}_{4\pi T}\{N_R + N_L - 2a\}}$$

$$N_R \equiv \sum_1^\infty \alpha_{-m}\cdot\alpha_m = \sum_1^\infty m \underbrace{a_m^\dagger a_m}_{\text{Number operator of level } m} \qquad a = \frac{d-2}{24}$$

One sees that (as long as one buys the claim $a = 1 > 0$) that the ground state, non-oscillating quantum string, is unstable, $M^2 < 0$.

To find proper excited states one has to learn to count physical states correctly $\Rightarrow$ LCG.

photon: A^μ has 4 dofs, $A^\mu \to A^\mu + \partial^\mu \chi$ eliminates one, current conservation $\partial_\mu j^\mu = 0$, $k_\mu J^\mu = 0$, another, leaves 2 transverse polarisations for real photons



$M_\mu k^\mu = 0$

Believing in the above closed string mass formula, the 1st excited state $N_R = N_L = 1$ looks exciting: it has $M = 0$!

$$\boxed{\alpha^{\mu}_{-1}\,\bar{\alpha}^{\nu}_{-1}\,|0;k^{\alpha}\rangle}$$

$$A_{\mu\nu} \equiv \underbrace{\tfrac{1}{2}\Big(A_{\mu\nu}+A_{\nu\mu}-\tfrac{2}{d}\,\mathrm{Tr}A\cdot\eta_{\mu\nu}\Big)}_{\substack{\text{symmetric traceless}\\ \frac{1}{2}d(d+1)-1\\ G_{\mu\nu}\\ \text{graviton}\\ \text{spin } 2}}+\underbrace{\tfrac{1}{2}\big(A_{\mu\nu}-A_{\nu\mu}\big)}_{\substack{\text{antisymm}\\ \frac{1}{2}d(d-1)\\ B_{\mu\nu}\\ \text{antisymm}\\ \text{spin } 1}}+\underbrace{\tfrac{1}{d}\,\mathrm{Tr}A\cdot\eta_{\mu\nu}}_{\substack{\text{trace}\\ 1\\ \phi\,\eta_{\mu\nu}\\ \text{dilaton}\\ \text{spin } 0}}$$

$$A^{\alpha}{}_{\alpha} = \eta^{\alpha\mu}A_{\mu\alpha}$$

$$d^2 = \tfrac{1}{2}d(d+1)-1 + \tfrac{1}{2}d(d-1) + 1$$

The above has too many dofs; let us see how constraints eliminate them.

- Compare photons again:

$A_\mu(x) = \mathrm{Re}\,\varepsilon_\mu(p)\,e^{-ip\cdot x}$ $\quad \Box A_\mu = 0 \;:\; p^2 = 0$ $\qquad$ 4 dofs

gauge transf: $\quad \varepsilon^\mu \to \varepsilon^\mu + \eta p^\mu$ $\qquad$ 3 "

current cons $\quad p\cdot\varepsilon = 0$ $\qquad$ 2 dofs

or gauge fixing $\partial_\mu A^\mu = 0$ $\quad \varepsilon^\mu = (0,0,\varepsilon^2,\varepsilon^3)$, $\; p = (E,E,0,0)$

- linearised Einstein gravity in $d=4$:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \qquad g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha}h_\alpha{}^\nu + \dots$$

$h_{\mu\nu}(x) = \mathrm{Re}\,\varepsilon_{\mu\nu}(p)\,e^{-ip\cdot x}$ $\quad \Box h_{\mu\nu} = 0 \quad p^2 = 0$ $\qquad$ $10 = \frac{d(d+1)}{2}$ dofs

gauge fixing $\quad p_\alpha \varepsilon^\alpha{}_\mu = \frac{1}{2}p_\mu \varepsilon^\alpha{}_\alpha$ $\qquad$ 6 dofs

inv. under $\quad \varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \eta_\mu p_\nu + \eta_\nu p_\mu$ $\qquad$ 2 dofs = massless graviton

$$\Rightarrow \varepsilon_{\mu\nu}(p = (E,E,\underset{\sim}{0})) = \left(\begin{array}{cc|cc} 0 & 0 & \multicolumn{2}{c}{0} \\ 0 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & \varepsilon_{22} & \varepsilon_{23} \\ & & \varepsilon_{23} & -\varepsilon_{22} \end{array}\right) \leftarrow \text{tensor}$$

graviton polarisation tensor

Now introduce $\varepsilon_{\mu\nu}(k)$ and work out the effect of

(1) $$L_1 = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{1-m}\cdot\alpha_m = \alpha_0\cdot\alpha_1 + \sum_{2}^{\infty}\alpha_{1-m}\cdot\alpha_m$$
$$= \alpha_0\cdot\alpha_1 + \alpha_{-1}\cdot\alpha_2 + \alpha_{-2}\cdot\alpha_3 + \cdots$$

on $$\varepsilon_{\mu\nu}(k)\,\alpha_{-1}^{\mu}\,\bar{\alpha}_{-1}^{\nu}\,|0;k^\alpha\rangle \qquad \alpha_0^\mu = \frac{1}{2} l_s p^\mu$$

For a physical state this should vanish. The terms in $\sum_2^\infty$ commute with $\alpha_{-1}\ \bar\alpha_{-1}$ and annihilate the state

$$L_1|\text{state}\rangle = \alpha_0\cdot\alpha_1|\text{state}\rangle = \varepsilon_{\mu\nu}(k)\frac{1}{2}l_s k_\alpha \underbrace{\alpha_1^\alpha\,\alpha_{-1}^\mu}\bar\alpha_{-1}^\nu|0;k^\alpha\rangle$$

$$= \frac{1}{2}l_s \underbrace{k^\mu\varepsilon_{\mu\nu}(k)}_{=0 \text{ to make } L_1|\varphi\rangle=0}\bar\alpha_{-1}^\nu\,|0,k^\alpha\rangle \qquad \alpha_{-1}^\mu\underbrace{\alpha_1^\alpha}_{\text{annihilates}} + \eta^{\mu\alpha}$$

gauge fixing $$\boxed{k^\mu\varepsilon_{\mu\nu}(k) = k^\nu\varepsilon_{\mu\nu}(k) = 0} \qquad 2d \text{ conditions } (\nu,\mu=1,\dots,d)$$

(2) One cannot demand $L_{-1}|\text{state}\rangle = 0$ (φ) $\qquad L_1L_{-1} - L_{-1}L_1 = 2L_0$

but $\langle\varphi|L_{-1}|\varphi\rangle = \underbrace{\langle\varphi|L_1^+}_{(L_1|\varphi\rangle)^+=0}|\varphi\rangle = 0 \qquad \Downarrow\ \ \|L_{-1}|\varphi\rangle\|^2 = 2a$

Thus $L_{-1}|\varphi\rangle$ is orthogonal to any physical state $\equiv$ spurious state. Here $L_{-1} = L_1^+ = \alpha_0\cdot\alpha_{-1} + \alpha_1\cdot\alpha_{-2} + \cdots$. Apply L_{-1} to the state $\bar\alpha_{-1}^\nu|0,k^\alpha\rangle$;

$L_{-1}\,\bar\alpha_{-1}^\nu|0,k^\alpha\rangle = \frac{1}{2}l_s k_\mu\,\underbrace{\alpha_{-1}^\mu\,\bar\alpha_{-1}^\nu|0,k^\alpha\rangle}$ = spurious state and so is the linear combination $\qquad$ graviton!

$$\bar\eta_\nu\,L_{-1}\bar\alpha_{-1}^\nu|0\,k^\alpha\rangle = \frac{1}{2}l_s\,k_\mu\bar\eta_\nu\,\overbrace{\alpha_{-1}^\mu\,\bar\alpha_{-1}^\nu|0,k^\alpha\rangle}$$

$$\Rightarrow \boxed{\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \eta_\mu k_\nu + \bar\eta_\nu k_\mu} \qquad k\cdot\varepsilon = 0 \Rightarrow \overset{2}{k\cdot\eta} = \overset{2}{k\cdot\bar\eta} = 0$$

gauge transf; freedom of choosing $\varepsilon_{\mu\nu}$ $\qquad d + d - 2\cdot2 = 2(d-2)$ cond's

Total $d^2 - 2d - 2(d-2) = (d-2)^2$ dofs remain

<u>Light cone gauge</u>

$$\eta_{\mu\nu} = \left(\begin{array}{c|c} -1 & \\ & 1 \\ \hline & \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} \end{array}\right) \Rightarrow \bar{\eta}_{\mu\nu} = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ -1 & 0 & \\ \hline & 0 & \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \end{array} \end{array}\right)$$

$0 \quad 1 \quad \underbrace{\qquad\qquad}_{\text{transverse}}$

$$S_p = -\frac{T}{2}\int d^2\sigma \sqrt{-h}\, h^{ab}\partial_a X\cdot\partial_b X$$
$$= \frac{T}{2}\int d^2\sigma\, \partial_+ X\cdot\partial_- X$$

using $h_{ab} = e^{\phi}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ reparam
$= \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ Weyl
and $\sigma^{\pm} = \tau \pm \sigma$

However, there is still some reparametrisation freedom:

$$\sigma^+ \rightarrow \tilde{\sigma}^+ = f(\sigma^+)$$
$$\sigma^- \rightarrow \tilde{\sigma}^- = g(\sigma^-)$$
$$d^2\sigma = \frac{\partial(\sigma^+\sigma^-)}{\partial(\tilde{\sigma}^+\tilde{\sigma}^-)}\, d^2\tilde{\sigma} = \frac{d^2\tilde{\sigma}}{\begin{vmatrix}\partial_+ f & 0\\ 0 & \partial_- g\end{vmatrix}} = \frac{d^2\tilde{\sigma}}{\partial_+ f\cdot\partial_- g}$$
$$\frac{\partial X^\mu}{\partial\sigma^+} = \frac{\partial X^\mu}{\partial\tilde{\sigma}^+}\partial_+ f \quad \text{etc}$$
$$\int d^2\sigma\, \partial_+ X\cdot\partial_- X = \int d^2\tilde{\sigma}\, \partial_+ X(\tilde{\sigma})\cdot\partial_- X(\tilde{\sigma}) = \text{invariant}$$

One can choose a new time $\begin{cases}\tilde{\tau} = f(\sigma^+) + g(\sigma^-)\\ \partial_+\partial_-\tilde{\tau} = 0\end{cases}$

so that $X^+ = x^+ + \frac{1}{2} l_s^2 p^+ \tau$; automatically $\partial_+\partial_- X^+ = 0$

τ: $\tilde{\tau}$; now $0<\sigma<2\pi$

Constraints:

$$\dot{X}^2 + X'^2 = -2\dot{X}^+\dot{X}^- - 2X'^+X'^- + \dot{\underset{\sim}{X}}^2 + \underset{\sim}{X}'^2 = 0$$
$$\dot{X}\cdot X' = -\dot{X}^+X'^- - \dot{X}^-X'^+ + \dot{\underset{\sim}{X}}\cdot\underset{\sim}{X}' = 0$$

where now $\dot{X}^+ = \frac{1}{2}l_s^2\, p^+$, $X'^+ = 0$

$$\Rightarrow \boxed{\begin{aligned} l_s^2 p^+ \dot{X}^- &= \dot{\underset{\sim}{X}}^2 + \underset{\sim}{X}'^2 \\ \tfrac{1}{2} l_s^2 p^+ X'^- &= \dot{\underset{\sim}{X}} \cdot \underset{\sim}{X}' \end{aligned}} \Rightarrow \underbrace{\dot{X}^- \pm X'^-}_{2\dot{X}^-_{L \text{ or } R}} = \frac{1}{l_s^2 p^+} \underbrace{\left(\dot{\underset{\sim}{X}} \pm \underset{\sim}{X}'\right)^2}_{4\partial_\pm \underset{\sim}{X} \cdot \partial_\pm \underset{\sim}{X} = 4\dot{\underset{\sim}{X}}^2_{L \text{ or } R}}$$

from which $X^-(\tau,\sigma)$ can be solved in terms of the transverse dofs $\underset{\sim}{X}(\tau,\sigma) = (X^2, \dots, X^{d-1})$,

$$X^i(\tau,\sigma) = x^i + \tfrac{1}{2} l_s^2 p^i \tau + i \frac{l_s}{2} \sum \frac{1}{n} \left(\alpha_n^i e^{-in\sigma^-} + \bar{\alpha}_n^i e^{-in\sigma^+}\right)$$

warning: here $0 < \sigma < 2\pi$! $\left[\begin{aligned} & P^{\tau i}_{(\sigma)} = T\dot{X}^i = \tfrac{1}{2} T l_s^2 p^i \\ & \Rightarrow P^{\tau i} = \int_0^{2\pi} d\sigma \cdots = p^i \end{aligned}\right.$

Now constraints are explicitly solved and only physical dofs x^i p^i α_n^i $\bar{\alpha}_n^i$ are quantised. Lorentz inv. is lost!

$\boxed{\alpha_n^\pm = ?}$

$$\begin{cases} X^+ = x^+ + \tfrac{1}{2} l_s^2 p^+ \tau \\ X^- = x^- + \tfrac{1}{2} l_s^2 p^- \tau + i \frac{l_s}{2} \sum\limits_{n \neq 0} \frac{1}{n} \left[\alpha_n^- e^{-in\sigma^-} + \bar{\alpha}_n^- e^{-in\sigma^+}\right] \end{cases} \Rightarrow \boxed{\alpha_0^+ = l_s p^+} \boxed{\alpha^+_{n \neq 0} = 0}$$

$$\dot{X}^i = \tfrac{1}{2} l_s^2 p^i + \frac{l_s}{2} \sum \left(\alpha_n^i e^{-in\sigma^-} + \bar{\alpha}_n^i e^{-in\sigma^+}\right)$$

To solve α_n^- from constraints separate L, R (p.37) $\quad X = \frac{X_L(\sigma^+) + X_R(\sigma^-)}{2}$

$$\dot{X}_L = \frac{2}{l_s^2 p^+} \dot{\underset{\sim}{X}}_L \cdot \dot{\underset{\sim}{X}}_L \qquad \text{same for } R$$

$$\int_0^\pi d\sigma^- e^{ik\sigma^-} \Bigg| \; \tfrac{1}{4} l_s^2 p^- + \frac{l_s}{2} \sum_n \alpha_n^- e^{-in\sigma^-} = \frac{2}{l_s^2 p^+} \left(\frac{l_s}{2}\right)^2 \sum_{n,m} \underset{\sim}{\alpha}_n \cdot \underset{\sim}{\alpha}_m e^{-i(n+m)\sigma^-}$$

$$\frac{l_s}{2} \pi \alpha_k^- = \frac{2}{4p^+} \sum_{-\infty}^{\infty} \underset{\sim}{\alpha}_n \cdot \underset{\sim}{\alpha}_m \underbrace{\int_0^\pi d\sigma^- e^{i(k-n-m)}}_{\pi \delta_{k,n+m}}$$

$$\Rightarrow \alpha_k^- = \frac{2}{l_s p^+} \underbrace{\frac{1}{2} \sum_{n \in \mathbb{Z}} \underset{\sim}{\alpha}_{k-n} \cdot \underset{\sim}{\alpha}_n}_{\equiv L_k^T}$$

$$\boxed{\alpha_k^- = \frac{2}{l_s p^+} L_k^T} \quad \boxed{\bar{\alpha}_k^- = \frac{2}{l_s p^+} \bar{L}_k^T} \quad \text{at least for } k \neq 0$$

For covariant L_0 we normal ordered

$$L'_{k=0} = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_m = \underbrace{\overbrace{\frac{1}{2}\alpha_0\cdot\alpha_0}^{\frac{1}{8}l_s^2k^2} + \overbrace{\sum_{m=1}^{\infty}\alpha_{-m}\cdot\alpha_m}^{N}}_{L_0} - a$$

Now

$$\dot{\underset{\sim}{X}}_L = \frac{1}{2}\cdot\frac{1}{2}l_s^2 p + \frac{l_s}{2}\sum' \underset{\sim}{\alpha}_m e^{-in\sigma^+} + \dots$$

$$\Rightarrow \underset{\sim}{\alpha}_0 = \frac{1}{2}l_s p = \bar{\underset{\sim}{\alpha}}_0$$

$$\frac{1}{2}\underset{\sim}{\alpha}_0\cdot\underset{\sim}{\alpha}_0 = \frac{1}{2}\frac{1}{4}l_s^2 p^2 = \frac{1}{2}\bar{\underset{\sim}{\alpha}}_0\cdot\bar{\underset{\sim}{\alpha}}_0$$

$$\boxed{L_0^T = \frac{1}{8}l_s^2 p^2 + N_R^T \quad , \quad \bar{L}_0^T = \frac{1}{8}l_s^2 p^2 + N_L^T}$$

$$\alpha_0^- = \frac{2}{l_s p^+}(L_0^T - a) = \frac{1}{2}l_s p^- \qquad \bar{\alpha}_0^- = \frac{2}{l_s p^+}(\bar{L}_0^T - a) = \frac{1}{2}l_s p^-$$

from $X^-_{L/R} = \frac{1}{2}x^- + \frac{1}{2}\cdot\frac{1}{2}l_s^2 p^- \tau + \dots$

$$L_0^T + \bar{L}_0^T - 2a = \frac{1}{4}l_s^2\cdot 2p^+p^-$$

$$\Rightarrow M^2 = 2p^+p^- - p^2 = \frac{4}{l_s^2}(L_0^T + \bar{L}_0^T - 2a) - p^2$$

↑ cancels!

$$\boxed{M^2 = \frac{4}{l_s^2}(N_R^T + N_L^T - 2a)}$$

$$N_R^T = \sum_{m=1}^{\infty}\underset{\sim}{\alpha}_{-m}\cdot\underset{\sim}{\alpha}_m$$

Now we do not have constraints of type $L_{n>0}|\phi\rangle = 0$, $(L_0 - a)|\phi\rangle = 0$; they are explicitly worked out!