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Pressure of hot QCD - is g^6 and beyond calculable?

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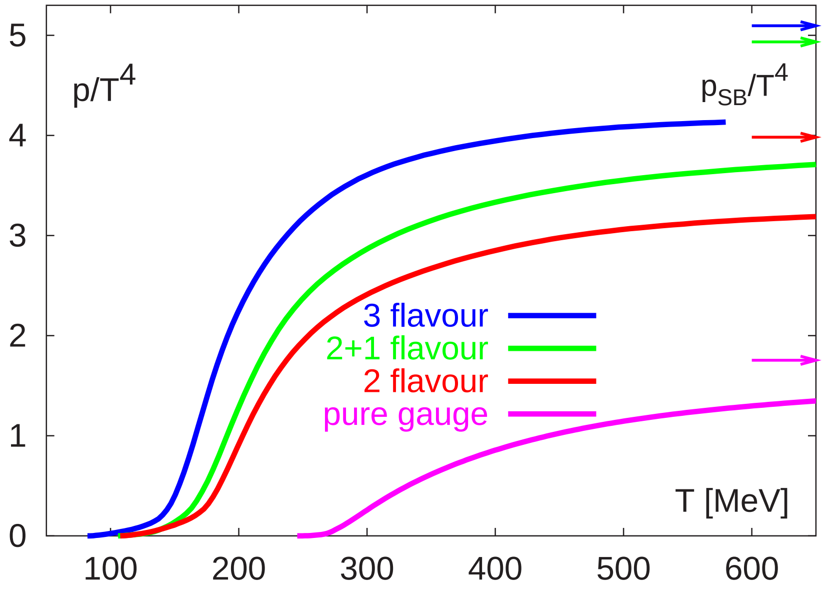
Here $p(T, \mu = 0)$, $\mu > 0$ has been done by Aleksi Vuorinen

We all know the problem: understand the ”experimental data”:

from Bielefeld



Basic reason of the difficulties: the magnetic sector of hot non-Abelian gauge theory is non-perturbative, confining, numerical.

The perturbative expression for the pressure is now known up to $g^6 \log g$:

$p/p_{\rm SB}$	$= 1$	Stefan-Boltzmann	
	$+g^2$	2-loop	(Shuryak 78)
	$+g^3$	resum 2-loop	(Kapusta 79)
	$+g^4 \ln 1/g$	resum 2-loop	(Toimela 83)
	$+g^4$	resum 3-loop	(Arnold, Zhai 94)
	$+g^5$	resum 3-loop	(Kastening, Zhai 95)
	$+g^6 \ln 1/g$	resum 4-loop	(KLRS02)
	$+g^6$	not computable in PT!	(Linde 80; this talk)
	$+g^7 \ldots$	. . .	(this talk)

$$p_{\rm SB} = \frac{\pi^2}{90}(16 + \frac{21}{2}N_f)T^4$$

Even though the g^6 coefficient is not calculable in PT, it is **calculable** - as is the non-perturbative sum starting g^7.

How?

Effective theory approach: Braaten-Nieto

$$p_{\rm QCD}(T)/T = (p_E + p_M + p_G)/T =$$
$$= T^3[1 + g^2(\Lambda) + g^4(\Lambda) + (\log\frac{\Lambda}{T} + a\log\frac{T}{\Lambda_E})g^6(\Lambda)] +$$
$$+m_E^3 + m_E^2 g_E^2 + m_E g_E^4 + (a\log\frac{\Lambda_E}{m_E} + b\log\frac{m_E}{\Lambda_M})g_E^6 +$$
$$+g_M^6\log\frac{\Lambda_M}{g_M^2}$$

Sequence of three theories:

- Full hot 4d QCD
- 3d gauge+adjoint Higgs theory, $S[A_i, A_0]$
- 3d gauge theory, $S[A_i] = \frac{1}{4}F_{ij}^2$

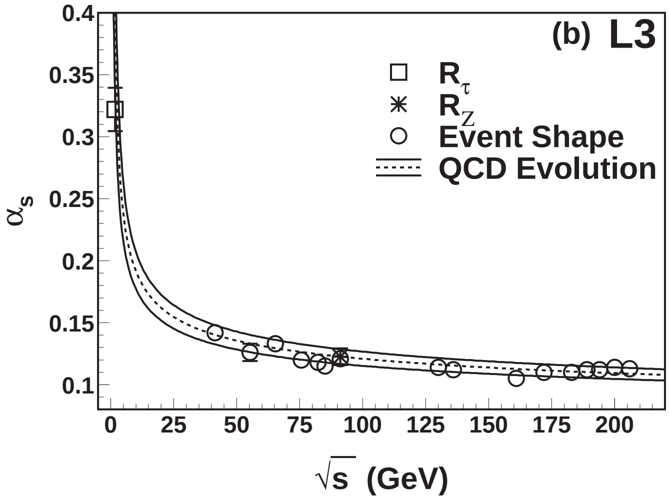
carefully matched in the UV. (Matching will be also the main point in the next talk by Vepsäläinen)

General remarks:

1. In the determination of $p(T)V = -F$ there is always an arbitrary "cosmological" constant. Fixed by

$$p(T \to 0) = 0 \quad \text{or} \quad p(T \to \infty) = p_{\text{SB}}(T).$$

2. For the calculation to be useful, renormalisation has to be carried out so that g is the $g(T/\Lambda_{\overline{\text{MS}}})$ experimentally determined in the $\overline{\text{MS}}$ scheme:



3. If

$$\mu \frac{\partial}{\partial \mu} g(\mu) = -\beta_0 g^3 - \beta_1 g^5 + \ldots$$

then

$$\mu \frac{\partial}{\partial \mu} \left[g^2(\mu) + 2\beta_0 \log(\mu) g^4(\mu) + (4\beta_0^2 \log^2(\mu) + 2\beta_1 \log(\mu)) g^6 \right]$$

is of order g^8.

Perturbative result (known for all N_f):

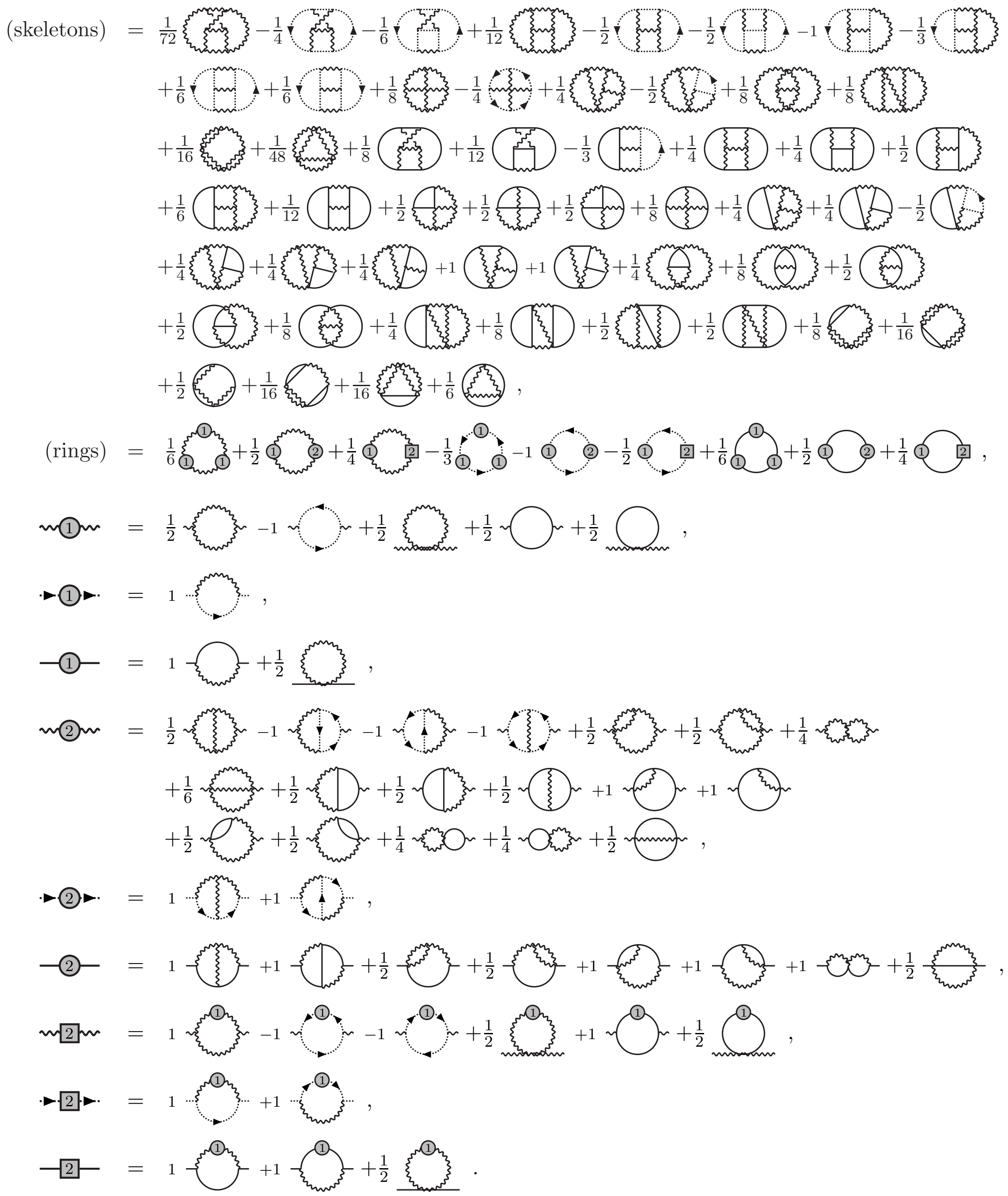
$$a \equiv \frac{\alpha_s(\bar\mu)}{\pi} \qquad N_f = 0$$

$$\begin{aligned}
p_{\text{QCD}}(T)/p_{\text{SB}}(T) &= 1 - \frac{15}{4}a + 30a^{3/2} + \\
+ & \left(237.2 + \frac{135}{2}\log a - \frac{11}{2}\frac{15}{4}\log\frac{\bar\mu}{2\pi T}\right)a^2 \\
+ & \left(-799.1 + \frac{495}{2}\log\frac{\bar\mu}{2\pi T}\right)a^{5/2} \\
+ & \left(-659.2 + 742.5\log\frac{\bar\mu}{2\pi T} - 475.6\right)a^3\log a \\
+ & \left(-\frac{1815}{16}\log^2\frac{\bar\mu}{2\pi T} + 2932.6\log\frac{\bar\mu}{2\pi T} + p_6\right)a^3 + \ldots
\end{aligned}$$

$$\begin{aligned}
-475.6 &= -\frac{17415}{16} + \frac{63585}{1024}\pi^2 \\
-659.2 &= -\frac{49005}{32} + \frac{198855}{2048}\pi^2 - \frac{1485}{2}(\log 2 - \gamma_E)
\end{aligned}$$

Better: do NOT expand $m_E^3/T^3 = (g^2+g^4+..)^{3/2} \sim g^3+g^5+..$!

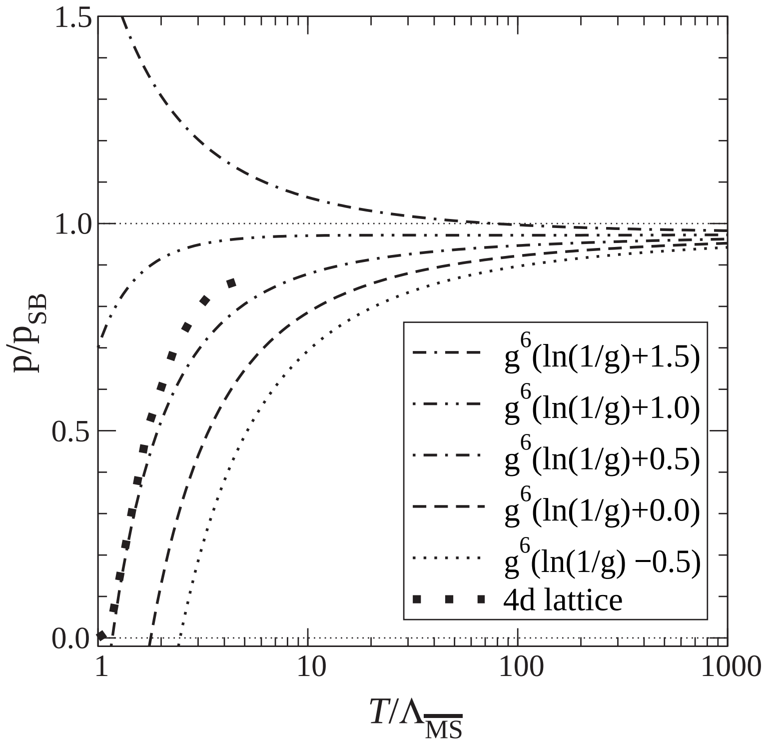
These were obtained by evaluating the 4loop graphs



Now that one has computed that

$$p(T)/p_{\text{SB}} = \ldots + 0.03738 g^6 \log \frac{1}{g} + \ldots$$

one can at least fit the g^6 coefficient:



$\log(1/g) + 0.7$ gives a good fit, but is the 0.7 only g^6?

Pure 3d SU(3)

$$Z = \exp[\frac{p_G(T)}{T} V_3]$$
$$= \int \mathcal{D}A_i \exp(-\int \mathrm{d}^d x \, \frac{1}{4} F^a_{ij} F^a_{ij}) \quad \overline{\mathbf{MS}}$$
$$= \int \mathcal{D}U_i \exp(-\beta \sum [1 - \Box]) \quad \text{lattice}$$

$\overline{\text{MS}}$ scheme: KLRS, hep-ph/0211321, Schröder, hep-ph/0211288

$$\frac{p_{\text{G}}(T)}{T\mu^{-2\epsilon}} = \frac{d_A C_A^3}{(4\pi)^4} g_{\text{M}}^6 \left[(\frac{43}{96} - \frac{157\pi^2}{6144})(\frac{1}{\epsilon} + 8 \ln \frac{\bar{\mu}}{2C_A g_{\text{M}}^2}) + \beta_G + .. \right]$$

For $\overline{\text{MS}}$ with no scale $1/\epsilon_{\text{UV}} - 1/\epsilon_{\text{IR}} = 0$. To separate the UV divergence, shield IR by introducing m_{gluon}, use arbitrary ξ. Then $\beta_G = \beta_G(m_{\text{gluon}}, \xi)$.

Lattice: Take derivative of $p_G(T)/T$ w.r.t β, determine $\langle \Box \rangle$, integrate back.
Remarkable recent progress using stochastic perturbation theory Di Renzo, Mantovi, Miccio, Schröder, hep-lat/0309111.
Determination of β_G and p_6 seems feasible! Avoid doing analytically 4-loop lattice perturbation theory!

After integration:

$$\begin{aligned}\frac{p_G(T)}{T} &= 3c\frac{\log g_3^2 a}{a^3} + \qquad c = d_A/3 \\ &+\frac{c_1}{2}\frac{g_3^2}{a^2} + \qquad c_1^{\text{Heller-Karsch}} = 1.94862 \\ &+\frac{c_2}{24}\frac{g_3^4}{a} + \qquad c_2 = 6.7 \pm 0.2 \\ &+\frac{g_3^6}{216}\left(c_3 \log\frac{6}{g_3^2 a} + \frac{1}{3}c_3 + \tilde{c}_3\right)\end{aligned}$$

$$c_3 = 0.9 \pm 0.4 \qquad \tilde{c}_3 = 23 \pm 5$$

The value of c_3 agrees with the analytic one!! Next

- Do $\overline{\text{MS}}$ in finite V or
- Do stochastic perturbation theory with m_{gluon}, ξ

to get the Linde coefficient β_{G}.

3d SU(3) gauge + adjoint Higgs theory:

$$\exp[\frac{p_M(T)}{T}V_3] = \int \mathcal{D}A_i^a \mathcal{D}A_0^a \exp\{-\int d^3x \times$$

$$[\frac{1}{4}F_{ij}^a F_{ij}^a + \frac{1}{2}(D_i A_0)^a (D_i A_0)^a +$$

$$+ \underbrace{\frac{1}{16\pi^2}(22\log\frac{5.371T}{\Lambda_{\overline{\rm MS}}} + 9)}_{\equiv m_E^2/g_E^4 = y \sim 1/g^2} \frac{1}{2}A_0^a A_0^a +$$

$$+ \underbrace{\frac{3}{44\log(5.371T/\Lambda_{\overline{\rm MS}})}}_{\equiv \lambda_E/g_E^2 \equiv x} \frac{1}{4}(A_0^a A_0^a)^2 + \frac{\cdots}{T}]\}$$

A_i, A_0,**x**, dimensionless, $D_i = \partial_i + iA_i$

Here m_E^2, g_E^2, λ_E are matched to 4d theory using next-to-leading-order optimised perturbation theory.

Both m_E^2 and λ_E^2 are given by $T/\Lambda_{\overline{\rm MS}}$.

$\overline{\text{MS}}$: KLRS, hep-ph/0304048

$$\frac{p_{\mathrm{M}}(T)}{T\mu^{-2\epsilon}} = m_E^3 + g_E^2 m_E^2 + g_E^4 m_E + $$
$$+\frac{d_A C_A^3}{(4\pi)^4} g_{\mathrm{E}}^6 \Big[\Big(\frac{43}{32} - \frac{491\pi^2}{6144}\Big)\Big(\frac{1}{\epsilon} + 8\ln\frac{\bar{\mu}}{2m_{\mathrm{E}}}\Big) + \beta_M + \mathcal{O}(\epsilon)\Big]$$

$$\begin{aligned}\beta_M &= -\frac{311}{256} - \frac{43}{32}\ln 2 - \frac{19}{6}\ln^2 2 + \frac{77}{9216}\pi^2 - \\ &\quad -\frac{491}{1536}\pi^2 \ln 2 + \frac{1793}{512}\zeta(3) + \gamma_{10} \\ &= -1.562519 + \gamma_{10} = -1.391512\ldots\end{aligned}$$

Lattice: Take derivative of $p_M(T)/T$ w.r.t m_E^2, determine $\langle A_0^a A_0^a \rangle$, integrate back.
Now we know 1-,2-,3-,4-loop perturbative results analytically (for $\langle \Box \rangle$ only 1- and 2-loop)! If

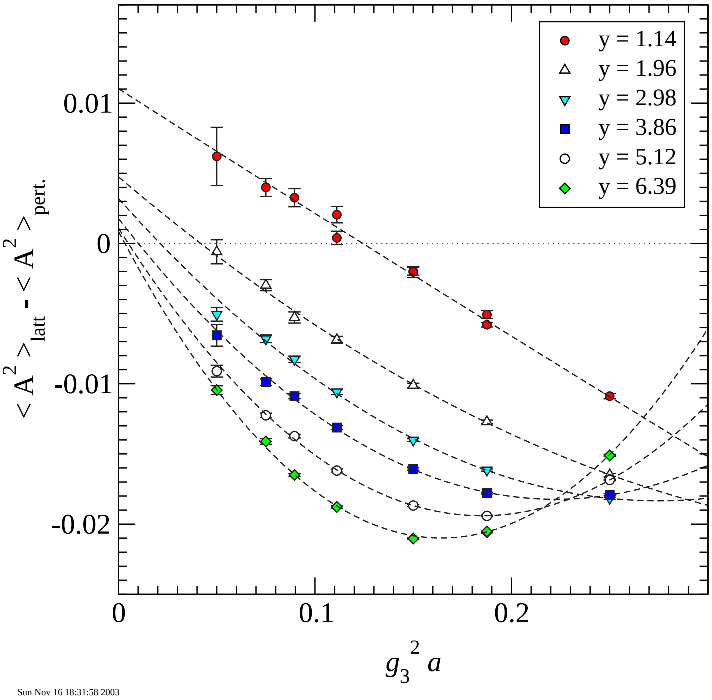
$$\frac{p_M}{T} \sim m_E^3 + g_E^2 m_E^2 + g_E^4 m_E + g_E^6 \Big(\log\frac{\bar{\mu}}{m_E} + \beta_M\Big) + \frac{g_E^8}{m_E} + \ldots$$

then (dimless terms; β_M disappears!))

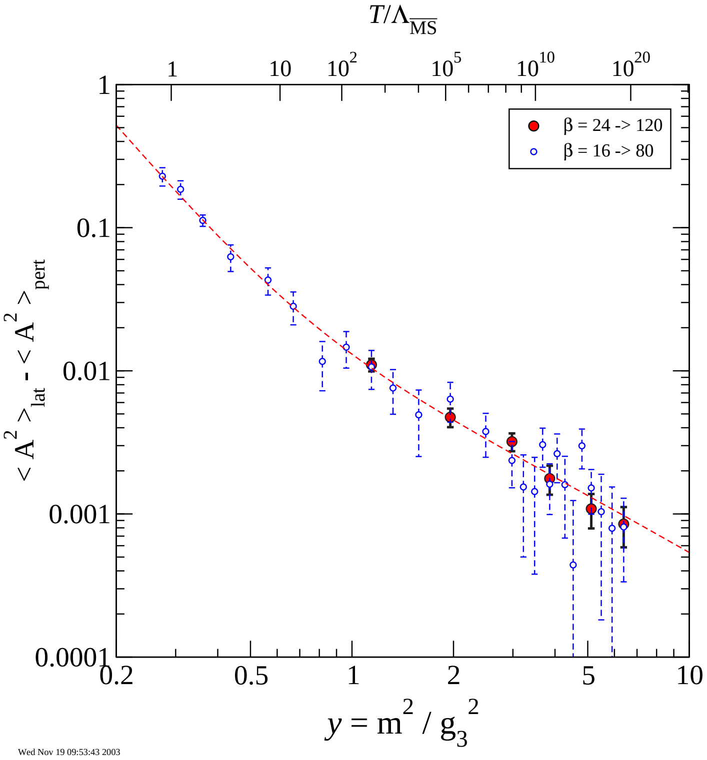
$$\Big\langle \frac{A_0^2}{g_E^2} \Big\rangle = -\frac{m_E}{\pi g_E^2} + 1 + \frac{g_E^2}{m_E} + \frac{g_E^4}{m_E^2} + \frac{g_E^6}{m_E^3} + \ldots$$

It is the last term +. . . we want to measure!

Continuum extrapolation: $\beta_G = 6/g_E^2 a \to \infty$.
Choose $y = m_E^2/g_E^4 \sim 1/g^2$ = 1.14 – 6.39, corresponding to $T \sim 100$ – $10^{20}\Lambda_{\overline{\rm MS}}$. Leading term was $-\sqrt{y}/\pi$.



Non-perturbative part of the condensate:



$$\langle \frac{A_0^2}{g_E^2} \rangle = \ldots + \frac{0.019}{y} + \frac{0.013}{y^{1.45}}$$

(0.019/y is the 4loop result, subtracted in fig).

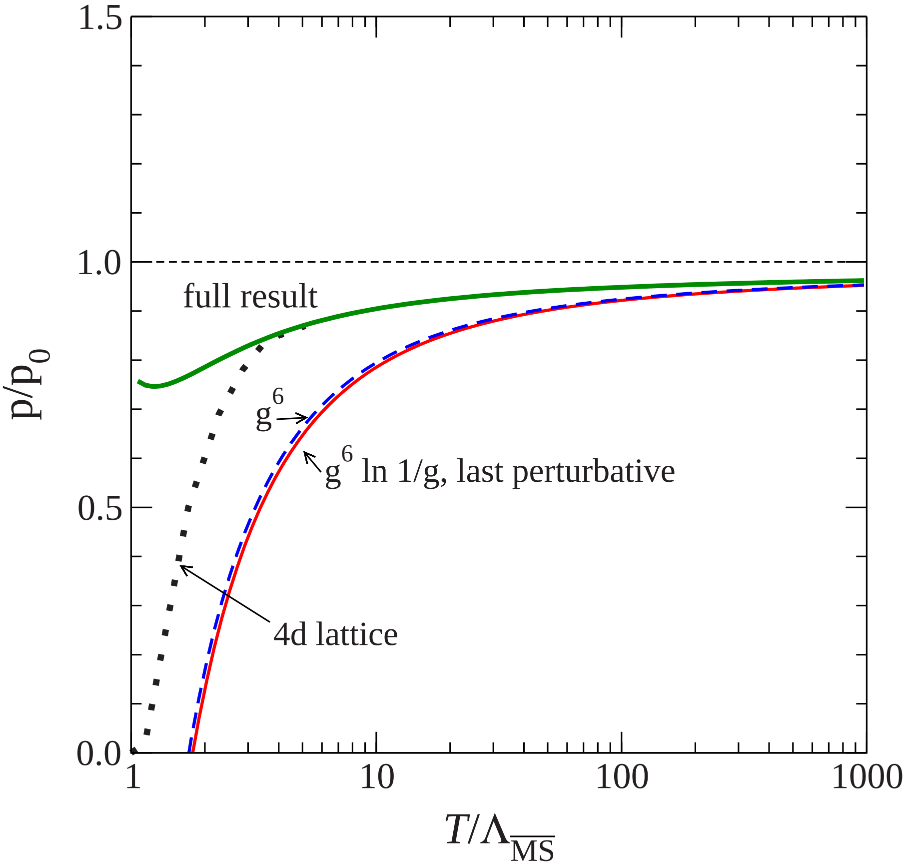
Integrating the nonperturbative additional term in $\langle A_0^2 \rangle$ over $y = m_E^2/g_E^4 = 1/g^2$ one obtains the additional free energy

$$-\frac{0.013}{0.45 y^{0.45}}$$

which adds to $p/p_{\rm SB}$ the last term in

$$\frac{p(T)}{p_{\rm SB}} = \ldots + 0.03738 g^6 \log\frac{1}{g} + 0.01645 g^{6.9}$$
$$= \ldots + 0.03738 g^6 \left(\log\frac{1}{g} + 0.44 g^{0.9}\right)$$

$1.7 > g > 1.3$
when
$2 < T/\Lambda < 10$
Thus
$0.44 g^{0.9}$
simulates 0.7!



That

$$+0.03738g^6\Big(\log\frac{1}{g}+0.44g^{0.9}\Big)$$

gives a good fit to data indicates that

- the order g^6 term is small
- perturbation theory + non-perturbative 3d effects describe physics down to $\approx 3T_c$. And this is a first-principle computation in field theory.

Why is the g^6 coefficient small? Big Linde term β_G which is effectively cancelled by -big terms from matching?

One should work out the g^6 coefficient!

Small corrections from $\langle A_0^4\rangle$ (start at g^6), higher dimensional terms truncated in the effective theory $S[A_i, A_0]$, say, $A_0^2F_{ij}^2$, (start at g^7)...

What remains to be done for g^6?

$$
\begin{aligned}
&+\frac{g^6}{(4\pi)^4}\Big\{\beta_{\mathrm{E1}} - \frac{1}{4}d_A\alpha_{\mathrm{E4}}\Big[(d_A+2)\beta_{\mathrm{E4}} + \frac{2d_A-1}{N_c}\beta_{\mathrm{E5}}\Big] \\
&\quad - d_A C_A\Big[\frac{1}{4}(\alpha_{\mathrm{E6}} + \alpha_{\mathrm{E5}}\alpha_{\mathrm{E7}} + 3\alpha_{\mathrm{E4}}\alpha_{\mathrm{E7}} + \beta_{\mathrm{E2}} + \alpha_{\mathrm{E4}}\beta_{\mathrm{E3}}) \\
&\quad + (\alpha_{\mathrm{E6}} + \alpha_{\mathrm{E4}}\alpha_{\mathrm{E7}})\Big(\frac{1}{4\epsilon} + \ln\frac{\bar\mu}{2gT\alpha_{\mathrm{E4}}^{1/2}}\Big)\Big] \\
&+ d_A C_A^3\Big[\beta_{\mathrm{M}} + \beta_{\mathrm{G}} + \alpha_{\mathrm{M}}\Big(\frac{1}{\epsilon} + 8\ln\frac{\bar\mu}{2gT\alpha_{\mathrm{E4}}^{1/2}}\Big) + \alpha_{\mathrm{G}}\Big(\frac{1}{\epsilon} + 8\ln\frac{\bar\mu}{2g^2TC_A}\Big)\Big]\Big\}
\end{aligned}
$$

β_{E1}: Calculate in full 4d theory in the $\overline{\mathrm{MS}}$ scheme the order g^6 term for the pressure. Need 4-loop sum-integrals, should be doable. IR $1/\epsilon$ poles appear which precisely cancel those above.

$\beta_{\mathrm{E2}}, \beta_{\mathrm{E3}}$: Calculate in full 4d theory the order ϵ 2-loop terms in m_E^2, g_E^2.

β_{G}: Use stochastic perturbation theory in pure SU(3) to relate $\overline{\mathrm{MS}}$ and lattice and to find this Linde coefficient.

Conclusions

- The perturbative computation of the pressure $p(T,\mu)$ of hot QCD has been driven as far it can be
- There exists a definite and realistic scheme for computing the order g^6 term. One needs both numerical lattice and analytic computations.
- Inclusion of 3d nonperturbative effects beyond g^6 extends the agreement with present lattice data down to $3T_c$.
- Can one ever have such accurate and controllable approximations for time-dependent kinetic problems?