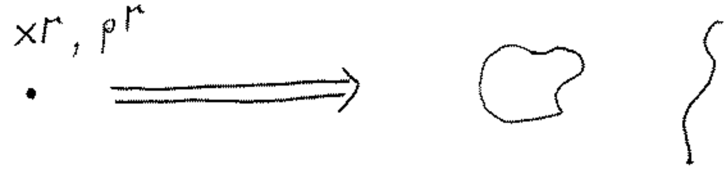
# String Theory [ies], Spring 2006

## 1. Idea




Symbolises the entire Standard Model

$=$ Quantum Mechanics $+$ Relativity

$\hbar$ $\qquad$ $c$

$xp = \hbar$ $\qquad$ $x^\mu = (ct, x)$

$p^\mu = (\frac{E}{c}, \vec{p})$

$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ invariant

Quantum Mechanics: $\psi(x) \Rightarrow \psi_i(x)$

one or more complex #s

Q Field Th: $\psi(x) \equiv \psi_x$

complex # at each $x^\mu$

$$Z = e^{-W[J]} = \int \mathcal{D}\psi(x)\, e^{-S[\psi(x)] - J\cdot\psi(x)}$$

$\mathcal{D}\psi(x) \approx \prod_x \int_{-\infty}^{\infty} d\psi_x$

Physics is in

$$\langle \mathcal{O}(\psi(x)) \rangle = \frac{1}{Z(J=0)} \int \mathcal{D}\psi(x)\, \mathcal{O}(\psi(x))\, e^{-S[\psi]}$$

"Quantize" "strings",
keep principles of QM & Rel.

Find massless spin 2 "modes" $\Rightarrow$ gravity

While a particle has mass, $p^2 = m^2$,
a string has tension

$$\boxed{T = \frac{\text{energy}}{\text{length}} = \frac{1}{\text{length}^2} \equiv \frac{1}{\pi l_s^2} \equiv \frac{1}{2\pi\alpha'} = \text{energy}^2}$$

$m$ characterises representations of the Poincaré group$(\Lambda, a)$,
No such fundamental characteris. for $T$

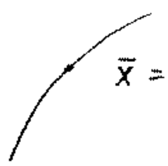
Started in 1970 from QCD & $q$ confinement:

$$E(R) = \sigma R + \text{const} - \frac{c}{R} + \cdots$$

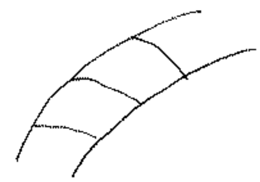
$q \sim\sim\sim \bar{q}$ $\qquad \sigma \approx \frac{\text{GeV}}{\text{fm}} \sim \text{GeV}^2$

# 2. Quantising a particle

## 2.1 Hamiltonian mechanics, Lagrangian, Action



$\bar{x} = \bar{x}(t)$ $\quad\quad \underbrace{\bar{F}}_{\substack{\text{caused by} \\ \text{fields (el.mag, gravity, ...)}}} = m\bar{a} = m\ddot{\bar{x}} = \frac{d\bar{p}}{dt} \quad \overset{?}{\Longrightarrow}$



Any more fundamental formulation than this Newton's law?

$$\text{Lagrangian:} \quad L(q, \dot{q}) = T - V = \underset{\text{kinetic}}{T(\dot{q})} - \underset{\text{potential}}{V(q)}$$

$q$: any coordinate degree of freedom

$\dot{q} \equiv \frac{dq}{dt}$ = velocity

$T(\dot{q}) - V(q)$: simplest possibility

$$\text{Action} \quad S = \int_{t_1}^{t_2} dt \, L(q, \dot{q})$$

$$\dim S = s \cdot J = \dim \bar{r} \times \bar{p} = \dim(\text{angular momentum})$$

$$s \cdot Nm \qquad\qquad = \dim \hbar$$

$$\hbar \sim 10^{-35} \, Js$$

$\frac{1}{\hbar} S$ is dimless: important for quantisation!

Principle of minimal action, Hamilton's principle:

> Classical paths are given by extrema of $S$, i.e. $\frac{\delta S}{\delta q(t)} = 0$

Lagrange equations of motion (EOM):


$$S[q] = \int_{t_1}^{t_2} dt \; L\left(q(t), \frac{dq(t)}{dt}\right)$$

$$S(q(t) + \delta q(t)) = \int_{t_1}^{t_2} dt \; L\left(q(t) + \delta q(t), \frac{d}{dt}q(t) + \frac{d}{dt}\delta q(t)\right)$$

$$= \int_{t_1}^{t_2} dt \left[ L(q, \dot{q}) + \frac{\partial L}{\partial q} \cdot \delta(q) + \underbrace{\frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q(t)} + \dots \right.$$

key point: to factor out $\delta q(t)$ do one partial integration

$$\int dt \; \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}\delta q = \underbrace{\Big| \frac{\partial L}{\partial \dot{q}} \delta q}_{\substack{=0 \text{ since} \\ \delta q(t_1) = \delta q(t_2) = 0}} - \int dt \; \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \, \delta q$$

$$= S + \int_{t_1}^{t_2} dt \underbrace{\left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right)}_{=0 \text{ if } \delta S = 0} \delta q(t)$$

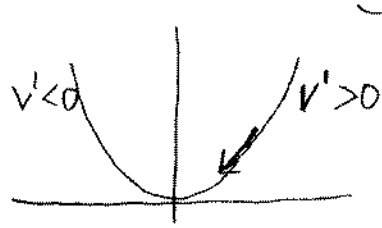
Class. EOM $\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0}$ $\quad q \to q_1, \dots, q_N$

one 2nd order eq/dof

from partial integration

Ex: $L = \frac{1}{2} m \dot{q}^2 - V(q) \quad \Rightarrow \quad -V'(q) - \frac{d}{dt} m \dot{q} = 0$

$$\underbrace{-V'(q)}_{F} = m\ddot{q} \equiv ma$$




$F < 0$

# Hamiltonian

Define $p = \frac{\partial L(q,\dot{q})}{\partial \dot{q}}$ and replace $\dot{q} \Rightarrow p$

$$H = p\dot{q} - L(q,\dot{q}) \qquad L(q,\dot{q}) \Rightarrow H(p,q)$$

$\partial$

"Legendre transformation"

$$F(T\,V\,N) = E(S,V,N) - TS \quad \text{etc}$$

EOM:
$$dH(p,q) = \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial q}dq$$

$$\underset{\dot{q}=\dot{q}(p)}{=} \dot{q}\,dp + \underbrace{p\,d\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q}}_{\text{cancel}}$$

$$\Rightarrow \underline{\dot{q} = \frac{\partial H}{\partial p}} \qquad -\frac{\partial L}{\partial q} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \underline{-\dot{p} = \frac{\partial H}{\partial q}}$$

Two 1st order eqs in time/dof

Ex $\quad L = \frac{1}{2}m\dot{q}^2 - V(q) \Rightarrow H = m\dot{q}^2 - \frac{1}{2}m\dot{q}^2 + V = T + V$

$p = m\dot{q}$

Ex: $\quad f = f(p,q) \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q} = \frac{\partial f}{\partial p}\left(-\frac{\partial H}{\partial q}\right) + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p}$

$$\frac{df}{dt} = \{H, f\} = \text{Poisson bracket}$$

Ex: Action principle now becomes

$$\delta S = \delta\int_{t_1}^{t_2}(p\dot{q} - H)\,dt = \delta\int_{t_1}^{t_2}(p\,dq - H\,dt) = 0$$

2.2. Relativistic particle (relativity is crucial for strings, too!)

Non-rel : $t$ & $\bar{x}$ separate

Relativity : $x^\mu = (ct, \bar{x}) \equiv (x^0, x^i) = (x^0, x^1, \dots, x^d)$

$$(ds)^2 \equiv ds^2 = -c^2 d\tau^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$= -c^2 dt^2 + dx^2 + dy^2 + dz^2 \qquad \text{Flat space}$$

Lorentz transformations:

$c\,d\tau = dt\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$

$\dot{x}^\mu = \frac{dx^\mu}{dt}$

$$x^\mu \to x^{\mu'} = \Lambda^\mu{}_\nu\, x^\nu$$

leave $ds^2$ invariant if

$$ds'^2 = \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta\, dx^\alpha dx^\beta = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$\Rightarrow \quad \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \qquad \text{"pseudo-orthogonal"}$$

We need: - $S$ is invariant
- $\dim S = \hbar$ = $\boxed{= -mc\int_{t_1}^{t_2} dt \sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}$

$$\Rightarrow \boxed{S = -mc^2 \int_{\tau_1}^{\tau_2} d\tau = -mc^2 \int_{t_1}^{t_2} dt \sqrt{1 - \frac{1}{c^2}\left(\frac{d\bar{x}}{dt}\right)^2}}$$

Normalisation: take NR limit, $\sqrt{1-\frac{1}{c^2}\bar{v}^2} = 1 - \frac{\bar{v}^2}{2c^2}$

$$S = -mc^2 \int_{t_1}^{t_2} dt \left(1 - \frac{\bar{v}^2}{2c^2}\right) = \text{const} + \int_{t_1}^{t_2} dt\, \underbrace{\tfrac{1}{2} m \bar{v}^2}_{=T}$$

OK!

Manifestly "reparametrisation invariant"

EOM: $$L = L(\dot{x}^\mu) = -mc\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \equiv -mc\sqrt{-\dot{x}^2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\mu} = \frac{d}{dt}\frac{(-mc)(-\dot{x}_\mu)}{\sqrt{-\dot{x}^2}} = \boxed{\frac{d}{dt}\frac{mc\dot{x}_\mu}{\sqrt{-\dot{x}^2}} = 0}$$

Why does Zwiebach make this so long & messy?

$$\frac{\partial}{\partial \dot{x}^\mu}(-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) = -\eta_{\alpha\beta}\,\delta_{\alpha\mu}\,\dot{x}^\beta - \eta_{\alpha\beta}\,\dot{x}^\alpha\,\delta_{\beta\mu}$$
$$= -2\dot{x}_\mu$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{mc\,\dot{x}_\mu}{\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} = \frac{mc\,dx_\mu}{dt\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} = m\frac{dx_\mu}{d\tau}$$

$$\boxed{p^\mu = m\frac{dx^\mu}{d\tau} = 0}$$

Trick: The $\sqrt{\phantom{x}}$ is unpleasant and what is $L$ if $m = 0$?
A common way of resolving these problems is introducing an auxiliary field $e(t)$ (an "einbein", $g_{\mu\nu}(x) = e^a_\mu(x)\,e^b_\nu(x)\,\eta_{\mu\nu}$ ← vierbein):

$$L = L(e(t), \dot{x}^\mu(t)) = \frac{1}{2}\left[\frac{1}{e(t)}\dot{x}^2 - m^2 e(t)\right] \qquad \dot{x}^2 \equiv -\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

EOM
$$\begin{cases} 2\dfrac{\partial L}{\partial e} = -\dfrac{\dot{x}^2}{e^2} - m^2 = 0 \qquad \Rightarrow -\dot{x}^2 = m^2e^2,\ \sqrt{-\dot{x}^2} = me \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}^\mu} = \dfrac{d}{dt}\dfrac{\dot{x}_\mu}{e} = \underbrace{\dfrac{d}{dt}\dfrac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}} = 0}_{\text{classically the same EOM}} \end{cases}$$

classically the same EOM and $L$

$$L = \frac{1}{2}\left(\frac{m}{\sqrt{-\dot{x}^2}}\dot{x}^2 - m\sqrt{-\dot{x}^2}\right) = -m\sqrt{-\dot{x}^2}$$