2.3 Fields

Generalize $q \to q_i = \{q_1, \cdots, q_N\} \to \underset{\substack{\uparrow \\ \text{continuum spatial index}}}{q_{\bar{x}}} \equiv \varphi(\bar{x})$

(meaning = ?)

$$q(t) \to \varphi(t, \bar{x}) = \underset{\substack{\uparrow \\ x^\mu}}{\varphi(x)}$$

$$L = \sum_i \to \int d^3x \, \underset{\substack{\uparrow \\ \text{Lagrangian density}}}{\mathcal{L}}(\varphi(t,x), \partial_\mu \varphi(t,x)) \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}$$

$$S = \int_{t_1}^{t_2} dt \, L \to S = \int_{\varphi(t_1,\bar{x}) = \varphi_1(\bar{x})}^{\varphi(t_2,\bar{x}) = \varphi_2(\bar{x})} d^4x \, \mathcal{L}(\varphi(x), \partial_\mu \varphi(x))$$

path goes from one configuration to another

EOM: $\int d^4x \, \mathcal{L}(\varphi(x) + \delta\varphi(x), \partial_\mu \varphi + \partial_\mu \delta\varphi(x))$

$$= \mathcal{L}(\varphi, \partial_\mu \varphi) + \frac{\partial \mathcal{L}}{\partial \varphi} \cdot \delta\varphi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\mu \delta\varphi(x)$$

move here by partial integration

$$\frac{\delta S}{\delta \varphi} = 0 \Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} = 0}$$

$$x^\mu = (x^0, x^1, \cdots, x^d)$$

- Scalar field: "(an)harmonic oscillator"

$$V_\mu = \eta_{\mu\alpha} V^\alpha \Rightarrow \begin{cases} V_0 = -V^0 \\ V_i = V^i \end{cases}$$

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - V(\varphi) \qquad (\eta_{\mu\nu} = (- + + + \cdots))$$

$$= \frac{1}{2} (\overset{\dot{q}^2}{\partial_0 \varphi})^2 - \frac{1}{2} (\bar{\partial} \varphi)^2 - V(\varphi) \qquad \frac{1}{2} \dot{q}^2 - V(q)$$

$$H = p\dot{q} - L \Rightarrow \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \partial_0 \varphi} \partial_0 \varphi - \mathcal{L} = \frac{1}{2} (\partial_0 \varphi)^2 + \frac{1}{2} (\bar{\partial} \varphi)^2 + V(\varphi)$$

$$\left(\frac{1}{2} m \dot{q}^2 + \frac{1}{2} m \omega^2 q^2\right) \qquad \frac{1}{2} m^2 \varphi^2 + \frac{1}{4} \lambda \varphi^4$$

- Electrodynamics

(You all know Maxwell!)

$$A^\mu = (\varphi, cA^i) \quad = (\varphi, A^i) \qquad \left(\text{put } c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = 1\right)$$

$$V \quad \tfrac{m}{s}\cdot\tfrac{Vs}{m} \qquad A_\mu = (-\varphi, A_i)$$

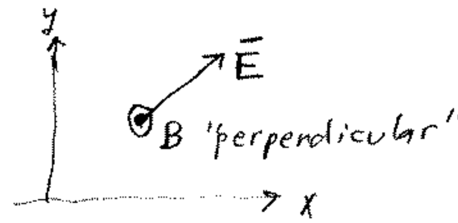
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \text{want } \bar{E} = -\bar{\nabla}\varphi - \frac{\partial \bar{A}}{\partial t}$$

$$= -F_{\nu\mu} \qquad\qquad E^i = -\partial_i\varphi - \partial_0 A^i$$

$$-E^i = F_{0i} = \partial_0 A_i - \partial_i A_0 = +\partial_0 A_i + \partial_i \varphi \qquad E^i = F_{i0}$$

$$\Rightarrow F_{\mu\nu} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix} \end{array} \begin{array}{l} \leftarrow 3 \text{ here} \\ \left.\vphantom{\begin{matrix} a\\b\\c \end{matrix}}\right\} 3 \text{ here} \end{array} \qquad \bar{B} = \bar{\nabla}\times\bar{A}$$

$$\begin{array}{l} 3d \\ 1+2d \end{array} \quad F_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 \\ E^1 & 0 & B \\ E^2 & -B & 0 \end{pmatrix} \longleftarrow B \text{ is a "scalar}$$



$$1+4 \quad F_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 & -E^4 \\ & 0 & F_{23} & F_{24} & F_{25} \\ & & 0 & F_{34} & F_{35} \\ & & & 0 & F_{45} \\ & & & & 0 \end{pmatrix} \begin{array}{l} \text{— 4 here} \\ \left.\vphantom{\begin{matrix} a\\b\\c \end{matrix}}\right\} 6 \text{ here} \end{array}$$

$$\boxed{\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}} = -\frac{1}{2}\left(F^{0i}F_{0i} + \sum_{i>j} F^{ij}F_{ij}\right) = \frac{1}{2}\left(\bar{E}^2 - \bar{B}^2\right)$$

contains $\sum_{\mu,\nu=0}^{3}$

coupling with a current $\quad \mathcal{L} = -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} - J_\mu A^\mu$

current 4-vector

Elmag EOM:

$$\underbrace{\frac{\partial \mathcal{L}}{\partial A_\mu}}_{-J^\mu} - \partial_\nu \underbrace{\frac{\partial \mathcal{L}}{\partial \partial_\nu A_\mu}} = -J^\nu - \partial_\nu F^{\mu\nu} = 0$$

you can work this out directly but simpler:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \Rightarrow \delta\mathcal{L} = -\frac{1}{4} F^{\mu\nu} \underbrace{\delta F_{\mu\nu}}_{\delta(\partial_\mu A_\nu - \partial_\nu A_\mu)} = + F^{\mu\nu} \delta(\partial_\nu A_\mu)$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\nu A_\mu} = F^{\mu\nu}$$

$\bar{\nabla}\cdot\bar{E} = \frac{1}{\varepsilon_0}\rho$

$\nabla\times\bar{B} - \frac{\partial \bar{E}}{c^2 \partial t} = \mu_0 \bar{J}$

$$\boxed{\partial_\nu F^{\mu\nu} = J^\nu} \qquad \Big(= \frac{1}{\varepsilon_0}(em, em\bar{v}) \Big)$$

number density

Also: $\boxed{\partial_\lambda F^{\mu\nu} + \partial_\mu F^{\nu\lambda} + \partial_\nu F^{\lambda\mu} = 0}$

$$\begin{cases} \nabla\times\bar{E} + \frac{\partial \bar{B}}{\partial t} = 0 \\ \bar{\nabla}\cdot\bar{B} = 0 \end{cases}$$

no J^μ !!

Or: Define dual of $F^{\mu\nu}$:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

$\varepsilon^{0123} = +1$, totally antisymmetric

$= F^{\mu\nu}(\bar{E}\to c\bar{B},\ c\bar{B}\to -\bar{E})$

$$\Big(\begin{array}{l} 3d:\ \tilde{F}^\mu = \varepsilon^{\mu\alpha\beta}F_{\alpha\beta} \\ 2d:\ \tilde{F} = \varepsilon^{\alpha\beta}F_{\alpha\beta} \end{array} \Big)$$

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\boxed{\partial_\nu \tilde{F}^{\mu\nu} = 0}$ ($= J^\nu_{mag}$ which do not exist)

$$-\frac{1}{4}\tilde{F}^{\mu\nu}\cdot F_{\mu\nu} = c\,\bar{E}\cdot\bar{B}$$

Where are magnetic monopoles?

Further topics which should be discussed:

Gauge invariance (local, $U(1)$)

$$\mathcal{L}(\underset{\text{real}}{\varphi(x)}) \Rightarrow \mathcal{L}(\underset{\text{complex}}{\varphi(x)} = \varphi_1 + i\varphi_2) \quad \text{invariant under}$$

$$\varphi \to e^{i\theta}\varphi$$

(dep. only on $\varphi^*\varphi$)

Demand locality, $\theta = \theta(x)$:

$$\varphi(x) \to e^{i\theta(x)}\varphi(x)$$

But then $\quad \partial_\mu\varphi \to \partial_\mu(e^{i\theta(x)}\varphi) \to i\partial_\mu\theta \cdot e^{i\theta(x)}\varphi + e^{i\theta(x)}\varphi$

$(\partial_\mu\varphi)^* \partial^\mu\varphi$ is NOT invariant

But defining $D_\mu = \partial_\mu + ieA_\mu$

and demanding $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\theta$

$$D_\mu\varphi \to (\partial_\mu + ieA_\mu - i\partial_\mu\theta)e^{i\theta(x)}\varphi$$

$$= \underbrace{i\partial_\mu\theta\, e^{i\theta(x)}\varphi + e^{i\theta}\partial_\mu\varphi + e^{i\theta}\, ieA_\mu\varphi - i\partial_\mu\theta\, e^{i\theta}\varphi}_{\text{cancel}}$$

$$\boxed{D_\mu\varphi \to e^{i\theta(x)} D_\mu\varphi}$$

and $\quad (D_\mu\varphi)^* D^\mu\varphi \quad \underline{\underline{\text{IS}}}$ invariant

For the EM field $\quad \boxed{\begin{aligned} A_\mu &\to A_\mu - \tfrac{1}{e}\partial_\mu\theta \\ F_{\mu\nu} &\to F_{\mu\nu} \end{aligned}}$

Energy-momentum tensor:

$$T_{\mu\nu} = \begin{array}{cc} 0 & i \end{array}$$
$$\left(\begin{array}{c|c} \text{en density} & \text{energy flux}_i \\ \hline \text{symm} & \text{momentum } p^i \text{ flux}_j \end{array}\right)$$

$$= F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}\eta_{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}$$

Einstein:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

for $\eta = (+ - - -)$:

$$T_{\mu\nu} = +F_\mu{}^\alpha \underbrace{F_{\alpha\nu}}_{\text{NOTE!}} + \frac{1}{4}\eta_{\mu\nu} F^2$$

Check:

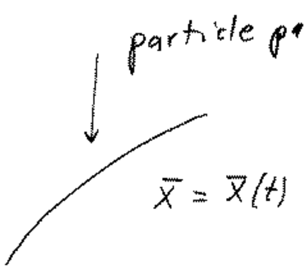
$$T_{00} = F_0{}^i F_{0i} + \eta_{00}\mathcal{L}$$

$$= +\bar{E}^2 - \frac{1}{2}(\bar{E}^2 - \bar{B}^2) = \frac{1}{2}(\bar{E}^2 + \bar{B}^2) \equiv \frac{1}{2}\epsilon_0(\bar{E}^2 + c^2\bar{B}^2)$$

$$T^\nu{}_\nu = \underbrace{F^{\nu\alpha}F_{\nu\alpha}}_{\equiv F^2} - \frac{1}{4}\eta^\mu{}_\mu F^2 = 0 \qquad \eta^\mu{}_\mu = +1+1+1+1 = 4$$

2.4 Action of a classical particle in EM field

$$\mathcal{L} = -\frac{1}{4}F^2 - \overbrace{\underbrace{J_\mu(x)}_{\text{need still the current}} A^\mu}^{\text{coupling}} \qquad = \text{density}$$



$$J^\mu = (en, en\bar{v})$$

$$= \left(e\,\delta^3(\bar{x} - \bar{x}(t)),\ e\,\delta^3(\bar{x} - \bar{x}(t))\frac{d\bar{x}}{dt}\right)$$

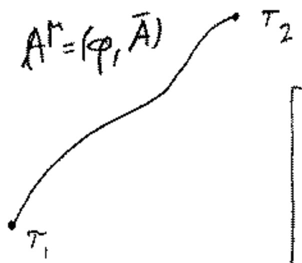
$$= e\,\delta^3(\bar{x} - \bar{x}(t))\frac{dx^\mu}{dt}$$

$$x^0 = -x_0 = t$$
$$A^0 = -A_0 = \varphi$$

$$\Rightarrow S = -m\int_{\tau_1}^{\tau_2} d\tau - \int dt \int d^3x\, e\,\delta^3(\bar{x} - \bar{x}(t))\frac{dx^\mu}{dt}A_\mu(x)$$

do $\bar{x}$-integral

$$-e\int_{\text{path}} dx^\mu A_\mu(x) = -e\int_{\text{path}} dt\,(\underbrace{A_0}_{-\varphi} + \bar{v}\cdot\bar{A})$$



$d\tau = dt\sqrt{1-\bar{v}^2}$

$$\boxed{\begin{aligned} S &= -m\int_{\tau_1}^{\tau_2} d\tau \; - e\int_{\text{path}} dx^\mu A_\mu(x) \quad \left(+ \int d^4x \left(-\tfrac{1}{4}F^2\right) \right) \\ &= \int_{\tau_1}^{\tau_2} d\tau \left[-m - e\frac{dx^\mu}{d\tau} A_\mu(x) \right] \\ &= \int_{t_1}^{t_2} dt \left[-m\sqrt{1-\bar{v}^2} + e\left(\varphi - \bar{v}\cdot\bar{A}\right) \right] \end{aligned}}$$

Assume $A_\mu(x)$ is a given background field. Then

$$L = -m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} - e\,\dot{x}^\alpha A_\alpha(x^\mu) = L(\dot{x}^\mu, x^\mu)$$

$$\frac{\partial L}{\partial x^\mu} = -e\,\dot{x}^\alpha \partial_\mu A_\alpha(x) \qquad \frac{\partial L}{\partial \dot{x}^\mu} = \underbrace{\frac{m\,\dot{x}_\mu}{\sqrt{-\dot{x}^2}}}_{p_\mu} - eA_\mu(x)$$

EOM: $\displaystyle \frac{\partial L}{\partial x^\mu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\mu} = -e\dot{x}^\alpha\partial_\mu A_\alpha + e\,\partial_\alpha A_\mu \cdot \dot{x}^\alpha - \frac{d}{dt}p_\mu$

$$\Rightarrow \boxed{\frac{dp_\mu}{d\tau} = e\,\frac{dx^\alpha}{d\tau} F_{\alpha\mu} \equiv f_\mu} \qquad \text{Lorentz force}$$

Gauge invariance: $\displaystyle \int dx^\mu A_\mu \to \int dx^\mu A_\mu + \underbrace{\int dx^\mu \partial_\mu\theta}_{\theta_f - \theta_i = 0}$

<u>Note:</u> Strings may couple like

$$\underset{\text{point}}{\int dx^\mu A_\mu} \Rightarrow \underset{\text{string}}{\int d^2\sigma \; \varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu \cdot B_{\mu\nu}} \Rightarrow \cdots$$

$F_2 = dA \qquad\qquad F_3 = dB$