4. Quantum String

use always harmonic gauge, $h_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ ($\tau\ \sigma$) or $\begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$ ($\sigma^+\ \sigma^-$)

String dofs are

$X^\mu(\tau, \sigma)$ $\quad L = \frac{T}{2}\int d\sigma (\dot{X}^2 - X'^2)$ $\quad P^\mu = \frac{\partial L}{\partial \dot{X}_\mu} = T\dot{X}^\mu$

$$\mathcal{H} = P_\mu \frac{\partial L}{\partial \dot{X}_\mu} - \mathcal{L} = \frac{T}{2}(\dot{X}^2 + X'^2)$$

or in the expansion

$\underbrace{x^\mu \quad p^\mu}_{\text{real}}$ $\quad \underbrace{\alpha^\mu_m}_{\alpha^{\mu *}_m = \alpha^\mu_{-m}}$ $\quad \bar{\alpha}^\mu_m$

Write commutators, generalising $[x, p] = i\hbar$: $\quad (-+++\dots)$

$$[X^\mu(\tau, \sigma), P^\nu(\tau, \sigma')] = T[X^\mu(\tau, \sigma), \dot{X}^\nu(\tau, \sigma')] = i\,\delta(\sigma - \sigma')\,\eta^{\mu\nu}$$

Insert the expansions here:

$$\boxed{[x^\mu, p^\nu] = i\eta^{\mu\nu} \qquad [\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m,-n}\,\eta^{\mu\nu} \qquad \text{same for } \bar{\alpha}\bar{\alpha},\ [\alpha, \bar{\alpha}] = 0}$$

$$\alpha_m \cdot \alpha_n - \alpha_n \cdot \alpha_m = m d\,\delta_{m,-n}$$

Harmonic oscillator: $[a, a^\dagger] = 1 \quad \rightarrow [a_k, a^\dagger_\ell] = \delta_{k\ell}$

$$(aa^\dagger - a^\dagger a)|0\rangle = |0\rangle \Rightarrow aa^\dagger|0\rangle = |0\rangle \qquad a|0\rangle = 0$$

$N = a^\dagger a = aa^\dagger - 1 \qquad$ if $a^\dagger a|n\rangle = n|n\rangle$ then

$$N a|n\rangle = (aN + \underbrace{Na - aN}_{a^\dagger a a - a a^\dagger a = -a^\dagger a - 1})|n\rangle = (aN - a)|n\rangle = (n-1)a|n\rangle$$

$$N a^\dagger|n\rangle = \quad \dots \quad = (n+1)a^\dagger|n\rangle$$

Normalised states: $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$ $\quad \begin{cases} a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \\ a|n\rangle = \sqrt{n}\,|n-1\rangle \end{cases}$

coherent states: $a|z\rangle = z|z\rangle$ $\quad \Rightarrow |z\rangle = e^{za^\dagger - \frac{1}{2}|z|^2}|0\rangle$

solving this, $a = \frac{1}{\sqrt{2}}\left(\frac{x}{L} + iL\frac{p}{\hbar}\right)$, gives

$$\psi_z(x) = \frac{1}{\sqrt{L\sqrt{\pi}}}\exp\left[-\frac{1}{2L^2}(x - x_0)^2 + i\frac{p_0}{\hbar}x\right] \quad z = \frac{1}{\sqrt{2}}\left(\frac{x_0}{L} + iL\frac{p_0}{\hbar}\right) \quad L^2 = \frac{\hbar}{m\omega}$$

minimum uncertainty state

$$\left[\underbrace{\alpha^\mu_{-m}}_{=\alpha^{\mu+}_m},\ \alpha^\nu_n\right] = -m\,\delta_{m,n}\,\eta^{\mu\nu} \qquad \alpha^\mu_{-m}\alpha^\nu_m - \alpha^\nu_m\alpha^\mu_{-m} = -m\,\eta^{\mu\nu}$$

$$\alpha_m\cdot\alpha_{-m} - \alpha_{-m}\cdot\alpha_m = m\,d$$

$$\Rightarrow \left[\underbrace{\frac{1}{\sqrt{m}}\alpha^\nu_n}_{a^\nu_n},\ \underbrace{\frac{1}{\sqrt{m}}\alpha^{\mu+}_m}_{\equiv a^{\mu+}_m}\right] = \delta_{mn}\,\eta^{\mu\nu}$$

$$\Rightarrow \left[a^\nu_n,\ a^{\mu+}_m\right] = \delta_{mn}\,\eta^{\mu\nu} \qquad \eta = (-\ +++\cdots)$$

$\uparrow$ wrong sign!!

$a^{\mu+}_m|0\rangle$ = state with osc. mode μ, m.

$$\big|a^{0+}_m|0\rangle\big| = \langle 0|\underbrace{a^0_m\, a^{0+}_m}_{a^{0+}_m a^0_m - 1}|0\rangle = -1 \quad ! \qquad \text{ghost state}$$

Including x^μ, p^μ: $|0\rangle \to |p\rangle \qquad \hat{p}^\mu|p\rangle = p^\mu|p\rangle$

states: $|p\rangle \quad a^\mu_{-1}|p\rangle \quad a^\mu_{-1}a^\nu_{-1}\,a^\lambda_{-2}|p\rangle \quad \cdots$ etc. Too many states; try to select physical ones by $L_k|\text{phys}\rangle = 0$ (cannot have $L_k = 0$!).

Operators: class $\to$ quantum

how does one order non-commuting α's ?

$$L_k = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha^\mu_{k-n}\,\alpha_{n\mu} = \frac{1}{2}\Big\{\alpha^\mu_k\alpha_{0\mu} + \alpha^\mu_{k-1}\alpha_{1\mu} + \alpha^\mu_{k+1}\alpha_{-1\mu} + \cdots$$

all commute for $k \neq 0$! no problem!

$$L'_0 = \frac{1}{2}\alpha^\mu_0\alpha_{0\mu} + \frac{1}{2}\sum_{m=1}^{\infty}\alpha^\mu_{-m}\alpha_{m\mu} + \frac{1}{2}\sum_{m=-1}^{-\infty}\alpha_{-m}\cdot\alpha_m$$

$$= \frac{1}{2}\Big[\alpha_0\cdot\alpha_0 + \alpha_{-1}\cdot\alpha_1 + \underbrace{\alpha_1\cdot\alpha_{-1}}_{\eta^\mu_\mu} + \alpha_{-2}\cdot\alpha_2 + \underbrace{\alpha_2\cdot\alpha_{-2}}_{\alpha_{-2}\cdot\alpha_2 + 2d} + \cdots\Big]$$

normal order these $\alpha_{m<0}\cdot\alpha_{m>0}$

$$\alpha_m\cdot\alpha_{-m} = \alpha_{-m}\cdot\alpha_m + m\,d$$

$$= \frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_{m=1}^{\infty}\alpha_{-m}\cdot\alpha_m + \frac{1}{2}d\,\underbrace{(1+2+3+\cdots)}_{\zeta(-1) = -\frac{1}{12}}$$

$$= L_0 - a$$

The correct one is $\frac{d-2}{24}$ (only $d-2$ transv. physical dofs) and this must be $= 1$ $\Rightarrow d = 26$ to have $m=0$ spin 1 states

$$\Rightarrow H(\text{closed}) = L_0 + \bar{L}_0 - 2a \qquad (p. 35)$$

$P = \bar{L}_0 - L_0$ generates translations in σ

$$\overbrace{\lambda a + \mu b}^{C \text{ or } R}$$

Algebra $\hat{=}$ vector space with product $a \cdot b$

Grassman: needed for superstrings! $\qquad c_i c_i = 0$

m-dim vector space, basis $c_1, \dots, c_m$, product $c_i c_j = -c_j c_i$

Elements: $1 \quad c_i \quad c_i c_j \quad c_i c_j c_k \quad \cdots \quad c_1 c_2 \cdots c_m$

$$1 + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \cdots + \binom{m}{m} = (1+1)^m = 2^m$$
elements

Derivative: $\frac{\partial c_i}{\partial c_j} = \delta_{ij} \qquad \frac{\partial}{\partial c_1} c_3 c_1 c_4 = -c_3 c_4$ etc

Integral: $\int dc_i = 0 \qquad \int dc_i c_j = -\int c_j dc_i = \delta_{ij}$

$$\int dc\, f(c) = \int dc \left[f(0) + c f'(0) + (c^2 = 0) \cdot \right] = \frac{\partial f}{\partial c} \quad (!)$$

$$\int d\bar\psi_1 d\bar\psi_2 d\psi_1 d\psi_2 = e^{\sum_{ij} \bar\psi_i A_{ij} \psi_j} = A_{11} A_{22} - A_{12} A_{21} = \det A$$

$m = 4$, 16 elements

$$\int d\bar\psi\, d\psi\, e^{-\bar\psi A \psi + \bar\psi \eta + \bar\eta \psi} = \det A \; e^{\bar\eta A^{-1} \eta} \qquad \psi = (\psi_1, \dots, \psi_m) \text{ etc}$$

Lie: $[T_a, T_b] = i f_{abc} T_c \quad T_c^+ = T_c, \; U = e^{i\theta_a T_a} = (U^+)^{-1} \quad c = 1, \dots,$ finite
(real: f_{abc})

Graded Lie: $[B, B] = B \quad [B, F] = F \quad [F, F] = B$

Virasoro $\quad [L_m, L_n] = (m-n) L_{m+n} + \underbrace{\frac{c}{12} m (m^2 - 1) \delta_{m,-n}}_{\text{central extension, } c = \text{const}}$

$m, n \in \mathbb{Z}$ $\qquad$ commutes with all L_n

Ex: Show that the Jacobi identity
$$[[L_m, L_n], L_p] + [[L_n, L_p], L_m] + [[L_p, L_m], L_n] = 0$$
is satisfied by the above form

Kac-Moody: $[T_m^a, T_n^b] = i f^{abc} T_{m+n}^c + k \cdot m\, \delta^{ab} \delta_{m,-n}$

k: commutes with all T_n^a

Now work out

$$[L_k, L_p] = \frac{1}{4}\sum_{n,m=-\infty}^{\infty}[\underset{a}{\alpha^\mu_{k-n}}\,\underset{b}{\alpha^\nu_n},\ \underset{c}{\alpha^\lambda_{p-m}}\,\underset{d}{\alpha^\rho_m}]\,\eta_{\mu\nu}\eta_{\lambda\rho}$$

$[ab, cd] = abcd - cdab$ note that $[a,c]$ etc commute : $[\alpha^\mu_n, \alpha^\nu_m] = n\delta_{n,-m}\eta^{\mu\nu}$

$$= ac[b,d] + ad[b,c] + cb[a,d] + db[a,c]$$

$$= \frac{1}{4}\sum_{n,m}\Big[\alpha^\mu_{k-n}\alpha^\lambda_{p-m}\, n\eta^{\nu\rho}\delta_{n,-m} + \alpha^\mu_{k-n}\alpha^\rho_m\, n\eta^{\nu\lambda}\delta_{n,p-m}$$
$$+ \alpha^\lambda_{p-m}\alpha^\nu_n (k-n)\eta^{\mu\rho}\delta_{k-n,-m} + \alpha^\rho_m\alpha^\nu_n(k-n)\eta^{\mu\lambda}\delta_{k-n,m-p}\Big]\eta_{\mu\nu}\eta_{\lambda\rho}$$

$$= \frac{1}{2}\sum_n\left[n\,\alpha_{k-n}\cdot\alpha_{p+n} + (k-n)\,\alpha_{p+k-n}\cdot\alpha_n\right] \overset{k\neq -p}{=} \sum (k-n)\alpha_{p+k-n}\cdot\alpha_n$$

$n = k-n'$ $\quad = \frac{1}{2}\left[\text{same} - \text{same}(k \leftrightarrow p)\right] = (k-p)L_{k+p}$

$$\Rightarrow [L_k, L_p] = (k-p)\,L_{k+p} \qquad \text{if } p+k \neq 0$$

$$[L_k, L_{-k}] = \frac{1}{2}\sum_{-\infty}^{\infty}\left[n\,\alpha_{k-n}\cdot\alpha_{-k+n} + (k-n)\,\alpha_{-n}\cdot\alpha_n\right]$$

again the ordering problem since (see p. 43)

$$\alpha_n\cdot\alpha_{-n} - \alpha_{-n}\cdot\alpha_n = n\,d \ !$$

$$\Rightarrow \boxed{[L_k, L_p] = (k-p)L_{k+p} + \frac{d}{12}\,k(k^2-1)\,\delta_{k+p,0}}$$

Constraints $T_{++} = 0$ $\big(\overset{\text{class:}}{\tilde{L}_k = 0}\big)$ $\quad T_{--} = 0$ $\big(\overset{\text{class}}{L_k = 0}\big)$

can be expressed as

$$L_k|\text{phys}\rangle = 0 \quad \tilde{L}_k|\text{phys}\rangle = 0 \qquad k>0$$
$$(L_0 - a)|\text{phys}\rangle = 0 \quad (\tilde{L}_0 - a)|\text{phys}\rangle = 0 \qquad k=0$$

but due to central term

$$\langle ph|\,L_kL_{-k} - L_{-k}\underbrace{L_k\,|ph\rangle}_{=0} = 2k\langle 0|L_0|0\rangle + (\text{const} \neq 0) \neq 0$$

$$\langle ph|L_k \rightarrow L_k^\dagger|ph\rangle = L_{-k}|ph\rangle \neq 0$$

Working out the central term:

Try $k=2$:

$$\begin{cases} L_2 = \frac{1}{2}\sum \alpha_{2-m}\cdot\alpha_m = \frac{1}{2}\left[\alpha_4\cdot\alpha_{-2} + \alpha_3\cdot\alpha_{-1} + \alpha_2\cdot\alpha_0 + \alpha_1\cdot\alpha_1 + \alpha_0\cdot\alpha_2 + \alpha_{-1}\cdot\alpha_3 + \cdots\right] \\ \quad = \frac{1}{2}\alpha_1\cdot\alpha_1 + \left[\alpha_0\cdot\alpha_2 + \alpha_{-1}\cdot\alpha_3 + \alpha_{-2}\cdot\alpha_4 + \cdots\right] \\ L_{-2} = \frac{1}{2}\alpha_{-1}\cdot\alpha_{-1} + \left[\alpha_{-2}\cdot\alpha_0 + \alpha_{-3}\cdot\alpha_1 + \alpha_{-4}\cdot\alpha_2 + \cdots\right] \end{cases}$$

$$[L_2, L_{-2}] = 4L_0 + \frac{d}{2}$$

The interesting part is $\qquad \alpha_m\cdot\alpha_n - \alpha_n\cdot\alpha_m = m\,d\,\delta_{m+n,0}$

$$\frac{1}{4}\left[\alpha_1\cdot\alpha_1, \alpha_{-1}\cdot\alpha_{-1}\right] \qquad \alpha_1\cdot\alpha_{-1} - \alpha_{-1}\cdot\alpha_1 = d$$

$$= \frac{1}{4}\left\{\left[\alpha_1\cdot\alpha_1, \alpha_{-1}^{\mu}\right]\alpha_{-1\mu} + \alpha_{-1\mu}\left[\alpha_1\cdot\alpha_1, \alpha_{-1}^{\mu}\right]\right\}$$

$$= \qquad \left[\alpha_1^{\alpha}\alpha_{1\alpha}, \alpha_{-1}^{\mu}\right] = \alpha_1^{\alpha}\left[\alpha_{1\alpha}, \alpha_{-1}^{\mu}\right] + \left[\alpha_{1\alpha}, \alpha_{-1}^{\mu}\right]\alpha_1^{\alpha}$$

$$\hat{=}\ 2\alpha_1^{\alpha}\underbrace{\left[\alpha_{1\alpha}, \alpha_{-1}^{\mu}\right]}_{\eta_\alpha^{\mu}} = 2\alpha_1^{\mu}$$

$$= \frac{1}{2}\Big(\underbrace{\alpha_1\cdot\alpha_{-1}}_{d+\alpha_{-1}\cdot\alpha_1} + \underbrace{\alpha_{-1}\cdot\alpha_1}_{\text{normal ordered}}\Big)$$

$$= \underbrace{\frac{d}{2}}_{\text{this creates the central term!}} + \alpha_{-1}\cdot\alpha_1$$

Cross terms $\left[\alpha_1\cdot\alpha_1, [\ \ ]\right] = 0$

$\left[[\ \ ], [\ \ ]\right]$ create $4L_0$ (with $\alpha_{-1}\cdot\alpha_1$)

Any k: Exercise (Zwiebach p. 225, Bailin-Love p. 162)