So now we have, for example, a classical particle-like string state

Spin $J_{12} = \frac{1}{2\pi T\hbar c} M^2 \equiv \alpha' M^2$

looks like point see from distance

(perhaps size $\sim l \sim \frac{1}{\omega} \sim \frac{1}{M_{Pl}}$, $T \simeq M_{Pl}^2$, $E \sim M \sim M_{Pl}$
$\Rightarrow J_{12} = \mathcal{O}(1)$)

Classically J, M are continuous, what about the spectrum in quantum mechanics? Quantisation? 3 methods:

(1) Covariant canonical quantisation

$$[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \qquad \eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & \ddots & \\ & & & +1 \end{pmatrix}$$ ↓ Problem!

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

oscillation modes of the string (coming soon!)

$$= \omega\Big(\sqrt{\tfrac{m\omega}{2}}\,\hat{x} - i\tfrac{1}{\sqrt{2m\omega}}\,\hat{p}\Big)\underbrace{\Big(\sqrt{}\,\hat{x} + i\tfrac{\hat{p}}{\sqrt{2m\omega}}\Big)}_{=a} + \tfrac{1}{2}\hbar\omega$$

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 = \omega a^\dagger a + \frac{1}{2}\hbar\omega$$

$$[x, p] = i\hbar \qquad [a, a^\dagger] = \hbar$$

Constraints $T_{ab} = 0$ become operator eqs!

(2) Light cone quantisation:

Solve first constraints classically and quantize remaining physical variables.

Like our $X^0 = \tau$ but it is better to use light cone coordinates:

$p^2 = -p_0^2 + p_z^2 + \underline{p}^2 = -m^2 \Rightarrow p_0 = \pm\sqrt{p_z^2 + \underline{p}^2 + m^2}$ bad

$= -2p^+p^- + \underline{p}^2 = -m^2 \Rightarrow p^+ = \frac{1}{2p^-}(\underline{p}^2 + m^2)$ good

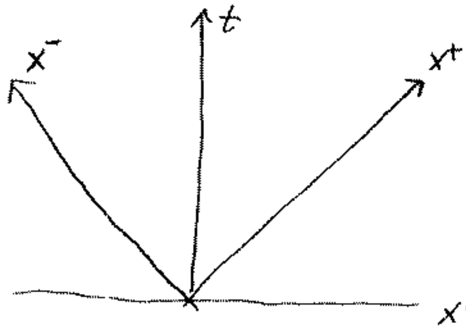
(3) Path integral quantisation

$$\underbrace{\int \mathcal{D}h_{ab}(\sigma)\,\mathcal{D}X(\sigma)}_{\substack{\text{measure breaks, in general,}\\ \text{symmetries of } S}}\, e^{\frac{i}{\hbar}S[h_{ab}(\sigma), X^\mu(\sigma)]}$$

Parenthesis: Light cone coordinates
(useful when $v \to c$!)

$$x^\mu = (t, x^1, x^2, \dots, x^d) \qquad x^\pm = \frac{x^0 \pm x^1}{\sqrt{2}} \qquad \begin{cases} x^0 = \dfrac{x^+ + x^-}{\sqrt{2}} \\ x^1 = \dfrac{x^+ - x^-}{\sqrt{2}} \end{cases}$$

same for any 4-vector



$$x \cdot y = \eta_{\mu\nu} x^\mu y^\nu$$
$$= -x^0 y^0 + x^1 y^1 + \cdots$$
$$= -\frac{1}{2}(x^+ + x^-)(y^+ + y^-) + \frac{1}{2}(x^+ - x^-)(y^+ - y^-) + \cdots$$

$$\boxed{x \cdot y = -x^- y^+ - x^+ y^- + \underset{\sim}{x} \cdot \underset{\sim}{y}} \quad \begin{cases} x^2 = -2x^+x^- + \underset{\sim}{x}^2 \\ p^2 = -2p^+p^- + \underset{\sim}{p}^2 = -m^2 \end{cases}$$

transverse to x^1

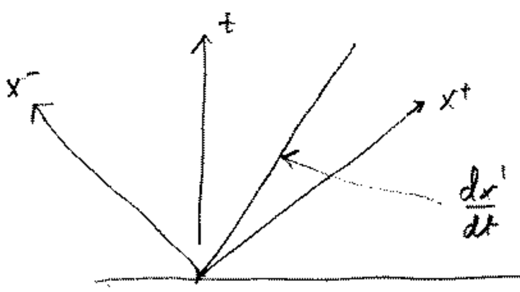
$$\Rightarrow \hat{\eta}_{\mu\nu} = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ -1 & 0 & \\ \hline 0 & & \begin{matrix} 1 & & \\ & 1 & \\ & & \ddots \; 1 \end{matrix} \end{array} \right) \qquad \equiv \tilde{\eta}_{\mu\nu}\, dx^\mu dx^\nu$$

$$\boxed{\begin{aligned} x^+ &= -x_- \\ x^- &= -x_+ \end{aligned}}$$

nicer for $\eta = (+ - - - - \cdots)$:
$$\hat{\eta} = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1, 1 \end{pmatrix}$$
$$x^+ = x_- \qquad x^- = x_+$$

$$p \cdot x = -p^0 x^0 + p^1 x^1 + \cdots$$

$$e^{-i p \cdot x} = e^{i p^0 t - i p^1 x^1 - i \underset{\sim}{p} \cdot \underset{\sim}{x}} = e^{-i \underbrace{p^-}_{\text{call this LC energy}} \underbrace{x^+}_{\text{call this LC time}} - i p^+ x^- - i \underset{\sim}{p} \cdot \underset{\sim}{x}}$$



$$\frac{dx^1}{dt} = v \Rightarrow \frac{dx^-}{dx^+} = \frac{1-v}{1+v}$$

Derivatives: Scalar field equation $\partial^2 \varphi = m\varphi \,(+\lambda\varphi^3)$

$$\partial^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu = -\partial_t^2 + \vec{\nabla}^2 = -2\partial_+\partial_- + \underset{\sim}{\partial}^2$$

$$\varphi = \varphi(\underbrace{x^+ \; x^- \; \underset{\sim}{x}}_{\text{Fourier}}) = \int_{-\infty}^{\infty} \frac{dp^+}{2\pi} \int \frac{d^{d-1}p_T}{(2\pi)^{d-1}} e^{-ix^-p^+ - i\underset{\sim}{x}\cdot\underset{\sim}{p}} \, \phi(x^+, p^+, \underset{\sim}{p})$$

$$\Rightarrow \boxed{\left[i\frac{\partial}{\partial x^+} - \frac{1}{2p^+}(\underset{\sim}{p}^2 + m^2) \right] \varphi(x^+, p^+, \underset{\sim}{p}) = 0}$$

1st order in x^+ !
Looks like Schrödinger

- LC in τ, σ :

choose no $\frac{1}{\sqrt{2}}$ here!

$$\sigma^a = (\tau, \sigma) \Rightarrow \begin{cases} \sigma^+ = \tau + \sigma \\ \sigma^- = \tau - \sigma \end{cases} \qquad \tau = \frac{1}{2}(\sigma^+ + \sigma^-) \qquad \sigma = \frac{1}{2}(\sigma^+ - \sigma^-)$$

$$ds^2 = \eta_{ab}\, d\sigma^a d\sigma^b = -d\tau^2 + d\sigma^2 = -d\sigma_+ d\sigma_-$$

$$\eta_{ab} = \begin{array}{c} \tau \;\; \sigma \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \Rightarrow h_{ab} = \begin{array}{c} + \;\;\; - \\ \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \end{array} \qquad h^{ab} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

$$v \cdot u = -\frac{1}{2} v_+ u_- - \frac{1}{2} v_- u_+ \qquad v^2 = -v_+ v_- \qquad \begin{cases} \partial_+ = \frac{\partial}{\partial \sigma^+} = \frac{1}{2}(\partial_\tau + \partial_\sigma) \\ \partial_- = \frac{\partial}{\partial \sigma^-} = \frac{1}{2}(\partial_\tau - \partial_\sigma) \end{cases}$$

$$\partial^2 = -\partial_+ \partial_-$$

$$S_p = \frac{-T}{2} \int d^2\sigma \underbrace{\sqrt{-h}}_{\frac{1}{2}}\, \underbrace{h^{ab}}_{-2} \partial_a X \cdot \partial_b X = \int d^2\sigma\, \frac{T}{2}\, \partial_+ X \cdot \partial_- X$$

$$\underset{EOM}{\Rightarrow} \partial_+ \partial_- X^\mu = 0$$

- Relativistic particle in LC :

We had: $S = -m \int d\tau = -m \int dt \frac{d\tau}{dt} = -m \int dt \sqrt{-\dot{x}^2}$

$$d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu \qquad = \sqrt{\left(\frac{dx^0}{dt}\right)^2 - \left(\frac{d\mathbf{x}}{dt}\right)^2}$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m \dot{x}_\mu}{\sqrt{-\dot{x}^2}} \qquad \text{EOM:} \quad \frac{dp^\mu}{dt} = 0 \qquad = \sqrt{1 - v^2}$$

constraint $\quad p^2 = \frac{m^2 \dot{x}^2}{-\dot{x}^2} = -m^2 \quad \Rightarrow p^\mu = \text{const} \quad$ if $x^0 = t \quad$ path $\mathbf{X}(t)$

$$= -2p^+p^- + \mathbf{p}^2 \Rightarrow \boxed{p^- = \frac{1}{2p^+}(\mathbf{p}^2 + m^2)} \quad (= -p_+ \;!)$$

Choose parametrisation not as $x^0 = t$ but as

$$\boxed{x^+ = \frac{p^+}{m} t}$$

Then $p^+ = \frac{m\dot{x}^+}{\sqrt{-\dot{x}^2}} = \frac{p^+}{\sqrt{-\dot{x}^2}} \qquad$ implies $\dot{x}^2 = -1 \quad \Rightarrow p^\mu = m\, \dot{x}^\mu$

and we can solve for

$$\begin{cases} \dot{x}^- \equiv \frac{dx^-}{dt} = \frac{1}{m} p^- \Rightarrow \\ \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \frac{1}{m} \mathbf{p} \Rightarrow \end{cases} \boxed{\begin{array}{l} x^-(t) = x_0^- + \frac{p^-}{m} t \\ \mathbf{x}(t) = \mathbf{x}_0 + \frac{\mathbf{p}}{m} t \end{array}}$$

3.5. Mode expansions, for $h_{ab} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$, X^μ any:

$$\begin{cases} \int dt\, e^{i\omega t} \Leftrightarrow \int \frac{d\omega}{2\pi} e^{-i\omega t} \\ \int dx\, e^{-ikx} \Leftrightarrow \int \frac{dk}{2\pi} e^{ikx} \end{cases}$$

conventions

$\eta = + - - -$; $\quad \phi(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip\cdot x} \phi(p)$

$$\phi(p) = \int d^4x\, e^{ip\cdot x} \phi(x)$$

$$\phi^*(p) = \phi(-p) \quad \text{if } \phi(x) = \text{real}$$

$$\int d^d x\, \varphi^m(x) = \int \prod_1^m \frac{d^d p_i}{(2\pi)^d} (2\pi)^d \delta^d(p_1 + \cdots + p_m) \varphi(p_1) \cdots \varphi(p_m)$$

for $\eta = - + + + \cdots$ $\quad \phi(x) = \int \frac{d^d p}{(2\pi)^d} e^{ip\cdot x} \phi(p) \qquad e^{ip\cdot x} = e^{-i\omega t + i\vec{p}\cdot\vec{x}}$

Do Fourier in $u = \tau + \sigma$ or $\tau - \sigma$ due to EOM $\partial_+ \partial_- X^\mu = 0$

Answer for closed: $\boxed{T = \frac{1}{2\pi\alpha'} = \frac{1}{\pi l_s^2}}$ $\boxed{2\alpha' = l_s^2}$

some constants

closed: $X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n} \Big[\underbrace{\alpha_n^\mu e^{-i2n(\tau-\sigma)}}_{\text{"right mover"}} + \underbrace{\tilde{\alpha}_n^\mu e^{-i2n(\tau+\sigma)}}_{\text{"left mover"}} \Big]$

open: $X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma$

"right" & "left" connected by BC

X^μ real $\Rightarrow \boxed{\begin{matrix} (\alpha_n^\mu)^* = \alpha_{-n}^\mu \\ \text{same for } \tilde{\alpha} \end{matrix}}$ $\quad + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu \left[e^{-in(\tau-\sigma)} + e^{-in(\tau+\sigma)} \right]$

$n = \pm 1, \pm 2, \ldots$ $\quad$ one set of α_n^μ

(1) EOM $\begin{cases} \ddot{X}^\mu - X''^\mu = 0 \\ \partial_+ \partial_- X^\mu = 0 \end{cases}$ obviously satisfied by both $\longrightarrow X^\mu = F(\tau-\sigma) + G(\tau+\sigma)$

(2) BC closed

$$X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma) \qquad e^{\pm i2n\sigma} \text{ periodic}$$

(the period is often 2π;
then $e^{-in(\tau\pm\sigma)}$ in series)

$$\text{closed}\begin{cases} \dot X^\mu = l_s^2 p^\mu + l_s \sum\limits_{m\neq 0}\left[\alpha_m^\mu e^{-i2m(\tau-\sigma)} + \tilde\alpha_m^\mu e^{-i2m(\tau+\sigma)}\right] \\ X'^\mu = \qquad + l_s \sum\limits_{m\neq 0}\left[- \qquad\qquad + \qquad\qquad \right]\end{cases}$$

$$\text{open}\begin{cases} \dot X^\mu = l_s^2 p^\mu + l_s \sum\limits_{m\neq 0}\alpha_m^\mu e^{-im\tau}\cos m\sigma = l_s\sum\limits_{m\in Z}\alpha_m^\mu e^{-im\tau}\cos m\sigma \\ X'^\mu = -i l_s \sum\limits_{m\in Z}\alpha_m^\mu \sin m\sigma\, e^{-im\tau} \qquad \text{if } \alpha_0^\mu = l_s p^\mu \end{cases}$$

"Zero mode"

$$\Rightarrow \dot X^\mu \pm X'^\mu = l_s \sum_{n\in Z}\alpha_m^\mu e^{-im(\tau\pm\sigma)}$$

BC open:  $X'^\mu(\tau,\sigma=0) = X'^\mu(\tau,\sigma=\pi) = 0$

CM position of string

$$X^\mu_{CM} = \frac{1}{\pi}\int_0^\pi d\sigma\, X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau$$

CM momentum $\quad P^\mu_{CM} = \int_0^\pi d\sigma\, T\dot X^\mu = T l_s^2 p^\mu = p^\mu$

Angular momentum $\quad J_{\mu\nu}(\tau) = \int_0^\pi d\sigma (X_\mu P_\nu^\tau - X_\nu P_\mu^\tau) = T\int_0^\pi d\sigma (X_\mu \dot X_\nu - X_\nu \dot X_\mu)$

$$\Rightarrow J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{m=1}^{\infty}\left(\alpha^\mu_{-m}\alpha^\nu_m - \alpha^\nu_{-m}\alpha^\mu_m\right) + \underbrace{\alpha\to\tilde\alpha}_{\text{for closed}}$$

Classical string dynamics: CMS position x^μ, momentum p^μ

+ {oscillation / vibration} modes described by α^μ_m

$$H = \frac{T}{2}\int_0^\pi d\sigma\left(\dot X^2 + X'^2\right) = \begin{cases}\frac{1}{2}\sum\limits_{m\in Z}\left(\alpha_{-m}\cdot\alpha_m + \tilde\alpha_{-m}\cdot\tilde\alpha_m\right) & \text{closed} \\ \sum\limits_{m\in Z}\alpha_{-m}\cdot\alpha_m & \text{open}\end{cases}$$

zero mode $\alpha\equiv 0$: $\quad \frac{T}{2}\pi\, l_s^4 p^2 = \frac{1}{2} l_s^2 p^2$

$\frac{1}{2}$?

$$\alpha_0\cdot\alpha_0 = l_s^2 p^2 = \frac{1}{\pi} T p^2$$

Constraints.

We still have to implement $T_{ab}=0 \Rightarrow \dot{X}^2+X'^2=\dot{X}\cdot X'=0$

$$\begin{array}{c|cc} & \tau & \sigma \\ \tau & \frac{1}{2}(\dot{X}^2+X'^2) & \dot{X}\cdot X' \\ \sigma & \dot{X}\cdot X' & \frac{1}{2}(\dot{X}^2+X'^2) \end{array} \Rightarrow \begin{array}{cc} \tau+\sigma & \tau-\sigma \\ T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{array}$$

$\tau^+=\tau+\sigma$, $\tau^-=\tau-\sigma$ according to general $T_{ab}=T[\partial_a X\cdot\partial_b X-\frac{1}{2}h_{ab}h^{cd}\partial_c X\cdot\partial_d X]$:

$h_{ab}=\begin{pmatrix}0&-\frac{1}{2}\\-\frac{1}{2}&0\end{pmatrix}$

$h^{ab}=\begin{pmatrix}0&-2\\-2&0\end{pmatrix}$

$\partial_\pm=\frac{1}{2}(\partial_\tau\pm\partial_\sigma)$

$$\begin{cases} T_{++}=T\partial_+X\cdot\partial_+X=T\frac{1}{4}(\dot{X}+X')\cdot(\dot{X}+X') \\ T_{--}=T\partial_-X\cdot\partial_-X=T\frac{1}{4}(\dot{X}-X')\cdot(\dot{X}-X') \\ T_{+-}=0 \end{cases}$$

So the constraints $T_{ab}=0$ or $(\dot{X}\pm X')^2=0$ are

often T is not included here!

$$\boxed{T_{++}=T\partial_+X\cdot\partial_+X=0 \quad T_{--}=T\partial_-X\cdot\partial_-X=0} \ \boxed{T_{+-}=0}$$

Closed strings: separate σ^+ & σ^-:

$$\begin{cases} X^\mu_R(\sigma^-)=\frac{1}{2}x^\mu+\frac{1}{2}l_s^2p^\mu\sigma^-+\frac{i}{2}l_s\sum\frac{1}{n}\alpha^\mu_n e^{-i2n\sigma^-} \\ X^\mu_L(\sigma^+)=\frac{1}{2}x^\mu+\frac{1}{2}l_s^2p^\mu\sigma^++\frac{i}{2}l_s\sum\frac{1}{n}\tilde{\alpha}^\mu_n e^{-i2n\sigma^+} \end{cases}$$

$$\begin{cases} T_{++}=T\partial_+X_L\cdot\partial_+X_L=T\partial_\tau X_L\cdot\partial_\tau X_L(\sigma^+) & \partial_+=\frac{1}{2}(\partial_\tau+\partial_\sigma) \\ T_{--}\hat{=}T\partial_-X_R\cdot\partial_-X_R=T\partial_\tau X_R\cdot\partial_\tau X_R(\sigma^-) & \partial_-=\frac{1}{2}(\partial_\tau-\partial_\sigma) \end{cases}$$

Since we Fourier X^μ, let us Fourier $T_{\pm\pm}$, too:

If $f(\sigma)=f(\sigma+\pi)$: $\quad f_m=\int_0^\pi d\sigma\, f(\sigma)\, e^{2im\sigma} \Leftrightarrow f(\sigma)=\sum_{m\in\mathbb{Z}} e^{-2im\sigma} f_m = f(\sigma+\pi)$

$f^*(\sigma)=f(\sigma) \Rightarrow f_m^*=f_{-m}$

$$\bar{L}_k = \frac{1}{2}\int_0^{\pi} d\sigma\, T_{++}(\tau+\sigma)\, e^{2ik(\tau+\sigma)}$$

$$\partial_+ X_L^\mu(\sigma^+) = \frac{1}{2} l_s^2 p^\mu + l_s \sum_{n\neq 0}{}' \bar{\alpha}_n^\mu e^{-i2n\sigma^+} \equiv l_s \sum \bar{\alpha}_n^\mu e^{-i2n\sigma^+}$$

$$\bar{\alpha}_0^\mu = \frac{1}{2} l_s p^\mu$$

$$\Rightarrow \bar{L}_k = \frac{T}{2} l_s^2 \sum_{n,m} \bar{\alpha}_n \cdot \bar{\alpha}_m \underbrace{\int_0^{\pi} d\sigma^+ e^{2i(k-n-m)\sigma^+}}_{\pi \delta_{k,n+m}} \qquad T\pi l_s^2 = 1$$

$$\boxed{\bar{L}_k = \frac{1}{2}\sum_n \bar{\alpha}_n \cdot \bar{\alpha}_{k-n} \qquad L_k = \frac{1}{2}\sum_n \alpha_{k-n}\cdot\alpha_n}$$

$$T_{++}(\tau,\sigma) = 0 \Rightarrow \bar{L}_k = 0 \ \forall k \qquad T_{--} = 0 \Rightarrow L_k = 0$$

p. 36: $\quad H = \frac{1}{2}\sum_n (\alpha_n \cdot \alpha_{-n} + \bar{\alpha}_n \cdot \bar{\alpha}_{-n}) = L_0 + \bar{L}_0$

Classical string in terms of Poisson brackets:

$$[q_i, p_j]_{PB} = \sum\left(\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \dots\right) = \delta_{ij}$$

$$\Rightarrow \left[X^\mu(\tau,\sigma), T\dot{X}^\nu(\tau,\sigma')\right]_{PB} = \delta(\sigma-\sigma')\,\eta^{\mu\nu}$$

$\Longrightarrow$ insert mode exp

$$[x^\mu, p^\nu]_{PB} = \eta^{\mu\nu}$$

$$[\alpha_m^\mu, \alpha_n^\nu]_{PB} = -i\eta^{\mu\nu} m\, \delta_{m,-n} = (\alpha \to \bar{\alpha})$$

$$[\alpha\ \bar{\alpha}] = 0$$

Quantize: $\quad [\ ,\ ]_{PB} \Rightarrow -i[\ ,\ ]$ commutator