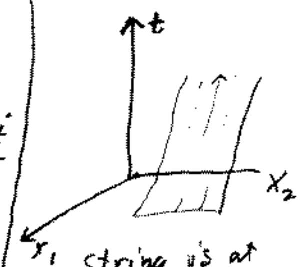
More interpretation: transverse velocity

Choose again $X^0(\tau,\sigma) = \tau = t$, i.e. $X^\mu = (\tau = t, \underbrace{\vec X(\tau,\sigma)}_{\text{all } X^i})$

we are in one fixed Lorentz frame x^μ:



string is at $x^\mu = X^\mu(\tau,\sigma)$ and we choose $\tau = x^0 = t$ "static gauge"

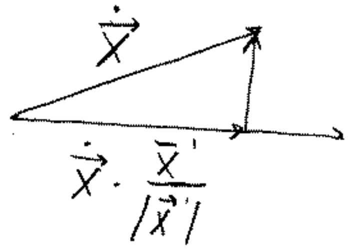
$$\Rightarrow \dot X^\mu = (1, \dot{\vec X}) \qquad \dot X^2 = -1 + \dot{\vec X}^2$$

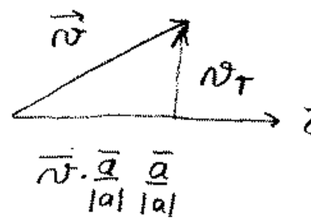
$$X'^\mu = (0, \vec X') \qquad X'^2 = \vec X'^2$$

$$\dot X \cdot X' = \dot{\vec X}\cdot \vec X' \equiv \frac{\partial X^i}{\partial \tau}\frac{\partial X_i}{\partial \sigma}$$

$$\sqrt{\ } = \sqrt{(\dot{\vec X}\cdot\vec X')^2 + (1-\dot{\vec X}^2)\vec X'^2}$$

$$= |\vec X'| \sqrt{1 - \left[\dot{\vec X}^2 - \left(\dot{\vec X}\cdot\frac{\vec X'}{|\vec X'|}\right)^2\right]}$$

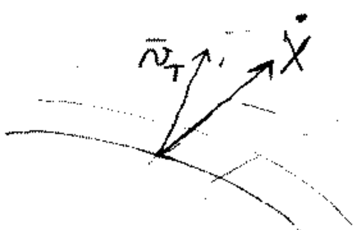




$$\bar v_T^2 = |\bar v - \hat a\,\bar v\cdot\hat a|^2$$
$$= \bar v^2 - 2(\bar v\cdot\hat a)^2 + |\bar v\cdot\hat a|^2$$
$$= \bar v^2 - (\bar v\cdot\hat a)^2$$

$$= \left|\frac{\partial\bar X}{\partial\sigma}\right|\sqrt{1-\bar v_T^2}$$

$$\bar v_T^2 = \left(\frac{\partial\bar X}{\partial t}\right)^2 - \left(\frac{\partial\bar X}{\partial t}\cdot\frac{\partial\bar X}{\partial\sigma}\frac{1}{\left|\frac{\partial\bar X}{\partial\sigma}\right|}\right)^2$$



$$m\int dt\sqrt{1-\bar v^2} \Longrightarrow -T\int_{t_i}^{t_f} dt\int_0^{\pi} d\sigma\left|\frac{\partial\bar X}{\partial\sigma}\right|\sqrt{1-\bar v_T^2}$$

Interpretation 2: motion of open string end points

$$P^\sigma_\mu = \frac{\partial L}{\partial X'^\mu} = -T\,\frac{\dot X\cdot X'\,\dot X_\mu - \dot X^2 X'_\mu}{\sqrt{}} = 0 \quad \text{at } \sigma = 0, \pi$$

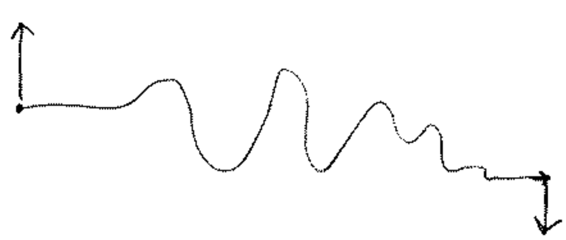
$$= -T\,\frac{\dot{\vec X}\cdot\vec X'\,\dot X_\mu + (1-\dot{\vec X}^2)\,X'_\mu}{|\vec X'|\sqrt{1-\vec v_T^{\,2}}}$$

$$\mu = 0: \quad P^\sigma_0 = +T\,\frac{\dot{\vec X}\cdot\hat{\vec X}'}{\sqrt{1-\vec v_T^{\,2}}} \qquad \Rightarrow \dot{\vec X}\cdot\hat{\vec X}' = 0 \quad \text{at endpoints}$$

$\vec v = \vec v_T = \dot{\vec X}$ motion tranversal

$$\Rightarrow P^\sigma_\mu = -T\,\underbrace{\sqrt{1-\dot{\vec X}^2}}_{\text{must vanish}}\;\underbrace{X'_\mu\,\frac{1}{|\vec X'|}}_{\text{unit vector for } \mu = i} = 0$$

$$\Rightarrow |\dot{\vec X}| = \frac{\partial \vec X}{\partial t} = 1 \quad \text{at end points}$$



3.4. Polyakov action

Do the analogue of $-m\int dt\,\sqrt{-\dot{x}^2} \to \frac{1}{2}\int dt\left[\frac{1}{e(t)}\dot{x}^2 - m^2 e(t)\right]$

–Remember in $D=1+d$ $\quad ds^2 = G_{\mu\nu}\,dx^\mu dx^\nu = \underbrace{\underset{\downarrow\,\eta_{\mu\nu}}{G_{\mu\nu}}\frac{dX^\mu}{d\sigma^a}\frac{dX^\nu}{d\sigma^b}}_{\downarrow} d\sigma^a d\sigma^b \equiv h_{ab}\,d\sigma^a d\sigma^b$

$$S = -T\int d^2\sigma\sqrt{-h} \Rightarrow -T\int d^2\sigma\sqrt{-h}\,\frac{1}{2}h^{ab}h_{ab}$$

$$h^{ab}h_{bc} = \delta^a_c$$

$$\boxed{S = -\frac{T}{2}\int d^2\sigma\sqrt{-h}\;h^{ab}\,\partial_a X^\mu\,\partial_b X^\nu\,\eta_{\mu\nu} = S\left[h_{ab},\partial_a X^\mu\right]}$$

$h = \det h_{ab}$

superstring: $\quad h^{ab}\partial_a X\cdot\partial_b X + \bar{\psi}^\mu i\gamma^a\partial_a\psi_\mu$

EOM:

$$\frac{\delta S}{\delta X^\mu} = 0 \qquad \partial_a\frac{\partial\mathcal{L}}{\partial\partial_a X^\mu} = 0 \Rightarrow \boxed{\partial_a\left(\sqrt{-h}\,h^{ab}\partial_b X^\mu\right) = 0 = \sqrt{-h}\,\nabla^2 X^\mu}$$

covariant Laplace

$\frac{\delta S}{\delta h_{ab}} = 0 \quad$ This gives the en-momentum tensor: $\quad$ basically massless scalar field eq.

Some general relativity: $\quad$ matter part

$$S(g_{\mu\nu}) = \int d^4x\left\{+\frac{\sqrt{-g}}{16\pi G}(\underset{g^{\mu\nu}R_{\mu\nu}}{R} + 2\Lambda) + \sqrt{-g}\,\mathcal{L}_m(g_{\mu\nu})\right\} = S_{EH} + S_m(g_{\mu\nu})$$

some computation

$$\Rightarrow \frac{\delta S}{\delta g_{\mu\nu}(x)} = +\frac{\sqrt{-g}}{16\pi G}\left[R_{\mu\nu} - \frac{1}{2}(R+2\Lambda)g_{\mu\nu} + 8\pi G\left(g_{\mu\nu}\mathcal{L}_m + 2\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}}\right)\right]$$

$$R_{\mu\nu} - \frac{1}{2}(R+2\Lambda)g_{\mu\nu} = 8\pi G\,T_{\mu\nu}$$

The rest is just <u>defined</u> as $T_{\mu\nu}$

$$\boxed{T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\left\{\underbrace{\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}}_{-\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}}\cdot\mathcal{L}_m + \sqrt{-g}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}\right\} = g_{\mu\nu}\mathcal{L}_m - 2\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}}}$$

In analogy:

$$S = \underbrace{\int d^2\sigma \sqrt{-h}\,(R + 2\Lambda)}_{2d \text{ gravity part, in } 2d} + \int d^2\sigma \sqrt{-h}\,\underbrace{\left(-\tfrac{T}{2}\right) h^{ab}\partial_a X^\mu \partial_b X_\mu}_{\mathcal{L}_m}$$

$\frac{\delta S}{\delta h^{ab}} \sim R_{ab} - \frac{1}{2} h_{ab} R = 0$ always, for any δh^{ab}! $\Rightarrow$ Topological invariant, Gauss-Bonnett

$$\Rightarrow T_{ab} = h_{ab}\mathcal{L}_m - 2\,\frac{\partial \mathcal{L}_m}{\partial h^{ab}}$$

$$= -\frac{T}{2}\left\{ h_{ab}\, h^{cd}\partial_c X\cdot\partial_d X - 2\,\partial_a X\cdot\partial_b X \right.$$

$$\frac{\delta S}{\delta h_{ab}} = 0 \Rightarrow \boxed{T_{ab} = T\left[\partial_a X\cdot\partial_b X - \frac{1}{2} h_{ab}\, h^{cd}\partial_c X\cdot\partial_d X\right] = 0}$$

$$\boxed{T^a_{\ a} = h^{ab} T_{ba} = 0}$$

Solve this for h_{ab}: write $\tilde{h}_{ab} \equiv \partial_a X\cdot\partial_b X$

$$\tilde{h}_{ab} = \frac{1}{2} h_{ab}\, h^{cd}\,\tilde{h}_{cd}$$

Solved by $h_{ab} = \lambda\tilde{h}_{ab}$: $1 = \frac{1}{2} h^{cd} h_{cd} = \frac{1}{2}\delta^c_c = 1$

$\Rightarrow$ classically $\boxed{h_{ab} = \partial_a X\cdot\partial_b X}$ = induced metric

and $S = -\frac{T}{2}\int d^2\sigma\sqrt{-h}\; h^{ab} h_{ab} = -T\int d^2\sigma\sqrt{-h} = S_{NG}$

But what about quantum string?

$$Z = \int \underbrace{\mathcal{D}h_{ab}}\,\mathcal{D}X^\mu\, e^{iS(h_{ab},\,\partial_a X^\mu)} \quad ?$$

note: lots of extra d.o.f.s like in gauge theories

$\int \mathcal{D}A^\mu \qquad A^\mu \to A^\mu - \frac{1}{e}\partial^\mu\theta$

$\int \mathcal{D}h_{ab} \qquad$ reparametrisations $h_{ab} \to h'_{ab}$

Symmetries of $S_{Pol} = S[h_{ab}, \partial_a X^\mu]$:

Remember symmetry $\Rightarrow$ currents:

If $L(\varphi_k, \partial_\mu \varphi_k)$ is invariant under $\varphi_k \to \varphi_k + \underbrace{\delta\varphi_k}_{\equiv \varepsilon \Delta\varphi_k}$ one must have $\quad \varepsilon \ll 1$

$$\delta L = L(\varphi_k + \delta\varphi_k, \partial_\mu(\varphi_k + \delta\varphi_k)) - L$$

$$= \underbrace{\left(\frac{\partial L}{\partial \varphi_k} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \varphi_k}\right)}_{=0 \text{ for solutions of classical EOM}} \delta\varphi_k + \partial_\mu \underbrace{\left(\frac{\partial L}{\partial \partial_\mu \varphi_k} \delta\varphi_k\right)}_{\text{must also vanish because of symmetry } \delta L = 0} = 0$$

surface term often neglected

$$\Rightarrow \partial_\mu J^\mu = 0 \quad J^\mu = \frac{\partial L}{\partial \partial_\mu \varphi_k} \Delta\varphi_k$$

1. Poincaré = Lorentz + Translations in target space

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu \qquad \delta h_{ab} = 0$$

or $\delta X^\mu = \omega^\mu{}_\nu X^\nu + \delta a^\mu$ infinitesimally

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \qquad \omega_{\mu\nu} = \eta_{\mu\alpha} \omega^\alpha{}_\nu$$

2. 2d reparametrisation invariance

$$\sigma^a \to \sigma'^a = \sigma'^a(\sigma^c) \Rightarrow \begin{cases} \tau' = f(\tau, \sigma) \\ \sigma' = g(\tau, \sigma) \end{cases}$$

$$\begin{cases} X'^\mu(\sigma') = X^\mu(\sigma) \\ h'^{ab}(\sigma') = \dfrac{\partial \sigma'^a}{\partial \sigma^c} \dfrac{\partial \sigma'^b}{\partial \sigma^d} h^{cd}(\sigma) \Rightarrow h'(\sigma') = \left| \det \dfrac{\partial \sigma'^a}{\partial \sigma^c} \right|^2 h(\sigma) \end{cases}$$

remember $\quad V'^a(x') = \underbrace{\dfrac{\partial x'^a}{\partial x^c}} V^c(x) \qquad V'_a(x') = \underbrace{\dfrac{\partial x^c}{\partial x'^a}} V_c(x)$

mnemonic use here upper indices

Note 1: a 2d metric

$$ds^2 = g_{11}\, dx^2 + 2 g_{12}\, dx\, dy + g_{22}\, dy^2$$

can always by a reparametrisation

$$u = u(x,y) \qquad x = x(u,v)$$
$$v = v(x,y) \qquad y = y(u,v)$$

be transformed to a "conformally flat" form

$$ds^2 = \pm \underbrace{e^{\phi(x,y)}}_{\text{some function of } x,y \text{ (or } u,v)} (du^2 - dv^2)$$

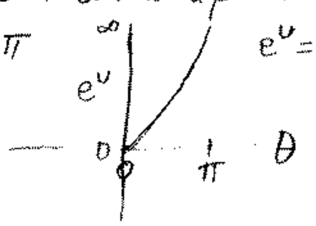
Loosely speaking, you have two partial diff. eqs

$$\begin{cases} g'_{12} = \partial_u x\, \partial_v x\, g_{11} + (\partial_u x\, \partial_v y + \partial_u y\, \partial_v x)\, g_{12} + \cdots\; g_{22} = 0 \\ g'_{11} = -g'_{22} \end{cases}$$

for the 2 functions $u(x,y)$, $v(x,y)$. (Proof: Nakahara, Geom. in Physics p. 238)

Ex $\quad ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 = \sin^2\theta\, (du^2 + dv^2) \qquad S^2$

$0 < \theta < \pi \qquad e^u = |\tan\frac{\theta}{2}|, \; v = \varphi$



$$ds^2 = dt^2 - dz^2 = d\tau^2 - \tau^2 d\eta^2$$

$$\begin{cases} t = \tau \operatorname{ch}\eta \\ z = \tau \operatorname{sh}\eta \end{cases}$$

Note 2:
Conformal transformations

$$g'_{\mu\nu} = \underbrace{\omega^2(x)}\, g_{\mu\nu} \qquad g_{\mu\nu} = \frac{1}{\omega^2} g'_{\mu\nu}$$

some function multiplies all the elements

often written as $e^{\omega(x)}$ or $e^{2\omega(x)}$

$$\{g'_{\mu\nu}\} = \begin{pmatrix} \omega^2 g_{11} & \omega^2 g_{12} & \cdots & \\ & & & \vdots \\ & & & \omega^2 g_{dd} \end{pmatrix} \qquad \begin{aligned} g^{\mu\alpha} g_{\alpha\nu} &= \delta^\mu{}_\nu \\ \underbrace{\frac{1}{\omega^2} g^{\mu\alpha}}_{= g'^{\mu\alpha}} \omega^2 g_{\alpha\nu} &= \delta^\mu{}_\nu \end{aligned}$$

$$\det g'_{\mu\nu} = (\omega^2)^d \det g_{\mu\nu}$$

$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$ is NOT invariant, this is not a reparametrisation, geometry changes, $R_{\mu\nu}$, R, etc change

$$R' = \frac{1}{\omega^2(x)} R + \text{covariant derivatives of } \omega$$

3. Weyl invariance

$$\delta X^\mu = 0 \qquad h'_{ab} = \omega^2(\sigma)\, h_{ab} \qquad h'_{ab} = e^{2\omega} h_{ab} \approx h_{ab} + \underbrace{2\omega h_{ab}}_{\equiv \delta h_{ab}}$$

$$\int d^2\sigma\, \underbrace{\sqrt{-h}\, h^{ab}}\, \partial_a X \cdot \partial_b X$$

$\Downarrow$ conformal

$$\omega^d \frac{1}{\omega^2} = 1 \text{ for } \boxed{d=2}\ !$$

This is not a symmetry of S_{NG} !

Distances measured with h_{ab} on world sheet have no physical significance!