Symmetry currents associated with Poincare:

S_{Pol} invariant under $X^\mu \to X^\mu(\tau\sigma) + \omega^\mu{}_\nu X^\nu(\tau\sigma) + a^\mu$

(1) Translations $\delta\varphi_k \to a^\mu$

$$\delta\mathcal{L} = -\frac{T}{2}\sqrt{-h}\,h^{ab}\Big(\partial_a X^\mu \underbrace{\partial_b a_\mu}_{=0} + \underbrace{\partial_a a_\mu}_{=0}\,\partial_b X^\mu\Big) = 0 \quad \text{trivially}$$

Current

$$a^\mu \frac{\partial\mathcal{L}}{\partial\partial_a X^\mu} = a^\mu P^a_\mu \Rightarrow \boxed{P_\mu^a = -T\sqrt{-h}\,h^{ab}\partial_b X_\mu}$$

$\partial_a P^a_\mu = 0$ is just EOM
$$= \frac{\partial P^\tau_\mu}{\partial\tau} + \frac{\partial P^\sigma_\mu}{\partial\sigma} = 0$$

(2) Lorentz: $\delta\varphi_k \to \omega_{\mu\nu}X^\nu(\tau\sigma)$ $\quad \omega_{\mu\nu} = -\omega_{\nu\mu}$

$$\omega^{\mu\nu} J^a_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_a X^\mu}\,\omega^{\mu\nu}X_\nu \Rightarrow J_{\mu\nu}{}^a \sim P^a_\mu X_\nu$$

$$\Rightarrow \boxed{\begin{aligned} J^a_{\mu\nu} &= -T\sqrt{-h}\,h^{ab}\big(X_\mu\partial_b X_\nu - X_\nu\partial_b X_\mu\big) \\ &= X_\mu P^a_\nu - X_\nu P^a_\mu \end{aligned}}$$

can by 'antisymmetrised'

Charges: $P_\mu(\tau) = \int_0^{\sigma_1} d\sigma\, P^\tau_\mu(\tau,\sigma)$

$\partial_\mu J^\mu = 0 \Rightarrow \int d^3x\, J^0$ is conserved

$$J_{\mu\nu}(\tau) = \int_0^{\sigma_1} d\sigma\,\big(X_\mu P^\tau_\nu - X_\nu P^\tau_\mu\big)$$

$$\dot{P}_\mu(\tau) = \int_0^{\sigma_1} d\sigma\,\frac{\partial P^\tau_\mu}{\partial\tau} = -\int_0^{\sigma_1} d\sigma\,\frac{\partial P^\sigma_\mu}{\partial\sigma}$$
$$= P^\sigma_\mu(0) - P^\sigma_\mu(\sigma_1)$$

$= 0$ for $\begin{cases}\text{closed} \\ \text{open with Neumann BC}\end{cases}$

3.5 String EOM:

$$\nabla^2 X^\mu = \frac{1}{\sqrt{-h}}\,\partial_a\left(\sqrt{-h}\,h^{ab}\partial_b X^\mu\right)=0$$

(1) Choose h_{ab}: : Lots of symmetries = redundant dofs. Now reduce them as much as you can so that finally only physical dofs remain!

Conformal gauge

$$h_{ab} \xrightarrow[\substack{\text{reparam. inv.}\\ \text{"diffeomorphism inv."}}]{} e^{2\Omega(\sigma)}\begin{pmatrix}-1&0\\0&1\end{pmatrix} \xrightarrow[\text{Weyl}]{} \begin{pmatrix}-1&0\\0&1\end{pmatrix}=\eta_{ab}$$

No problem classically! Affects quantisation!

$$\Rightarrow S=-\frac{T}{2}\int d^2\sigma\;\partial^a X^\mu\partial_a X_\mu=\int d^2\sigma\;\frac{T}{2}\left[\dot X^2-X'^2\right]$$

$$=\frac{T}{2}\int d^2\sigma\left(\partial_\tau X^\mu\partial_\tau X_\mu-\partial_\sigma X^\mu\partial_\sigma X_\mu\right)$$

really simple!

$$\boxed{\left(\partial_\tau^2-\partial_\sigma^2\right)X^\mu=0}\;\boxed{+\text{ boundary conditions}}$$

We chose $h_{ab}=\eta_{ab}$ but still the constraint $T_{ab}=0$ remains:

$$\left(\dot X^2\equiv\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X_\mu}{\partial\sigma_a}\ \text{etc}\right)\quad \partial_a X\cdot\partial_b X-\frac{1}{2}\eta_{ab}\,\partial^c X\cdot\partial_c X=0$$

$$T_{\tau\tau}=\dot X^2+\frac{1}{2}\left(-\dot X^2+X'^2\right)=\frac{1}{2}\left(\dot X^2+X'^2\right)=T_{\sigma\sigma}=0$$

$$T_{\sigma\tau}=\dot X\cdot X'=0$$

for solutions

$$\boxed{\begin{aligned}\dot X^2+X'^2&=0\\ \dot X\cdot X'&=0\end{aligned}}\;\boxed{\begin{aligned}P^\tau_\mu&=T\dot X_\mu\\ P^\sigma_\mu&=-T X'_\mu\end{aligned}}$$

$0=(\dot X\pm X')^2 \Leftarrow$

(2) Choose X^μ:

(a) $X^0(\tau,\sigma)=\tau=t \Longrightarrow \dot X^\mu=(1,\dot{\vec X})\quad X'=(0,\vec X')$

these imply

$$\boxed{\begin{aligned}-1+\dot{\vec X}^2+\vec X'^2&=0\\ \dot{\vec X}\cdot\vec X'&=0\\ \ddot{\vec X}-\vec X''&=0\end{aligned}}$$

$\Big\}\ T_{ab}=0$ (remaining) constraint after choosing $h_{ab}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$

$\}$ EOM of X^μ after $X^0=\tau$

also:

$$P_\mu(\tau)=\int_0^{\sigma_1}d\sigma\,P^\tau_\mu(\tau\sigma)$$

$$\Rightarrow E=P^0(\tau)=\int_0^{\sigma_1}d\sigma\,T\underbrace{\dot X^0}_{=1!}=T\sigma_1$$

$$\Rightarrow \sigma_1=\frac{E}{T}$$

$$\boxed{\vec X'(\tau,\sigma=0)=\vec X'(\tau,\sigma=\sigma_1)=0}\quad \text{open, NBC}$$

Interprete:

$\dot{\vec X}\cdot\vec X'=0 \Rightarrow \dot{\vec X}=\dfrac{\partial\vec X}{\partial t}\equiv\vec v_T$ transversal velocity

$|\vec X'|^2=\left|\dfrac{\partial\vec X}{\partial\sigma}\right|^2\equiv\left(\dfrac{ds}{d\sigma}\right)^2$ s = length parameter along string

Conformal gauge continued: can parametrize length along string by $ds = d\sigma\sqrt{1-\vec{v}_T^2}$

$$S = \int dt \int_0^{\sigma_1} d\sigma \, \frac{T}{2}\left(\dot{X}^2 - X'^2\right)$$

$$P_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = T\dot{X}_\mu \equiv P_\mu^\tau$$

$$\mathcal{H} = P_\mu \dot{X}^\mu - \mathcal{L} = T\dot{X}^2 - \frac{T}{2}\left(\dot{X}^2 - X'^2\right) = \frac{T}{2}\left(\dot{X}^2 + X'^2\right)$$

$$\Rightarrow \underset{\uparrow \text{conf gauge}}{H =} \frac{T}{2}\int_0^{\sigma_1} d\sigma \left(\dot{X}^2 + X'^2\right) \underset{\uparrow \text{for solutions}}{=} 0$$

$$P_\mu^\sigma = -TX'_\mu = -T\frac{\partial X_\mu}{\partial \sigma} = -T\sqrt{1-\vec{v}_T^2}\,\frac{\partial X_\mu}{\partial s}$$

Solve for motion of open strings:

EOM $\quad \ddot{\vec{X}} - \vec{X}'' = 0 \Rightarrow \vec{X}(\tau\sigma) = \frac{1}{2}\left[\vec{F}(\tau+\sigma) + \vec{G}(\tau-\sigma)\right]$

$$\vec{X}'(\tau,\sigma) = \frac{1}{2}\left[\vec{F}'(\tau+\sigma) - \vec{G}'(\tau-\sigma)\right]$$

End pts: $\begin{cases} \vec{X}'(\tau,0) = 0 \Rightarrow \vec{F}'(\tau) = \vec{G}'(\tau) \Rightarrow \vec{F} = \vec{G} \\ \vec{X}'(\tau,\sigma_1) = 0 \quad \vec{F}'(\tau+\sigma_1) = \vec{F}'(\tau-\sigma_1) \Rightarrow \vec{F}'(u) \text{ periodic with period } 2\sigma \end{cases}$

$$\Downarrow$$

$$\vec{F}(u+2\sigma_1) = \vec{F}(u) + 2\sigma_1\vec{v}$$ ↙ some const

$$(\dot{\vec{X}} \pm \vec{X}')^2 = 1 \Rightarrow \dot{\vec{X}} \pm \vec{X}' = \frac{1}{2}\left[\vec{F}'(\tau+\sigma) + \vec{F}'(\tau-\sigma)\right] \pm \frac{1}{2}\left[\vec{F}'(\tau+\sigma) - \vec{F}'(\tau-\sigma)\right]$$

$$= \vec{F}'(\tau\pm\sigma) \Rightarrow \left|\vec{F}'(u)\right|^2 = 1$$

Summary:

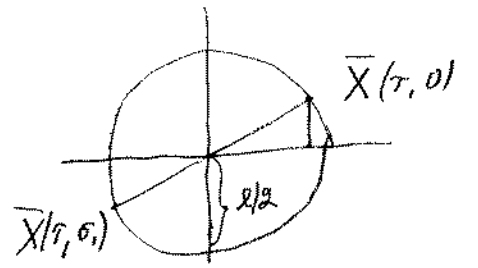
$$\boxed{\begin{aligned} &\vec{X}(\tau,\sigma) = \frac{1}{2}\left[\vec{F}(\tau+\sigma) + \vec{F}(\tau-\sigma)\right] \\ &\left|\vec{F}'(u)\right|^2 = 1 \qquad \vec{F}(u+2\sigma_1) = \vec{F}(u) + 2\sigma_1\vec{v} \end{aligned}}$$

$\sigma = 0$: $\quad \vec{X}(\tau,0) = \vec{F}(\tau)$ = motion of starting point

$$\vec{X}(\tau+2\sigma_1, 0) = \vec{F}(\tau+2\sigma_1) = \vec{X}(\tau,0) + 2\sigma_1\vec{v}$$

motion of $\sigma = 0$ gives the entire motion!

Special case: Rotating string



$$\vec{X}(t,0) = \frac{\ell}{2}(\cos\omega t,\ \sin\omega t) = \vec{F}(t)$$

$$\vec{F}'(t) = \frac{\ell}{2}(-\omega\sin\omega t,\ \omega\cos\omega t)$$

$$|\vec{F}'|^2 = 1 \Rightarrow \frac{\ell\omega}{2} = 1,\quad \ell = \frac{2}{\omega}$$

$\left(v = \omega\frac{\ell}{2} = 1 \right.$ end pts move with $\left. v = c = 1 \right)$

$$\bar{F}(t+2\sigma_1) = \bar{F}(t) \Rightarrow e^{i\omega(t+2\sigma_1)} = e^{i\omega t}$$

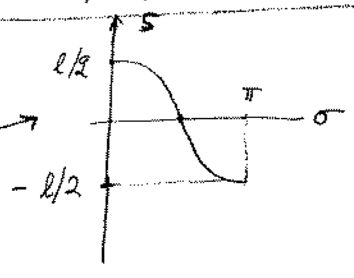
$$2\omega\sigma_1 = 2\pi n \Rightarrow \omega = \frac{\pi}{\sigma_1}\cdot n$$

$$\vec{X}(t,0) = \frac{\sigma_1}{\pi}\left(\cos\frac{\pi t}{\sigma_1},\ \sin\frac{\pi t}{\sigma_1}\right) \quad \ell = \frac{2}{\omega} = \frac{2\sigma_1}{\pi} \quad n = \pm1, \pm2, \dots \Rightarrow n = 1$$

$$\Rightarrow \boxed{\vec{X}(t,\sigma) = \underbrace{\frac{\sigma_1}{\pi}}_{=\frac{\ell}{2}}\cos\frac{\pi\sigma}{\sigma_1}\left(\cos\frac{\pi t}{\sigma_1},\ \sin\frac{\pi t}{\sigma_1}\right) \underset{\sigma_1=\pi,\ \ell=2}{=} \cos\sigma(\cos t,\ \sin t)}$$

Energy = ?
= Mass

parametrize $\sigma \to -\frac{\ell}{2} < s < \frac{\ell}{2}$



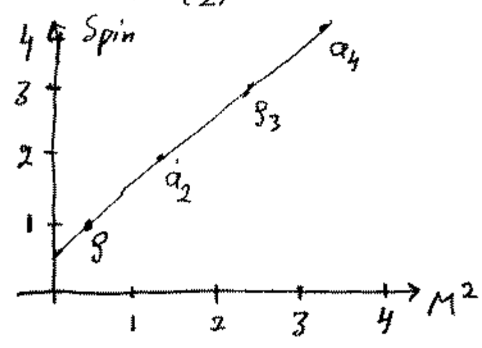
$$\mathcal{E}(s) = \frac{T}{\sqrt{1-\left(\frac{s}{\ell/2}\right)^2}} \quad \Rightarrow E = \frac{\pi}{2}T\cdot\ell = \pi T\cdot\frac{\ell}{2} \geq T\cdot\frac{\ell}{2}$$

$$\frac{ds}{d\sigma} = \sqrt{1-v_T^2} \Rightarrow H = T\int d\sigma = \int ds\,\frac{T}{\sqrt{1-v_T^2}}$$

Ang. momentum: $J_{12} = \frac{\pi}{2}T\left(\frac{\ell}{2}\right)^2 = \alpha' E^2 = \alpha'\pi^2T^2\left(\frac{\ell}{2}\right)^2$

if $\alpha' \equiv \frac{1}{2\pi T\hbar c}$ = Regge slope

Regge trajectory



Chew-Frautschi plot $I=1,\ S=0,\ B=0$ from PDG

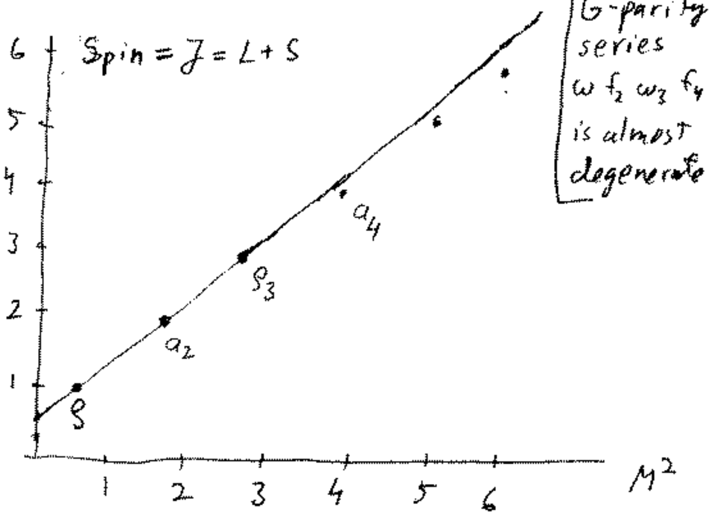
$^{2S+1}L_J$		m^2	J^{PC}	I^G	$\alpha' = (J-\frac{1}{2})/M^2$
$^3S_1$	ρ	0.59	1^{--}	1^+	0.85
$^3P_2$	a_2	1.74	2^{++}	1^-	0.86
$^3D_3$	ρ_3	2.86	3^{--}	1^+	0.87
$^3F_4$	a_4	4.16	4^{++}	1^-	0.84
	ρ_5	2.35^2 5.52	5^{--}	1^+	0.81
	a_6	2.45^2 6.00	6^{++}	1^-	0.92

PDG: "omitted from summary table" "needs confirmation"

search for $\pi^- p \to \underbrace{\underbrace{\pi^+\pi^-\pi^0}_{\omega}\ \pi^0}_{\rho_5} n$

Remarkably (?) good fit to

$$\boxed{\alpha(t) = \frac{1}{2} + 0.85 M^2}$$



the opposite G-parity series $\omega\ f_2\ \omega_3\ f_4 \ldots$ is almost degenerate

In 1970 Loma Koli Proceedings
p. 132 an experimental plot was given up to $J = 13$!
Real optimism! → But too early!

Baryons

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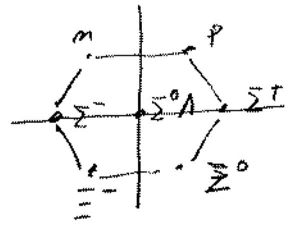

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Δ^{++}

$\Omega = sss$