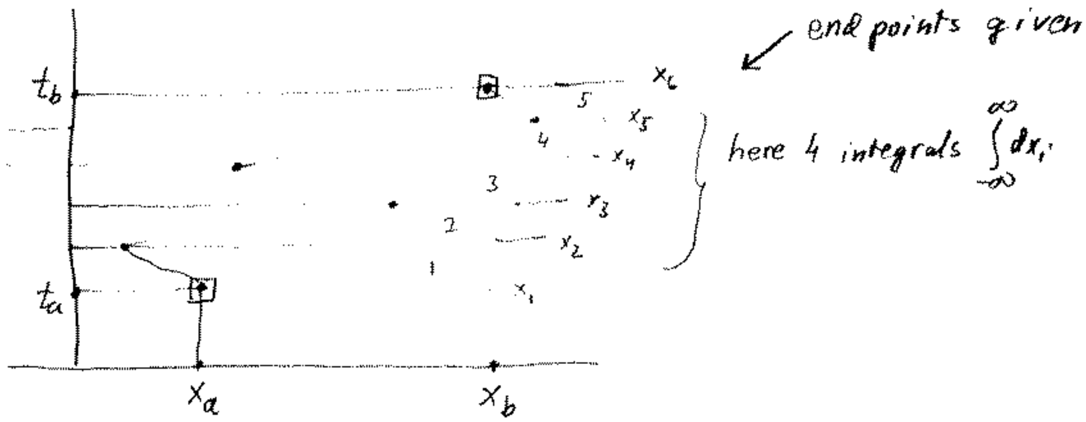
2.5 Quantising a particle; path integral

[Total amplitude for getting from x_a at t_a to x_b at t_b: Weight each path by $e^{iS/\hbar}$ and sum over paths]



$$\langle x_b t_b | x_a t_a \rangle = \underbrace{\int \mathcal{D}x(t)}\, \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt\, L(x(t), \dot{x}(t)) \right\}$$

has to be defined as a limiting process: $\int_{-\infty}^{\infty} dx_2\, dx_3\, dx_4\, dx_5 \cdot$ (weight factors)
$\downarrow$
∞

Long story! Lots of literature

3. Classical string actions

(Cosmic string parameters

neglect all couplings, constants ~1 :

GUTscale $v \sim 10^{15}\,\text{GeV}$ $\quad \sim$ fm

thickness $\frac{1}{v} \sim 10^{-15} \frac{1}{\text{GeV}} \sim 10^{-30}\,m$

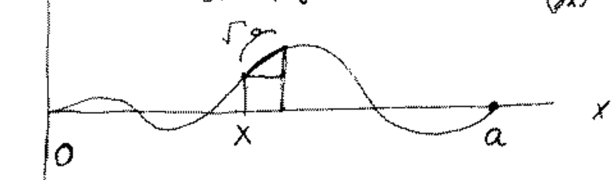
$$T = \frac{en}{length} \sim v^2 \sim 10^{30} \underbrace{\text{GeV}}_{m_p \sim 10^{-27} kg} \cdot \text{GeV} = 10^{3} kg \frac{1}{\text{fm}} \sim 10^{18} \frac{kg}{m}$$

$m_{earth} \sim 6 \cdot 10^{24} kg$)

3.1 Non-relativistic string

Parameters : $\begin{cases} \text{mass density} & \mu = \frac{\text{mass}}{\text{length}} \\ \text{tension} & T = \frac{\text{energy}}{\text{length}} \end{cases}$ $\qquad$ relativistically $T = \mu c^2$

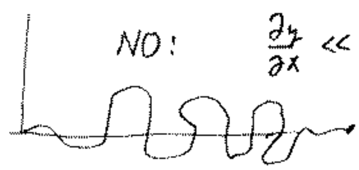
$$dl = \sqrt{dy^2 + dx^2} - dx \approx dx\, \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2$$



want $y(t,x)$ = displacement of point x at time t : only transverse small oscillations

$$T(x) = \frac{1}{2} \cdot dx\,\mu \cdot \left(\frac{\partial y}{\partial t}\right)^2$$

$$V(x) = \frac{1}{2} T\,dx \left(\frac{\partial y}{\partial x}\right)^2$$



$$\Rightarrow L = \int_0^a dx \left[\frac{1}{2}\mu\,\dot{y}^2 - \frac{1}{2} T\, y'^2\right] \qquad \boxed{\dot{y} = \frac{\partial y}{\partial t}, \; y' = \frac{\partial y}{\partial x}}$$

$$\boxed{S = \int_{t_i}^{t_f} dt \int_0^a dx \left(\frac{1}{2}\mu\,\dot{y}^2 - \frac{1}{2} T\, y'^2\right)}$$

Path

String EOM :

Using the general $\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 0 \qquad \mathcal{L} = \frac{1}{2}\mu \dot{y}^2 - \frac{1}{2} T y'^2$

$$\frac{\partial \mathcal{L}}{\partial y} = T \partial_x^2 y \qquad \text{NOTE: } \int dx (\partial_x y)^2 \equiv \int dx\, y(-\partial_x^2) y$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \mu \dot{y} \quad \Rightarrow \quad \boxed{\mu \ddot{y} - T y'' = 0}$$

$$\frac{\partial^2 y}{\partial t^2} - \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} = 0$$

$$y = F(t - \tfrac{x}{v}) \Rightarrow \partial_x^2 y = \frac{1}{v^2}\partial_t^2 F$$

$$\Rightarrow \boxed{v^2 = \frac{T}{\mu}} \qquad \text{wave}$$

However, consider the end points more carefully

$$\frac{1}{2}\mu\left(\frac{\partial}{\partial t}(y + \delta y)\right)^2 - \frac{1}{2}T\left(\frac{\partial}{\partial x}(y + \delta y)\right)^2$$

$$\simeq \mu \underbrace{\dot{y}\frac{\partial}{\partial t}\delta y}_{\frac{\partial}{\partial t}(\dot{y}\delta y) - \ddot{y}\cdot\delta y} - T \underbrace{y'\frac{\partial}{\partial x}\delta y}_{\frac{\partial}{\partial x}(y'\delta y) - y''\cdot\delta y} \qquad \text{linear terms}$$

$$\delta S = \int_{t_i}^{t_f} dt\, dx \left[\underbrace{\mu\frac{\partial}{\partial t}(\dot{y}\delta y)} - \underbrace{T\frac{\partial}{\partial x}(y'\delta y)} - \underbrace{(\mu\ddot{y} - Ty'')}_{=0;\ \text{EOM above}}\delta y\right]$$

$$\mu\int_0^a dx\left[\dot{y}(t_f,x)\underbrace{\delta y(t_f,x)}_{=0} - \dot{y}\cdot\underbrace{\delta y(t_i,x)}_{=0}\right] + T\int_{t_i}^{t_f} dt\left[y'(t,0)\,\delta y(t,0) - y'(t,a)\,\delta y(t,a)\right]$$

initial & final configs are fixed

$y'(t,0) = y'(t,a) = 0 \quad \forall t$ Neumann BC

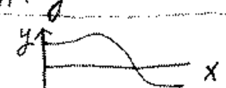
$\delta y(t,0) = \delta y(t,a) = 0$ Dirichlet BC

fixed end points

where would they be fixed?

D-branes!

String momentum:



$$p_y = \int_0^a dx\, \mu\frac{\partial y}{\partial t} \qquad \dot{p}_y = \int_0^a dx\, \mu\ddot{y} = \int_0^a dx\, T y''$$

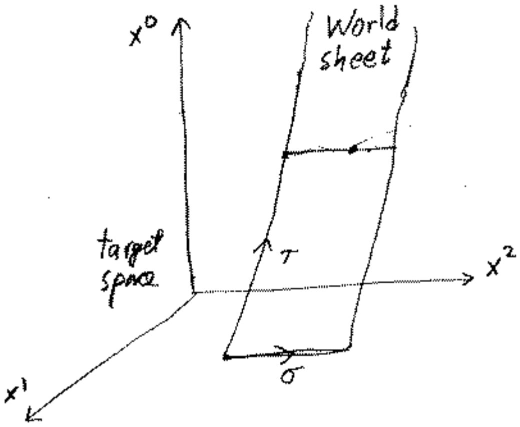
$$= T[y'(t,a) - y'(t,0)] \underset{NBC}{=} 0$$

$\dot{p}_y = 0$ for NBC

3.2 Relativistic string

Answer $\quad -m\int d\tau \implies T\int dA = S_{\text{Nambu-Goto}}$

What is dA? Give the answer, explain later.



String is at "string coordinates"

$$\begin{cases} x^0 = X^0(\tau,\sigma) \\ x^i = X^i(\tau,\sigma) \quad i = 1,\cdots,d \end{cases}$$

Suppose $G_{\mu\nu}$ is the metric in the target space. Then

$$ds^2 = G_{\mu\nu}\, dx^\mu dx^\nu \underset{\substack{\text{take } x^\mu \\ \text{on the world} \\ \text{sheet}}}{=} G_{\mu\nu} \frac{dX^\mu}{d\sigma^a} \frac{dX^\nu}{d\sigma^b} d\sigma^a d\sigma^b \equiv h_{ab}\, d\sigma^a d\sigma^b$$

$\sigma^1 = \tau \quad \sigma^2 = \sigma$

$$\Rightarrow \boxed{S_{NG} = -T\int d^2\sigma \sqrt{-\det h_{ab}}}$$

If $G_{\mu\nu} = \eta_{\mu\nu}$: $\quad h_{ab} = \frac{dX^\mu}{d\sigma^a}\frac{dX_\mu}{d\sigma^b} \equiv \frac{dX}{d\sigma^a}\cdot\frac{dX}{d\sigma^b}$

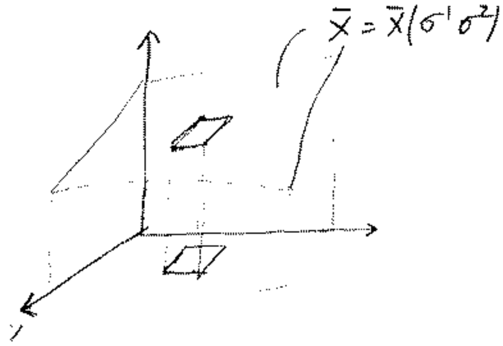
$$\Rightarrow h_{ab} = \begin{pmatrix} \dot{X}\cdot\dot{X} & \dot{X}\cdot X' \\ X'\cdot\dot{X} & X'\cdot X' \end{pmatrix} \qquad \dot{X}^\mu = \frac{\partial X^\mu}{\partial\tau} \quad X'^\mu = \frac{\partial X^\mu}{\partial\sigma}$$

$$\boxed{S_{NG} = -T\int d\tau\, d\sigma \sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 (X')^2} = S(\dot{X}^\mu, X'^\mu)}$$

$$\dot{X}\cdot X' = \frac{\partial X^\mu}{\partial\tau}\frac{\partial X_\mu}{\partial\sigma} = -\partial_\tau X^0 \partial_\sigma X^0 + \partial_\tau X^1 \partial_\sigma X^1 + \cdots$$

etc. $\qquad$ Why is $(\dot{X}\cdot X')^2 > \dot{X}^2 X'^2$?

1. How can we understand that form?



$$d\bar{v}_1 = \frac{\partial \bar{x}}{\partial \sigma_1} \qquad d\bar{v}_2 = \frac{\partial \bar{x}}{\partial \sigma_2}$$

$$dA = |d\bar{v}_1 \times d\bar{v}_2|$$

$$|\bar{a} \times \bar{b}| = \sqrt{(\bar{a} \times \bar{b}) \cdot (\bar{a} \times \bar{b})}$$

$$= \sqrt{\bar{a} \cdot \underbrace{\bar{b} \times (\bar{a} \times \bar{b})}_{\bar{a}\,\bar{b}^2 - \bar{b}\,\bar{a}\cdot\bar{b}}} = \sqrt{\bar{a}^2 \bar{b}^2 - (\bar{a}\cdot\bar{b})^2}$$

$$\Rightarrow \boxed{dA = d\sigma^1 d\sigma^2 \sqrt{\left(\frac{\partial \bar{x}}{\partial \sigma_1}\right)^2 \left(\frac{\partial \bar{x}}{\partial \sigma_2}\right)^2 - \left(\frac{\partial \bar{x}}{\partial \sigma_1} \cdot \frac{\partial \bar{x}}{\partial \sigma_2}\right)^2}}$$

Exactly the same structure (in Euclidian space)

2. Reparametrisation invariance under $x^\mu \to x'^\mu$ (Exercise)

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu = g'_{\mu\nu}(x')\, \underbrace{dx'^\mu}_{\frac{\partial x'^\mu}{\partial x^\alpha} dx^\alpha} dx'^\nu$$

$$= \underbrace{g'_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta}}_{= g_{\alpha\beta}} dx^\alpha dx^\beta$$

$$J_{\mu\alpha} \equiv \frac{\partial x'^\mu}{\partial x^\alpha} \text{ Jacobian}$$

$$g_{\alpha\beta} = g'_{\mu\nu} J_{\mu\alpha} J_{\nu\beta} = J^T_{\beta\nu} g'_{\nu\mu} J_{\mu\alpha} = (J^T g' J)_{\beta\alpha}$$

$$\det g = \det J^T \det g' \det J = \det g' (\det J)^2$$

$$d^d x' = \underbrace{\det \frac{\partial x'^\mu}{\partial x^\alpha}}_{\text{Jacobian}} \underbrace{d^d x}_{\det \frac{\partial x^\nu}{\partial x'^\beta} d^d x'} \Rightarrow \det \frac{\partial x'}{\partial x} \cdot \det \frac{\partial x}{\partial x'} = 1$$

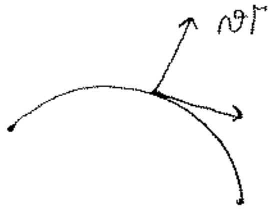
$$\int d^d x' \sqrt{|\det g'|} = \int d^d x \det \frac{\partial x'^\mu}{\partial x^\alpha} \cdot \frac{1}{\left|\det \frac{\partial x'^\mu}{\partial x^\alpha}\right|} \sqrt{|\det g|}$$

[$\int d^d x \sqrt{|\det g|}$ is reparametrisation invariant

$\int d^2\sigma \sqrt{|\det h|}$ — " —

Ex. Relate the full S_{NG} to the non-relativistic action on p. 14.

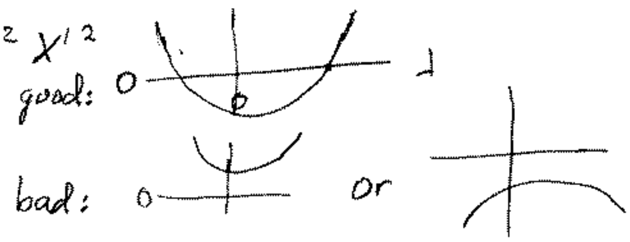
3. Why is $(\dot X\cdot X')^2 - \dot X^2 X'^2 > 0$?



String should have both a time-like ($v^2<0$) and spacelike ($v^2>0$) tangent vector. Tangent vectors are lin. combs

$$v^\mu = \partial_\tau X^\mu + \lambda\, \partial_\sigma X^\mu \equiv \dot X^\mu + \lambda X'^\mu$$

$$\Rightarrow v^2 = \dot X^2 + 2\lambda\, \dot X\cdot X' + \lambda^2 X'^2$$



need 2 real roots:

$$D = (\dot X\cdot X')^2 - \dot X^2 X'^2 > 0$$

3.3 EOM:

$$L = L(\dot X^\mu, X'^\mu) \quad , \text{much like particle, } L = L(\dot X^\mu)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial\tau}\underbrace{\frac{\partial L}{\partial \dot X^\mu}} + \frac{\partial}{\partial\sigma}\frac{\partial L}{\partial X'^\mu} = 0}$$

$$= \Pi_\mu \equiv P^\tau_\mu = -T\,\frac{\dot X\cdot X' X'_\mu - X'^2 \dot X_\mu}{\sqrt{(\dot X\cdot X')^2 - \dot X^2 X'^2}} \qquad P^\sigma_\mu = \frac{\partial L}{\partial X'^\mu} = \cdots$$

Constraints: $\quad \Pi\cdot X' = 0 \qquad \Pi^2 + T^2 X'^2 = 0$

(for free particle $p_\mu = \frac{m\dot x_\mu}{\sqrt{-\dot x^2}}$, $p^2+m^2=0$, $H = p_\mu \dot x^\mu - L = 0$)

$L = m\sqrt{-\dot x^2}$

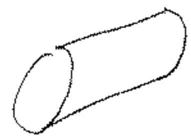
$$H = \int d\sigma\left(\Pi_\mu \dot X^\mu - L\right) = -T\int d\sigma\left(\sqrt{} - \sqrt{}\right) = 0 \;!$$

Classical dynamics (of free relativistic string) determined by constraints!

How can this be related to quantum gravity!?

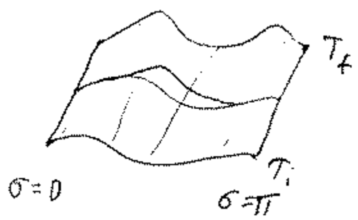
BC

Closed strings: No BC; let $0 \le \sigma < 2\pi$, e.g.



Open strings:

vary path: $X^\mu(\tau,\sigma) + \delta X^\mu(\tau,\sigma)$



again $\delta X^\mu(\tau_i, \sigma) = \delta X^\mu(\tau_f, \sigma) = 0$

$$L(\dot X^\mu \; X'^\mu) \to L(\dot X^\mu, \partial_\sigma(X^\mu + \delta X^\mu))$$

$$\Downarrow$$

$$\frac{\partial L}{\partial X'^\mu} \partial_\sigma \delta X^\mu$$

$$\Downarrow$$

partial int

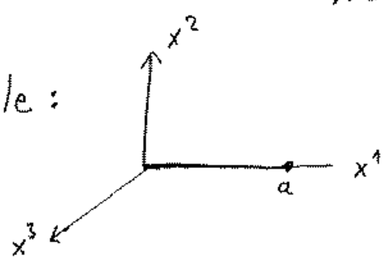
$$\int d\tau\, d\sigma\, \partial_\sigma\left(P^\sigma_\mu \,\delta X^\mu\right)$$

$$= \int_{\tau_i}^{\tau_f} d\tau \Big| _0^\pi \underbrace{P^\sigma_\mu(\tau\,\sigma)}_{\substack{\text{make this}\\ \text{vanish at endpts}\\ \Rightarrow \text{Neumann}}} \; \underbrace{\delta X^\mu(\tau,\sigma)}_{\text{make this vanish at endpoints}}$$

endpts fixed $\Rightarrow$ Dirichlet

implies also $P^\tau_\mu = \left.\frac{\partial L}{\partial \dot X^\mu}\right|_{\sigma = 0,\pi} = 0$

Example:



$f(0)=0 \quad f(\pi)=a$

$$X^\mu(\tau,\sigma) = (\tau, f(\sigma), 0, 0, \cdots, 0)$$

$$\Rightarrow \dot X^\mu = (1, 0, 0, 0, \cdots, 0)$$

$$X'^\mu = (0, f'(\sigma), 0, \cdots, 0)$$

$$S = -T\int_{\tau_i}^{\tau_2} d\tau \int_0^\pi d\sigma \sqrt{0 + (f')^2} = -T\int_{\tau_i}^{\tau_f} d\tau \left[f(\pi) - f(0)\right] = -Ta(\tau_f - \tau_i)$$

$$= \int d\tau (\underset{=0}{T} - V) \Rightarrow V = Ta \quad \text{explains } -T \text{ in def.}$$

EOM: $\underbrace{\partial_\tau P^\tau_\mu}_{\substack{=0\\ \text{no } \tau\text{-dep}}} + \partial_\sigma P^\sigma_\mu = \partial_\sigma\left[(-T)\frac{X'^\mu}{f'}\right] = 0$