String-theory inspired methods for hot QCD

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Strictly speaking, there are NO results for hot QCD. There are results from "AdS/CFT=AntideSitter/ConformalFieldTheory", "gravity/gauge theory duality", for $\mathcal{N} = 4$ hot supersymmetric gauge theory matter for $N_c \gg 1$, $g^2 N_c \gg 1$.[1]

One argues that hot QCD matter (QGP) observed at RHIC simulates this for $1.5T_c \lesssim T \lesssim \text{few} \times T_c$. It is "conformally invariant" (why not for $T_c \lesssim T \lesssim 1.5T_c$?), has $N_c = 3 \gg 1$ and is "strongly coupled", $g^2 N_c \approx 12 \gg 1$.

$\Rightarrow$ Excitement

These 3 lectures[1] assume familiarity with the QCD side of deriving properties of hot QCD matter using perturbation theory and/or lattice MC and aim at explaining AdS, CFT, $\mathcal{N} = 4$, gauge/gravity duality, 4d boundary, 5d bulk, black holes and giving an operational derivation of some of the results.

This is a complicated theoretical structure with no experimental confirmation. Great fun for a theorist.

[1] General lectures on this topic I have found useful are Lyng Petersen, hep-ph/9902131; D'Hoker-Freedman, hep-ph/0201253

To keep in mind:

The duality $\sim$ equality will be between

Quantum field theory (a special one!) in 4d

Classical gravity in 5d (for $N_c \gg 1$, $g^2 N_c \gg 1$)

1. Some Classical Gravity [2]

Einstein-Hilbert[3] (some coordinates $x^\mu = x^0, x^1, ..., x^{d-1}$, flat metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,..,1)$):

$$S[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^dx\sqrt{g}(R+2\Lambda), \qquad ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \quad g = |\det g_{\mu\nu}| \quad g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu,$$

$$g_{\mu\nu} \Rightarrow R^\alpha_{\mu\beta\nu},\; R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}, \quad R = g^{\mu\nu}R_{\mu\nu}, \quad \dim R = 1/\mathrm{length}^2 = \mathrm{GeV}^2, \quad \dim G = \mathrm{GeV}^{d-2}.$$

EOM from $\delta S/\delta g_{\mu\nu} = 0$:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = 0\left(= 8\pi G T_{\mu\nu},\; T_{\mu\nu} = \frac{-2}{\sqrt{g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}\right).$$

Varying conventions is a nuisance. Here for d-dimensional anti-de Sitter (AdS) space of radius $\mathcal{L}$:

$$\Lambda = \frac{(d-1)(d-2)}{\mathcal{L}^2} \Rightarrow R_{\mu\nu} = -\frac{(d-1)}{\mathcal{L}^2}g_{\mu\nu}, \quad R = -\frac{(d-1)d}{\mathcal{L}^2}, \quad R+2\Lambda = -\frac{2(d-1)}{\mathcal{L}^2} \tag{1}$$

The perfect fluid en-mom tensor $T_{\mu\nu} = (\epsilon+p)u_\mu u_\nu + pg_{\mu\nu}$ becomes for vacuum $T_{\mu\nu} = p_{\text{vac}}g_{\mu\nu} = -\epsilon_{\text{vac}}g_{\mu\nu}$ and since $\Lambda g_{\mu\nu}$ should behave as $+8\pi G T_{\mu\nu}$ we have $\epsilon_{\text{vac}} = -\Lambda/(8\pi G) < 0$ which really characterises anti-dS space.

Number of comps in d dim: $R_{\mu\nu\alpha\beta} : d^2(d^2-1)/12$, $R_{\mu\nu} : d(d+1)/2$.

[2] Sean Carroll, Spacetime and Geometry; S. Weinberg, Gravitation and Cosmology

[3] Interested in history? Read Ebner, How Hilbert has found the Einstein equations before Einstein and forgeries of Hilbert's page proofs, arXiv:physics/0610154

Special solutions

1. Black hole in our world, $d = 4$, solution of

$$R_{\mu\nu} = 0 \qquad \text{or of } R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

which is asymptotically flat ($\eta_{\mu\nu}$) and regular on and outside an event horizon (coordinates t, r, θ, ϕ):

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{1}{1 - r_s/r} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

$$T_{\mathsf{Hawk}} = \frac{1}{4\pi r_s} = \frac{\hbar c}{4\pi r_s} = \frac{\hbar c^3}{8\pi G M} = \frac{M_{\mathsf{Pl}}^2 c^2}{8\pi M}, \quad S_{\mathsf{BH}} = \frac{A}{4G} = \frac{c^3}{\hbar G}\frac{A}{4} = \frac{4\pi G M^2}{\hbar c} \equiv 4\pi \frac{M^2}{M_{\mathsf{Pl}}^2} \equiv \frac{A}{8\pi L_{\mathsf{Pl}}^2}.$$

2. AdS_5, a solution of [4]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{6}{\mathcal{L}^2} g_{\mu\nu} \tag{2}$$

With coordinates t, x^1, x^2, x^3, z

$$ds^2 = \frac{\mathcal{L}^2}{z^2}(-dt^2 + d\mathbf{x}^2 + dz^2) \quad z = 0 \text{ is } \textbf{boundary}, \ z > 0 \text{ is } \textbf{bulk}$$

$$= \frac{r^2}{\mathcal{L}^2}(-dt^2 + d\mathbf{x}^2) + \frac{\mathcal{L}^2}{r^2} dr^2, \quad r = \frac{\mathcal{L}^2}{z}, \quad r = \infty \text{ is } \textbf{boundary}$$

Note how a distance scale $\mathcal{L}$ has entered!

[4]For parametrisations of the Anti de Sitter metric, see Balasubramanian-Kraus-Lawrence, hep-th/9805171, Appendix A; Lyng Petersen, hep-th/9902131, section 2; Douglas-Randjbar-Daemi, hep-th/9902022, section 6. For a detailed analysis of de Sitter, see Kim-Oh-Park, hep-th/0212326

Symmetry of AdS_5

AdS_5 can be represented as the surface

$$-t_1^2 - t_2^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = -\mathcal{L}^2$$

in the flat 6 dimensional space with metric

$$ds^2 = -dt_1^2 - dt_2^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

Just like a 2d sphere S_2 is the surface $x_1^2 + x_2^2 + x_3^2 = R^2$ in the flat R_3 with metric $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$.

AdS_5 has the symmetry O(2,4)

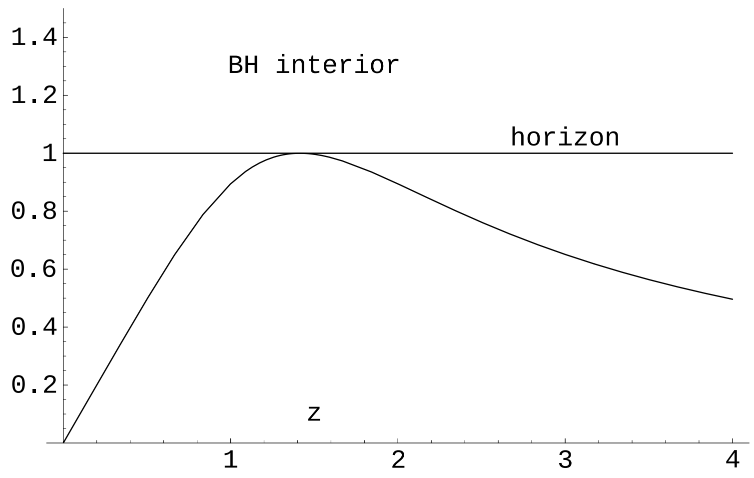
3. AdS$_5$ black hole

$$ds^2 = \frac{\mathcal{L}^2}{\tilde{z}^2}\left[-\left(1-\frac{\tilde{z}^4}{z_0^4}\right)dt^2 + d\mathbf{x}^2 + \frac{d\tilde{z}^2}{1-\tilde{z}^4/z_0^4}\right] \quad (3)$$

$$T_{\text{Hawk}} = \frac{1}{\pi z_0}, \qquad S = \frac{A}{4G_5} = \frac{1}{4G_5}\int d^3x \sqrt{\left(\frac{\mathcal{L}^2}{z_0}\right)^3} = V_3 \cdot \frac{\mathcal{L}^3}{G_5}\frac{1}{4}\left(\pi T_{\text{Hawk}}\right)^3 = V_3 \cdot \frac{\pi^2 N_c^2}{2} T^3.$$

Polchinski, cosmicvariance.com/2006/12/07/: *Physicists have found that some of the properties of this plasma are better modeled (via duality) as a tiny black hole in a space with extra dimensions than as the expected clump of elementary particles in the usual four dimensions of spacetime.*

Transform to a form with just ... $+ dz^2/z^2$ by $\tilde{z}^2 = z^2/(1+z^4/4z_0^4)$:



$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-\frac{(1-z^4/(4z_0^4))^2}{1+z^4/(4z_0^4)}dt^2 + \left(1+\frac{z^4}{4z_0^4}\right)(dx_1^2+dx_2^2+dx_3^2) + dz^2\right] \quad (4)$$

Some technology: use Mathematica (or similar) to do gravity algebra. Assume you want to show that the BH (3) satisfies 5d AdS eqs (2). Here is a piece of Math code: [5]

n=5 (fix the dimensionality; do not use i j k l s n for anything on your own!)

coord = {t, x1, x2, x3, z}   (fix the vector with coordinates used)

metric = {{-a[z] b^2/z^2, 0, 0, 0, 0}, {0, b^2/z^2, 0, 0, 0}, {0, 0, b^2/z^2, 0, 0}, {0, 0, 0, b^2/z^2, 0}, {0, 0, 0, 0, b^2/(a[z]z^2)}}

(define the matrix $g_{\mu\nu}$ by desired symmetries. Here we pretend one function a[z] is unknown)

inversemetric = Simplify[Inverse[metric]]   (compute $g^{\mu\nu}$)

affine := affine = Simplify[Table[(1/2)*Sum[(inversemetric[[i, s]])* (D[metric[[s, j]], coord[[k]]] + D[metric[[s, k]], coord[[j]]] - D[metric[[j, k]], coord[[s]]]), {s, 1, n}], {i, 1, n}, {j, 1, n}, {k, 1, n}]]

(compute the Christoffel symbols $\Gamma^i_{jk} = \frac{1}{2} g^{is}(\partial_k g_{sj} + \partial_j g_{sk} - \partial_s g_{jk})$)

riemann := riemann = Simplify[Table[D[affine[[i, j, l]], coord[[k]]] - D[affine[[i, j, k]], coord[[l]]] + Sum[affine[[s, j, l]] affine[[i, k, s]] - affine[[s, j, k]] affine[[i, l, s]], {s, 1, n}], {i, 1, n}, {j, 1, n}, {k, 1, n}, {l, 1, n}]]   (compute $R^i_{jkl} = ...$).

ricci := ricci = Simplify[Table[Sum[riemann[[i, j, i, l]], {i, 1, n}], {j, 1, n}, {l, 1, n}]]   (Ricci is $R_{jl} = R^i_{jil}$)

scalar = Simplify[Sum[inversemetric[[i, j]]ricci[[i, j]], {i, 1, n}, {j, 1, n}]]   ($R = g^{ij} R_{ij}$; not needed now)

einstein := einstein = Simplify[ricci - (1/2)scalar*metric]   (Einstein tensor also not needed now)

ads := ads = Simplify[ricci + (4/b^2)*metric]   (this should vanish, but does not, ads[[2,2]] gives)

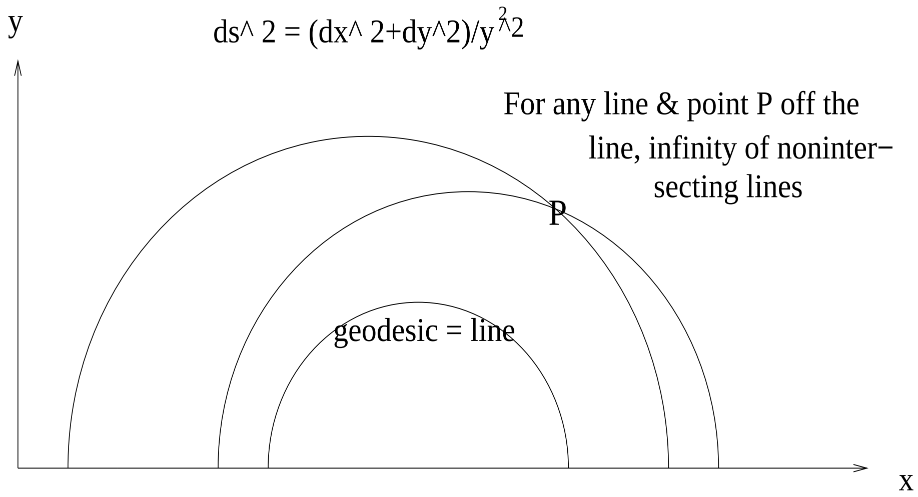
$(4 - 4a[z] + za'[z])/z^2$ from which you solve $a[z] = 1 + Cz^4$ and find that all comps vanish.

[5] Google curvature.nb or get diffgeo5.m or diffgeo4.m from M. Headrick: http://www.stanford.edu/~headrick

What is this $\mathcal{L}^2/z^2$ in the AdS metric?

Poincare plane: model of non-Euclidian geometry [6]

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$$



$$s = \int ds = \int dy \frac{1}{y}\sqrt{1 + x'(y)^2} \equiv \int dy L[x'(y)] \Rightarrow \frac{d}{dy}\left[\frac{x'(y)}{y\sqrt{1+x'^2}}\right] = 0 \Rightarrow (x-a)^2 + y^2 = c^2$$

Note: much simpler to pull out dy, use $L[x'(y)]$, not $L[y(x), y'(x)]$!

[6] http://en.wikipedia.org/wiki/Poincarémetric

IIB string theory on $G_{\mu\nu} \leftrightarrow$ AdS$_5\times$S^5 (=just (!!) a 2-dimensional nonlinear sigma model for the $d=10$ fields $X^0(\sigma^1,\sigma^2), ..., X^{d-1}(\sigma^1,\sigma^2)$):

$$S[X^\mu,\psi^\mu,..] \; = \; -\frac{T}{2}\int d^2\sigma\sqrt{-\mathrm{det}h_{ab}}\left[h^{ab}G_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu + \epsilon^{ab}B_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + ... \right.$$
$$\left. -G_{\mu\nu}(X)e^\alpha_a\bar{\psi}^\mu i\rho^a\partial_\alpha\psi^\nu + ...\right] \qquad X^\mu(\sigma^1,\sigma^2)$$

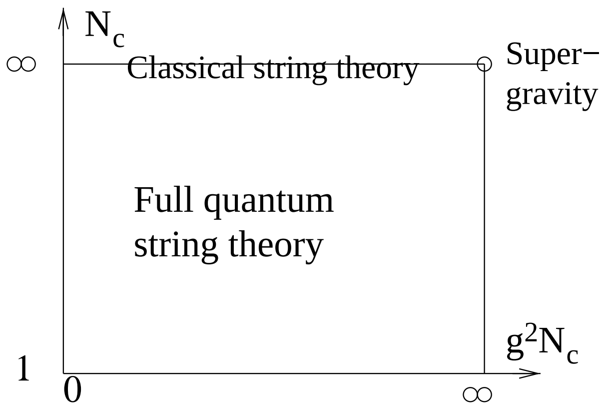
Parameters: the tension $T = 1/(2\pi\alpha')$ and radius $\mathcal{L}$ of AdS$_5$ and S^5.

Nobody knows what is the correct **background** $G_{\mu\nu}(X)$.

AdS/CFT duality: This theory is the same as $\mathcal{N}=4$ (number of SuSy generators) conformal supersymmetric SU(N_c) Yang-Mills theory living on the 4d boundary of AdS$_5$ if

$$\mathcal{L}^2 = \sqrt{g^2N_c}\alpha'$$

Calculable if $N_c \gg 1$,
$\lambda \equiv g^2N_c \gg 1$



String actions:

Particle action $= -m\int d\tau \Rightarrow$ String action $= -T\int dA$. $T = \frac{1}{2\pi\alpha'} = \textbf{Tension}$.

String $X^\mu(\tau,\sigma)$ moving in a space with metric $ds^2 = G_{\mu\nu}dx^\mu dx^\nu$ has the action (reparametrisation invariance!!):

$$S = -T\int d\tau d\sigma\sqrt{-\mathrm{det}h_{ab}}, \qquad h_{ab} = G_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}, \qquad \sigma^a = (\tau,\sigma).$$

Nambu-Goto: $G_{\mu\nu} = \eta_{\mu\nu}$

$$h_{ab} = \begin{pmatrix} \dot{X}\cdot\dot{X} & \dot{X}\cdot X' \\ X'\cdot\dot{X} & X'\cdot X' \end{pmatrix} \qquad X\cdot Y \equiv \eta_{\mu\nu}X^\mu Y^\nu \qquad \dot{} = \partial/\partial\tau,\ {}' = \partial/\partial\sigma.$$

Polyakov:

$$\begin{aligned} S &= -T\int d\tau d\sigma\sqrt{-\mathrm{det}h_{ab}} = -T\int d\tau d\sigma\sqrt{-\mathrm{det}h_{ab}}\,\frac{1}{2}\,h^{ab}h_{ab} \\ &= -\frac{1}{2}T\int d\tau d\sigma\sqrt{-\mathrm{det}h_{ab}}\,h^{ab}\,G_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b} \end{aligned}$$

String theory simplifies if $N_c \gg 1$ (string loops suppressed, corrections of order $1/N_c$ cannot be computed) and if also $g^2 N_c \gg 1$ (then $\alpha' \ll \mathcal{L}^2$ is "small"; corrections of order $1/\lambda^{3/2}$ can be computed):

String theory (for $G_{\mu\nu} = \eta_{\mu\nu}$) has a spectrum of excitations of type $M^2 = N/\alpha',\ N = 0, 1, 2, \ldots$ In the limit $\alpha' \to 0$ only massless excitations of spin $2, 1, 0, 3/2, \ldots$ are excited. The theory describing these excitations is supergravity[7]

$$S[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^{10}x\sqrt{g}e^{-2\Phi}[R + 4(\nabla\Phi)^2 - \frac{1}{4\cdot 5!}F_{\mu\nu\alpha\beta\gamma}F^{\mu\nu\alpha\beta\gamma} + \ldots]$$

$$S[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^{10}x\sqrt{g}[R - \frac{1}{2}(\nabla\Phi)^2 - \frac{1}{4\cdot 5!}F_5^2 + .. + \frac{\zeta(3)\alpha'^3}{8}e^{-\frac{3}{2}\Phi}W + \ldots]$$

where[8] the α' correction term is proportional to fourth power of the Weyl tensor (dimensionally the corrections could be $R + \alpha' R^2 + \alpha'^2 R^3 + \ldots$) and we are back to classical gravity field eqs with solutions of type

$$ds^2 = \frac{1}{\sqrt{1+\frac{\mathcal{L}^4}{r^4}}}\Big[-\Big(1-\frac{r_0^4}{r^4}\Big)dt^2 + d\mathbf{x}^2\Big] + \sqrt{1+\frac{\mathcal{L}^4}{r^4}}\Big[\frac{dr^2}{1-\frac{r_0^4}{r^4}} + r^2 d\Omega_5^2\Big], \quad \Phi = \mathrm{const}, \quad F_5^2 \sim \frac{\mathcal{L}^8}{r^{10}}$$

which for $\sqrt{1+\mathcal{L}^4/r^4} \to \mathcal{L}^2/r^2$ go to the 5d black hole.

[7]For string frame $\to$ Einstein frame, see Carroll p. 184
[8]Gubser-Klebanov-Tseytlin, hep-th/9805156

2. $\mathcal{N} = 4$ supersymmetric Yang-Mills

$$\text{QCD} \qquad S[A_\mu, \psi, \bar{\psi}] = \int d^d x \left[-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \sum_f \bar{\psi}_f \left(i\gamma^\mu D_\mu \right) \psi_f \right]$$

$$\mathcal{N} = 1 \text{ SuSy} \qquad S[A_\mu, \lambda] = \int d^d x \left[-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \frac{1}{2} \bar{\lambda}^a i \Gamma^\mu \left(D_\mu \lambda \right)^a \right]$$

$\mathcal{N} = 4$ SuSy in full glory (1 vector, 4 fermions, 6 scalars, all adjoint) [9]

$$S[A^a_\mu, \phi^a_i, \psi^a, \bar{\psi}^a] = \frac{1}{2g^2} \int d^4 x \left\{ \frac{1}{2} F^{a\,2}_{\mu\nu} + (\partial_\mu \phi^a_i + f_{abc} A^b_\mu \phi^c_i)^2 + \bar{\psi}^a i \gamma^\mu (\partial_\mu \psi^a + f_{abc} A^b_\mu \psi^c) \right.$$

$$\left. + i f_{abc} \bar{\psi}^a \Gamma^i \phi^b_i \psi^c - \sum_{i<j} f_{abc} f_{ade} \phi^b_i \phi^c_j \phi^d_i \phi^e_j + \partial_\mu \bar{c}^a (\partial_\mu c^a + f_{abc} A^b_\mu c^c) + \xi (\partial_\mu A^a_\mu)^2 \right\}$$

Here $\mu = 1, .., 4$, $i = 5, .., 10$, (γ^μ, Γ^i) are ten real 16×16 Dirac matrices, ψ^a is a 16-component spinor, also ghost and gauge fixing has been introduced.

Compute the beta function:

$$\mu \frac{\partial g(\mu)}{\partial \mu} = \beta(g)$$

find $\beta(g) = 0$! Conformally invariant on quantum level! Coupling does not run! [10]

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[9] Many forms in literature, this is from Erickson-Semenoff-Zarembo, hep-th/0003055

[10] D.R.T.Jones, Phys.Lett.B100(1977)199(2loop), Brink et al, 1983(anyloop)

What is this $\mathcal{N}$ in $\mathcal{N} = 4$?

Read[11]. $\mathcal{N} = 4$ is the number of SuSy generators Q_a^i, $a = 1, 2$ (Weyl spinor index, rep of Lorentz group), $i = 1, .., \mathcal{N}$.

$$[P_\mu, P_\nu] = 0,\ [P_\lambda, L_{\mu\nu}] = ..., \qquad \{Q_a^i, \bar{Q}_{\dot{a}}^j\} = \delta_{ij}\, 2\sigma^\mu_{a\dot{a}} P_\mu, \quad \{Q_a^i, Q_{\dot{a}}^j\} = 0,$$
$$[P_\mu, Q_a^i] = 0, \qquad [L_{\mu\nu}, Q_a^i] = -i(\sigma_{\mu\nu})_a^b Q_b^i.$$

Massless reps of Poincaré: $|p, h\rangle$. Including Q_a^i means (read[12]) also $|p, h - \frac{1}{2}\rangle$, etc are needed $\Rightarrow$ $1,\ \frac{1}{2},\ 0,\ -\frac{1}{2},\ -1$:

$\mathcal{N} = 4$ gauge multiplet:

A_μ, $h = \pm 1$, 2 states, λ_a^i, $h = \pm\frac{1}{2}$, 8 states, X^k, 6 scalars.

[11] 'd Hoker, Freedman, hep-th/0201253, Sections 2.1-4,3.1-3
[12] Bailin-Love, Sect. 1.4, 1.6

Symmetries of $\mathcal{N}=4$ SYM

Of course there is Lorentz, O$(1,3)\sim$ SL$(2$,C). Scale invariance (dimensionless coupling) $\Rightarrow$ conformal invariance, O$(2,4)\sim$ SU$(2,2)$. In QuantumCD this is broken ($g(\mu)$ runs), in $\mathcal{N}=4$ Quantum SYM coupling does not run.

$\mathcal{N}=4$ SYM has the symmetry O(2,4), just like AdS$_5$

Other symmetries, too: The $\mathcal{N}=4$ generators Q_i^a can be rotated $\Rightarrow$ SU(4)$\sim$ SO(6) "R symmetry". (Count dofs: $4^2-1=\frac{1}{2}6(6-1)$)

Including SuSy transformations $\Rightarrow$ an even larger supergroup SU(2,2|4).

SU(N_c)

Master formula for 4d gauge quantum field theory $\Leftrightarrow$ 5d classical gravity:

$$\langle \exp\left[\int d^4x\, O(x)\phi(x,0)\right]\rangle_{\rm FT} = \exp\left\{-\int d^4x \int_0^{z_0} dz\, \mathcal{L}_{\rm class}[\phi(x,z)]\right\}$$

$$x^\mu = (t, x^1, x^2, x^3) \qquad\qquad x^M = (t, x^1, x^2, x^3, z)$$

LHS: All there is in the field theory, all operator expectation values:

$$\text{e.g.,}\quad \frac{\delta^2 \rm LHS}{\delta\phi(x,0)\delta\phi(y,0)} = \langle O(x)O(y)\rangle_{\rm FT}$$

RHS: Find the field, current $\phi(x)$ to which the operator $\mathcal{O}$ couples ($\mathcal{O} = F^{a\,2}_{\mu\nu} \Rightarrow \phi(x)$, $\mathcal{O} = T_{\mu\nu} \Rightarrow \phi = g_{\mu\nu}$, etc). Then solve classical 5d gravity EOM for $\phi(x,z)$ with proper BC and compute the LHS. Approximation works when the coupling of LHS is large, non-perturbative!

Key issue: holography

Dofs can match since number of dofs for gravity $\sim$ area, not volume.

Application 1: pressure of hot SYM matter

In the ideal gas limit the pressure of $\mathcal{N} = 4$ SuSy YM would be

$$p(T) = (g_B + \frac{7}{8} g_F)\frac{\pi^2}{90}T^4 = (8+7)d_A\frac{\pi^2}{90}T^4 = \frac{\pi^2(N_c^2-1)}{6}T^4 \equiv aT^4,$$

(one vector = 2, six scalars = 6, four fermions = 8, all adjoint).

Weak coupling correction terms have been computed[13], a, b, c, d are coming:

$$a = N_c^2\,\frac{\pi^2}{6}\left[1 - \frac{3}{2\pi^2}\lambda + \frac{3+\sqrt{2}}{\pi^3}\lambda^{3/2} + a\lambda^2\log\lambda + b\lambda^2 + c\lambda^{5/2} + d\lambda^3\log\lambda + ...\right], \quad \lambda \equiv g^2N_c.$$

The plasmon term is

$$\frac{d_A}{12\pi}(m_E^3 + N_s m_S^3), \quad N_s = 6$$

where the two effective masses are[14]

$$m_E^2 = \left(2\lambda + \text{number}_E\cdot\lambda^2 + ...\right)T^2, \qquad m_S^2 = \left(\lambda + \text{number}_S\cdot\lambda^2 + ...\right)T^2.$$

For QCD:

$$m_E^2 = \frac{1}{3}\,N_c g^2(T)T^2 \equiv \frac{1}{3}\,\lambda(T)T^2,$$

the number "runs".

No phase transition, no "hadrons", in $\mathcal{N} = 4$ SuSy YM!

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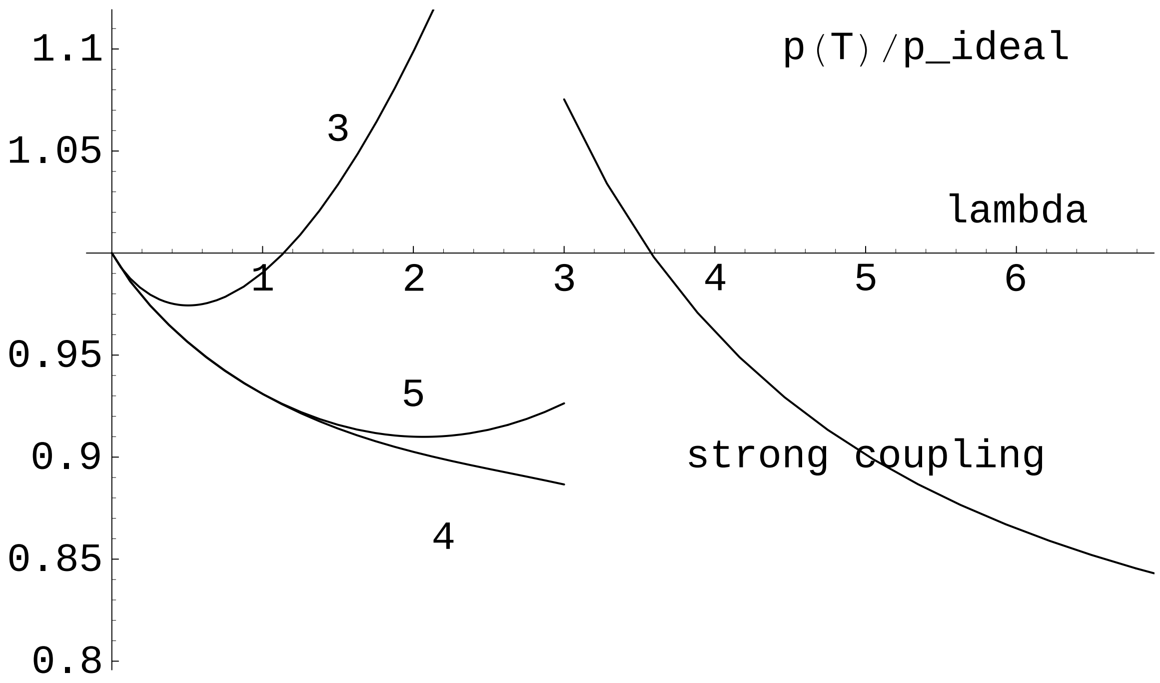
[13] E.g., Nieto-Tytgat, hep-th/9906147

[14] In the strong coupling limit $m_E = 3.4\pi T$, Bak-Karch-Yaffe, arXiv:0705.0994

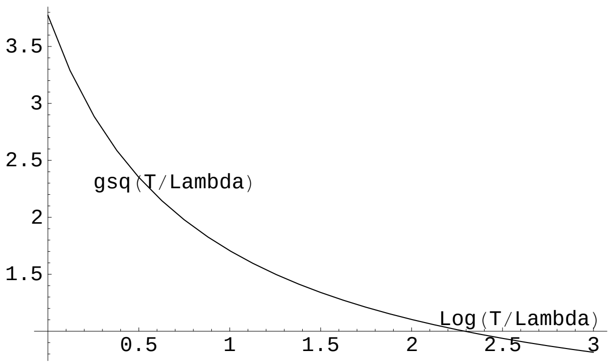
The result from $\mathrm{AdS}_5 \times \mathrm{S}_5$ is

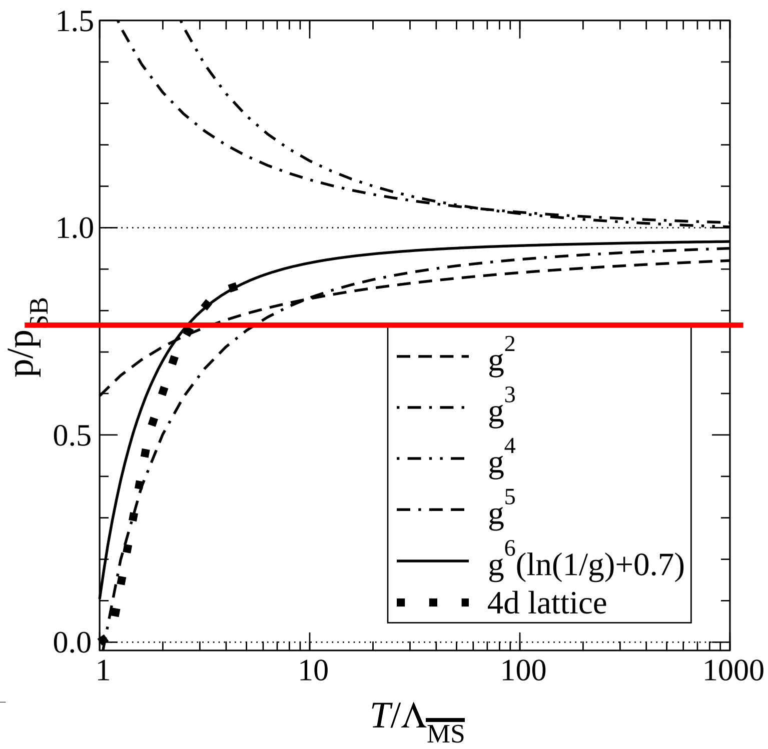
$$p(T) = \frac{\pi^2 N_c^2}{6} T^4 \left[\frac{3}{4} + \frac{45\zeta(3)}{32} \frac{1}{\lambda^{3/2}} + ... \right]$$

Joining weak (orders g^3, g^4, g^5, preliminary) and strong coupling:



"Experimental evidence for the 3/4"





Points: Lattice Monte Carlo, Curves: Perturbation theory, Red: The famous 3/4

Argument: Near T_c the gauge system is necessarily strongly interacting. For $T \gtrsim 2T_c$ $g^2(T)$ nearly constant, $N_c g^2 \gg 1$, $\epsilon - 3p \approx 0$, the system is $\approx$ conformally invariant. The 3/4 gives average behavior. Good fit!

How do you derive the result?

You want to get the energy-momentum tensor of a thermalised system of quanta of $\mathcal{N}=4$ 4d SYM boundary theory from solutions of 5d bulk gravity. Expect $T_{\mu\nu}(x)$ to be related to $g_{MN}(x,z)$.

Method [15]: write the 5d metric in the form

$$g_{MN} = \frac{\mathcal{L}^2}{z^2}\begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}$$

and expand near $z=0$:

$$g_{\mu\nu}(x,z) = g^{(0)}_{\mu\nu}(x) + g^{(2)}_{\mu\nu}(x)z^2 + g^{(4)}_{\mu\nu}(x)z^4 + ...$$

Then

$$T_{\mu\nu} = \frac{\mathcal{L}^3}{4\pi G_5}\left[g^{(4)}_{\mu\nu} - \frac{1}{8}g^{(0)}_{\mu\nu}[(\mathrm{Tr}\, g_{(2)})^2 - \mathrm{Tr}\, g^2_{(2)}] - \frac{1}{2}(g_{(2)}g^{-1}_{(0)}g_{(2)})_{\mu\nu} + \frac{1}{4}\mathrm{Tr}\, g_{(2)}\cdot g_{(2)\mu\nu}\right]$$

Here the boundary can even be curved; for us it is flat, $\eta_{\mu\nu}$, and only $g^{(4)}_{\mu\nu}$ contributes. Note its correct dimensionality, $1/z^4 \sim T^4$.

The gravity dual of hot boundary matter is the 5d AdS black hole

$$ds^2 = \frac{\mathcal{L}^2}{\tilde{z}^2}\left[-(1-\frac{\tilde{z}^4}{z_0^4})dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + \frac{1}{1-\tilde{z}^4/z_0^4}d\tilde{z}^2\right]$$

which was written in the form

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-\frac{(1-z^4/(4z_0^4))^2}{1+z^4/(4z_0^4)}dt^2 + \left(1+\frac{z^4}{4z_0^4}\right)(dx_1^2+dx_2^2+dx_3^2) + dz^2\right]$$

---

[15] For relating energy-momentum tensor on the boundary to the corresponding supergravity solution, see Brown-York, PRD47 (1993) 1407; Henningson-Skenderis, hep-th/9806087; Balasubramanian-Kraus, hep-th/9902121; Myers, hep-th/9903203; and, in particular, de Haro-Solodukhin-Skenderis, hep-th/0002230

Expanding:

$$\equiv \frac{\mathcal{L}^2}{z^2}\left\{\left[g_{\mu\nu}(x,0) + \underbrace{g^{(4)}_{\mu\nu}(x)}_{\sim T_{\mu\nu}} z^4 + \ldots\right] dx^\mu dx^\nu + dz^2\right\}$$

$$\Rightarrow g^{(4)}_{\mu\nu} = \mathrm{diag}(3,1,1,1)\frac{1}{z_0^4}, \qquad \frac{1}{z_0} = \pi T.$$

Magnitude [16]: Relating string theory $\rightarrow$ supergravity

$$16\pi G_{10} = (2\pi)^7 \alpha'^4 g_s^2, \qquad \text{nontrivial!!}$$

$g_s =$ closed string coupling, one handle costs g_s^2. Integral over the 5d S_5 can be separated $\Omega_5 = \pi^3$:

$$\frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{G} = \frac{1}{16\pi G_{10}}\int d^5x\sqrt{g}\int d^5y\sqrt{\gamma} = \frac{\mathcal{L}^5\pi^3}{16\pi G_{10}}\int d^5x\sqrt{g} = \frac{1}{16\pi G_5}\int d^5x\sqrt{g}.$$

$$\Rightarrow \frac{\mathcal{L}^3}{G_5} = \frac{2N_c^2}{\pi}$$

$$\Rightarrow T_{\mu\nu} = \frac{\mathcal{L}^3}{4\pi G_5} g^{(4)}_{\mu\nu} = \frac{N_c^2}{2\pi^2} g^{(4)}_{\mu\nu} = \begin{pmatrix} 3aT^4 & 0 & 0 & 0 \\ 0 & aT^4 & 0 & 0 \\ 0 & 0 & aT^4 & 0 \\ 0 & 0 & 0 & aT^4 \end{pmatrix} \qquad a = \frac{\pi^2 N_c^2}{8}$$

[16] Gubser-Klebanov-Peet, hep-th/9602135; Gubser-Klebanov-Tseytlin, hep-th/9805156

Application 2: Viscosity

Another celebrated result[17] is

$$\eta = \frac{\pi}{8} N_c^2 T^3 \left[1 + \frac{75\zeta(3)}{4\lambda^{3/2}} + ...\right] \qquad 1 + \left(\frac{8.0}{\lambda}\right)^{3/2}$$

$$\frac{\eta}{s} = \frac{\hbar}{4\pi} \left[1 + \frac{135\zeta(3)}{8\lambda^{3/2}} + ...\right] \qquad 1 + \left(\frac{7.4}{\lambda}\right)^{3/2}$$

obtained by evaluating the correlator:

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt d^3x \, e^{i\omega t} \langle T_{12}(x) T_{12}(0) \rangle \qquad \int d^4x \, T_1^2(x) g_2^1(x, z=0)$$

In particular, $\hbar/4\pi$ should be the lower limit for all physical systems:

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi}$$

where $=$ holds for all systems having a gravity dual (not proven, so far).

Reminder: η decribes the response of flow to shear, v_1 varying as a function of $x_2 \Rightarrow T_{12}$. Friction force on a spherical object moving in a fluid:

$$\mathbf{F}_{\text{friction}} = -6\pi R \cdot \eta \cdot \mathbf{v}$$

$\Rightarrow$ dimη=kg/(ms)=Js·m^3. Reynolds number Re=$\rho LV/\eta$. "Small" $\eta \Rightarrow$ turbulent flow; Navier-Stokes flow does not go to Euler flow when η is "small"; flow develops an internal length scale $\delta \ll L$!!

[17] Policastro-Son-Starinets, hep-th/0104066; Buchel-Liu-Starinets, hep-th/0406264

Air ($\eta \sim 10^{-5}$, $s = S/V \sim N/V \sim 1\mathrm{kg}/m_p/\mathrm{m}^3 \sim 10^{27}/\mathrm{m}^3$):

$$\frac{\eta}{s} \approx \frac{10^{-5}}{10^{27}} \gg \frac{10^{-35}}{4\pi}$$

Kinetic theory:

$$\frac{\eta}{s} \approx \frac{p\tau_c}{s} \approx \frac{T^4\tau_c}{T^3} = T\tau_c \gtrsim \hbar \qquad \text{uncertainty principle}$$

So the limit $\eta/s \gtrsim \hbar = 1$ is quite expected, but now $\gtrsim\ \rightarrow \geq$!

Experimental fact: QCD matter observed in heavy ion collisions at RHIC/BNL has T up to $5T_c$ (strongly coupled!!) and flows nearly ideally.

Seems paradoxical: weakly coupled fluid has a "large" viscosity!

Bjorken flow: $v(t, x) = x/t$,

$$T(\tau) = \left(T_i + \frac{1}{6\pi\tau_i}\right)\left(\frac{\tau_i}{\tau}\right)^{1/3} - \frac{1}{6\pi\tau}.$$

$A(t) = e^{iHt}A(0)e^{-iHt}$ and $B(t)$ are two operators, $\langle O \rangle = Z^{-1}\mathrm{Tr}e^{-\beta H}O$.

$$J_1(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\langle A(t)B(0)\rangle \qquad J_2(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\langle B(0)A(t)\rangle = e^{-\beta\omega}J_1(\omega).$$

$$G_R(t) = \langle\, i\,[A(t), B(0)]\,\theta(t)\,\rangle \quad G_R(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{\pi}\frac{\rho(\omega')}{\omega' - \omega - i\epsilon} = G_A^*(\omega).$$

$$\rho(\omega) = \frac{1}{2}(1-e^{-\beta\omega})J_1(\omega) = \mathrm{Im}G_R(\omega) = \frac{1}{2}\sum_{m,n} 2\pi\delta(\omega+E_n-E_m)\langle n|A(0)|m\rangle\langle m|B(0)|n\rangle(e^{-\beta E_n}-e^{-\beta E_m}),$$

$$G_\beta(\omega_n) = G_R(\omega + i\epsilon \to i\omega_n \equiv i2\pi nT) = \int_0^\beta d\tau\, e^{i\omega_n\tau}G_\beta(\tau),$$

$$G_\beta(\tau) = T\sum_{n=-\infty}^{\infty} e^{-i\omega_n\tau}G_\beta(\omega_n) = \int_{-\infty}^{\infty}\frac{d\omega}{\pi}\rho(\omega)\frac{\exp(-\omega\tau)}{1-\exp(-\beta\omega)} = \int_0^\infty \frac{d\omega}{\pi}\rho(\omega)\frac{\cosh(\frac{1}{2}\beta-\tau)\omega}{\sinh\frac{1}{2}\beta\omega}$$

"Lattice MC determination of η in QCD" means measuring $G_\beta(\tau)$ for T_{12} and somehow inverting the spectral representation and finding out the derivative of $\rho(\omega, T)$ at $\omega = 0$:

$$\eta = \rho'(0, T).$$

Is this even possible[18]?

[18] Aarts-Martinez Resco, hep-ph/0203177, Meyer, 0704.1801

In AdS/CFT the master formula permits one to compute even Minkowskian correlators!

Discuss on blackboard, if time permits!

Application 3: Expanding matter?

What would be the gravity dual of an expanding system? Can we see the effects of $\eta/s = 1/(4\pi)$ there?

A conformally invariant ($T^\mu_{\ \mu} = 0$) conserved ($D_\mu T^{\mu\nu} = 0$) $T_{\mu\nu}$ in the coordinates $ds^2 = -d\tau^2 + \tau^2 d\eta^2 + dx_2^2 + dx_3^2$ is

$$T^\mu_{\ \nu} = \begin{pmatrix} -\epsilon(\tau) & 0 & 0 & 0 \\ 0 & -\epsilon(\tau) - \tau\epsilon'(\tau) & 0 & 0 \\ 0 & 0 & \epsilon(\tau) + \frac{1}{2}\tau\epsilon'(\tau) & 0 \\ 0 & 0 & 0 & \epsilon(\tau) + \frac{1}{2}\tau\epsilon'(\tau) \end{pmatrix}$$

Here $\epsilon(\tau)$ is an unknown function, $p_L = p_T$ if $\epsilon \sim 1/\tau^{4/3}$. Any constraints from AdS/CFT?

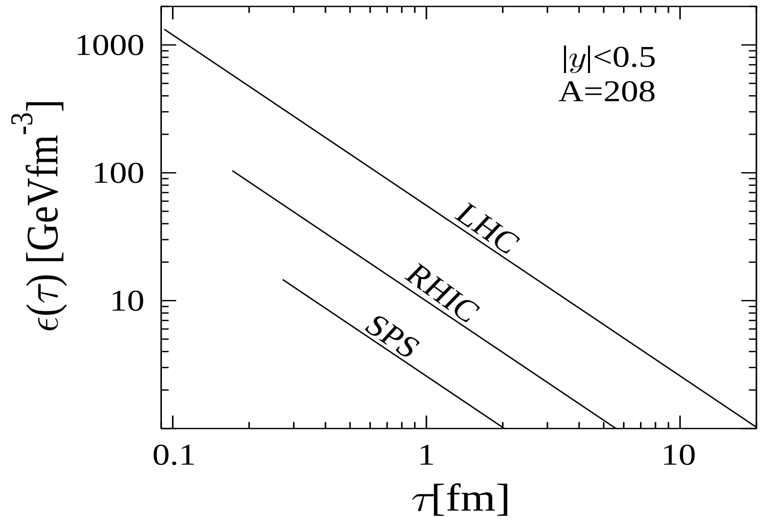
1+1+2d Bjorken similarity flow with AdS/CFT parameters:

$\epsilon(T) = 3p(T) = 3aT^4$, $\eta = p'(T)/(4\pi) = aT^3/\pi$, $a = \pi^2 N_c^2/8$, $\zeta = 0$

$$v(t,x) = \frac{x}{t} \equiv \tanh\Theta(\tau,\eta), \qquad \Theta(\tau,\eta) = \eta, \qquad u^\mu = \frac{x^\mu}{\tau},$$

$$T(\tau) = \left(T_i + \frac{1}{6\pi\tau_i}\right)\left(\frac{\tau_i}{\tau}\right)^{1/3} - \frac{1}{6\pi\tau}, \qquad \epsilon(\tau) = 3aT^4(\tau)$$

Adiabatic flow numbers for Pb+Pb:



$$1/\tau^{1/3}, 6\pi \quad \text{simple}, \quad T_i, \tau_i \text{ (very) hard}, \; T_i \tau_i \sim \hbar$$

Shuryak-Sin-Zahed

Search for time-dependent solutions of AdS$_{d+1}$ with proper symmetries [19]

$$R_{MN}-\frac{1}{2}Rg_{MN}-\frac{d(d-1)}{2\mathcal{L}^2}g_{MN}=0,\qquad (\tau,\eta,\mathbf{x}_T,z)$$

$$g_{MN}=\frac{\mathcal{L}^2}{z^2}\left(g_{\mu\nu}dx^\mu dx^\nu+dz^2\right)\qquad g_{\mu\nu}(x,z)=g^{(0)}_{\mu\nu}(x)+g^{(4)}_{\mu\nu}(x)z^4+z^6+...$$

$$T_{\mu\nu}=\#\cdot g^{(4)}_{\mu\nu}(x)$$

Heavy ion collision, boundary metric $g^{(0)}_{\mu\nu}$: $-dt^2+d\mathbf{x}^2=-d\tau^2+\tau^2d\eta^2+dx_2^2+dx_3^2$:

$$ds^2=\frac{\mathcal{L}^2}{z^2}\left[-e^{a(\tau,z)}d\tau^2+\tau^2e^{b(\tau,z)}d\eta^2+e^{c(\tau,z)}(dx_2^2+dx_3^2)+dz^2\right]$$

Small-z expansions start: $a(\tau,z)=z^4a_0(\tau)+z^6a_1(\tau)+...$, $e^a=1+z^4a_0(\tau)+z^6+..$

$$\Rightarrow\epsilon(\tau)=-\#\cdot a_0(\tau)$$

b_0, c_0 give longitudinal and transverse pressures(τ). So "just" need $a_0(\tau)$!

Obtain 5 non-linear 2nd order partial differential equations ($\tau\tau$, $\eta\eta$, TT, zz, τz components of Einstein) for $a(\tau,z)$, $b(\tau,z)$, $c(\tau,z)$ $\Rightarrow$ no analytic solution known $\Rightarrow$ global structure of soln unknown.

First appro idea: feed to Einstein the above expansions and solve a_0,a_1, etc. Not good enough: basically you reproduce the form of conformally invariant conserved $T_{\mu\nu}$ with one unknown function.

Need info on large z, the bulk; criteria for choosing correct soln

$$\text{Cosmology},\quad dt^2-r^2(t)d\mathbf{x}^2:\qquad ds^2=\frac{\mathcal{L}^2}{z^2}[-a(t,z)dt^2+b(t,z)d\mathbf{x}^2+dz^2]$$

[19] Janik-Peschanski hep-th/0512162, Heller-Janik hep-th/0703242, Nakamura-Sin hep-th/0607123, Kovchegov-Taliotis 0705.1234, Kajantie-Louko-Tahkokallio 0705.1791

Exact time dep solution for $d=2$, $-d\tau^2+\tau^2 d\eta^2+dz^2$: [20]

$$ds^2=\frac{\mathcal{L}^2}{z^2}\left[-(1-\frac{z^2}{v^2\tau^2})^2 d\tau^2+(1+\frac{z^2}{v^2\tau^2})^2\tau^2 d\eta^2+dz^2\right]\qquad \frac{1}{v^2}\equiv\frac{M-1}{4}$$

Suggests a horizon at $z=v\tau$ moving with velocity v. However, structure of AdS_3 is completely known (BTZ)! Transform $\tau,z\ \rightarrow\ V,U\ \rightarrow\ t,r$

$$V\ =\ \left(\frac{2\tau-\left(\sqrt{M}+1\right)z}{2\tau+\left(\sqrt{M}-1\right)z}\right)\left(\frac{\tau}{\mathcal{L}}\right)^{\sqrt{M}},\quad r=\mathcal{L}\sqrt{M}\left(\frac{1-UV}{1+UV}\right),\quad M\equiv 1+\frac{4}{v^2}=M_{\text{BH}}\cdot 8G_3$$

$$U\ =\ -\left(\frac{2\tau-\left(\sqrt{M}-1\right)z}{2\tau+\left(\sqrt{M}+1\right)z}\right)\left(\frac{\tau}{\mathcal{L}}\right)^{-\sqrt{M}},\qquad t=\frac{\mathcal{L}}{2\sqrt{M}}\ln\left|\frac{V}{U}\right|.$$

$$\Rightarrow\ ds^2=\mathcal{L}^2\left[-\frac{4}{(1-UV)^2}dV\,dU+M\left(\frac{1-UV}{1+UV}\right)^2 d\eta^2\right]$$

$$ds^2=-\left(\frac{r^2}{\mathcal{L}^2}-M\right)dt^2+\frac{dr^2}{r^2/\mathcal{L}^2-M}+r^2 d\eta^2$$

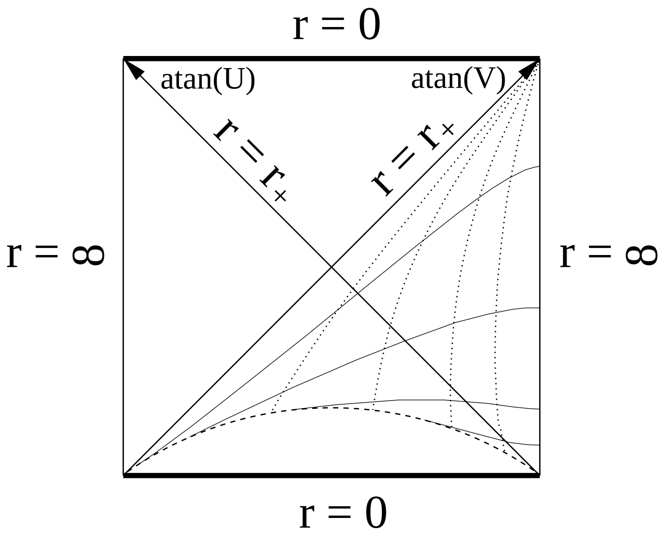
Completely static! $s,\ T$:

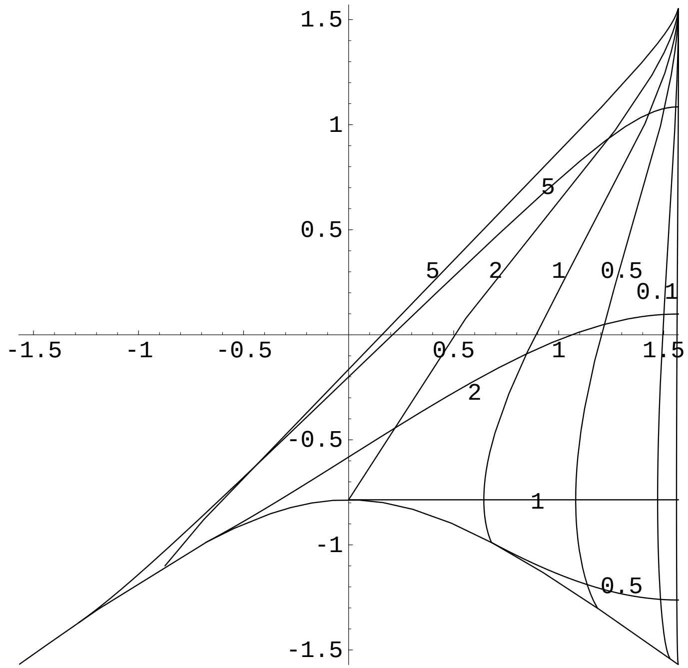
$$\frac{dS}{\mathcal{L}d\eta}=\frac{\sqrt{M}}{4G_3},\qquad T=\frac{\sqrt{M}}{2\pi\mathcal{L}},\quad M\geq 0.$$

$0<\eta<2\pi$ is unwrapped, no limits!

[20] Kajantie-Louko-Tahkokallio, arXiv:0705.1791[hep-th]

Global structure is completely known:





Boundary is $r = \infty$, $z = 0$!

The region $0 < z < v\tau =$

part of interior of while hole + exterior of black hole.

$$r_m = \frac{2\mathcal{L}}{v} = \mathcal{L}\sqrt{M-1} < r < r_+ = \mathcal{L}\sqrt{M} < r < \infty$$

Matter comes out of a white hole!

Energy-momentum tensor in boundary CFT

$$ds^2 = \frac{\mathcal{L}^2}{z^2}[g_{\mu\nu}dx^\mu dx^\nu + dz^2], \qquad g_{\mu\nu} = g^{(0)}_{\mu\nu}(\tau) + g^{(2)}_{\mu\nu}(\tau)z^2 + .. \qquad g^{(0)}_{\mu\nu} = \mathrm{diag}(-1, \tau^2)$$

$$T_{\mu\nu} = \frac{\mathcal{L}}{8\pi G_3}[g^{(2)}_{\mu\nu} - g^{(0)}_{\mu\nu}\mathrm{Tr}(g^{(2)}_{\mu\nu})] T^\mu_\nu = \frac{\mathcal{L}(M-1)}{16\pi G_3}\begin{pmatrix} \tau^{-2} & 0 \\ 0 & 1 \end{pmatrix} \qquad M_{\mathsf{fluid}} - 1_{\mathsf{vac}}$$

$$T^\mu{}_\nu = \begin{pmatrix} -\epsilon(\tau) & 0 \\ 0 & p(\tau) \end{pmatrix}, \qquad \epsilon(\tau) = p(\tau) = \frac{\mathcal{L}}{16\pi G_3}\frac{M}{\tau^2} = \frac{\pi\mathcal{L}}{4G_3}T^2(\tau)$$

Entropy density and T: $S = s(T)V = p'(T)V$:

$$s(\tau) = \frac{dS}{\tau d\eta} = \frac{1}{\tau}\frac{\sqrt{M}\mathcal{L}}{4G_3} = \frac{\pi\mathcal{L}}{2G_3}T(\tau)$$

Effectively: scale static T_H, s by $\mathcal{L}/\tau$,

$$T_{BTZ} = \frac{\sqrt{M}}{2\pi\mathcal{L}} \to \frac{\sqrt{M}}{2\pi\tau}$$

Works nicely, but gravitation in 3d is not dynamical!

Back to $d = 4$: The metric with $T_H = 1/\pi z_0$

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-\frac{(1 - z^4/(4z_0^4))^2}{1 + z^4/(4z_0^4)}dt^2 + \left(1 + \frac{z^4}{4z_0^4}\right)(dx_1^2 + dx_2^2 + dx_3^2) + dz^2\right]$$

is a solution $\Rightarrow$ clearly need large z! Solve iteratively in powers of $z/\tau^{1/3}$ or z/τ and demand regularity.

Large τ: If you take the time dependent JP metric

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-\frac{(1 - z^4/(4z_0(\tau)^4))^2}{1 + z^4/(4z_0(\tau)^4)}d\tau^2 + \left(1 + \frac{z^4}{4z_0(\tau)^4}\right)(\tau^2 d\eta^2 + dx_2^2 + dx_3^2) + dz^2\right]$$

$$z_0(\tau) \equiv 3^{1/4}\tau^{1/3}$$

you get

$$R_{MN} - \frac{1}{2}Rg_{MN} - \frac{6}{\mathcal{L}^2}g_{MN} = \begin{pmatrix} -\frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{7}{2}\tau^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \frac{8}{9\tau^2}\frac{z^4}{z_0^4(\tau)}$$

Not a soln but $\rightarrow 0 \quad \sim 1/\tau^2$ at fixed $z/\tau^{1/3}$. Further $\epsilon = -a_0(\tau) =\sim 1/z_0^4 \sim \tau^{-4/3}$.

The JP metric looks like the static metric with moving horizon, but for the factor $dx_1^2 \leftrightarrow \tau^2 d\eta^2$.

Small τ: take the KT metric ($a =$ unknown constant)

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-(1-az^4)d\tau^2 + (1-az^4)\tau^2 d\eta^2 + (1+az^4)dx_2^2 + (1+az^4)dx_3^2 + dz^2\right]$$

and find

$$R_{MN} - \frac{1}{2}Rg_{MN} - \frac{6}{\mathcal{L}^2}g_{MN} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 32a^2(1-a^2z^8)^{-2}z^6 \end{pmatrix}$$

This is a solution if $\tau \to 0$ at fixed z/τ! Implication:

$$g^{(4)}_{\mu\nu} = a\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad g^{\mu}{}_{\nu} = a\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{traceless like} \quad g^{\mu}{}_{\nu} = a\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In local thermal equilibrium $\epsilon/3 = p_L = p_T$, at $\tau = 0$ $\epsilon = -p_L = p_T$, negative p_T!

Can one interpolate? Is there some time τ_{th} when "$\tau = 0$" goes over to "$\tau \to \infty$"?

- For $\tau \to 0$ the system is in a quantum state and

$$\pi T = \frac{1}{\tau}$$

- For $\tau \to \infty$ the system is thermalised:

$$\pi T = \pi T_i \left(\frac{\tau_i}{\tau}\right)^{1/3}$$

- These are equal at $\tau = \tau_{\text{th}}$:

$$\tau_{\text{th}} = \tau_i \frac{1}{(\pi T_i \tau_i)^{3/2}}.$$

— Whatever this means! —-

- Theoretical status of time dependent systems? More exact solutions, understanding global structure?
- More fields? Scalars, form fields?
- Source terms?

Application 4: Expectation values of Wilson loops

$$\mathrm{P}\exp\left[ig\int_C A^\mu dx_\mu\right] \quad C = \text{closed loop}, \quad \mathrm{Tr}\ \text{ is gauge invariant}$$

Expectation value of a Wilson loop [21] in the boundary field theory = Action of the string hanging from the loop in the 5th dimension.

Take Q at $x = -L/2$, $\bar{Q}$ at $L/2$. How deep does the string connecting them hang in the z direction, i.e., what is $z = z(x,t) = z(x)$ for the extremal configuration (expected to be static, no t)?

Particle action $= -m\int d\tau \Rightarrow$ String action $= -T\int dA$. $T = \frac{1}{2\pi\alpha'} =$ **Tension**.

String $X^\mu(\tau,\sigma)$ moving in a space with metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ has the action:

$$S = -T\int d\tau d\sigma\sqrt{-\mathrm{det}h_{ab}}, \qquad h_{ab} = g_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}, \qquad \sigma^a = (\tau,\sigma).$$

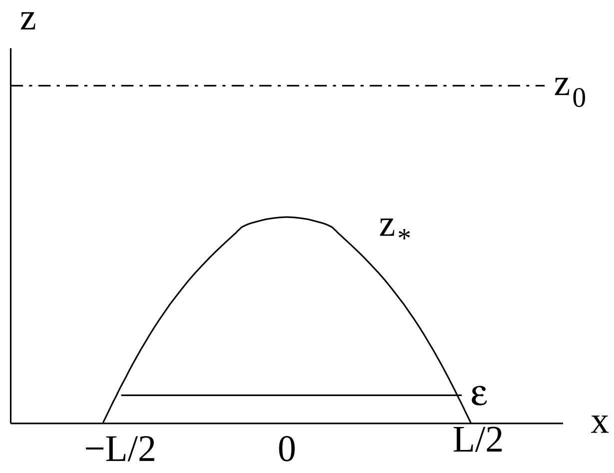
Nambu-Goto: $g_{\mu\nu} = \eta_{\mu\nu}$

$$h_{ab} = \begin{pmatrix} \dot{X}\cdot\dot{X} & \dot{X}\cdot X' \\ X'\cdot\dot{X} & X'\cdot X' \end{pmatrix} \qquad X\cdot Y \equiv \eta_{\mu\nu}X^\mu Y^\nu \qquad {}^{\cdot} = \partial/\partial\tau,\ {}' = \partial/\partial\sigma.$$

---

[21] For a discussion of a Wilson loop and its relation to a $Q-\bar{Q}$ (free)energy at $T=0$ and $T\neq 0$, and real $t \Leftrightarrow$ imaginary time $\tau = it$, see H.J. Rothe, Lattice gauge theories, Ch. 7 and 8 and Sect. 20.2.

String hanging [22] from a Q and a $\bar{Q}$ at $z = 0$ in the metric $ds^2 = g_{tt}(z)dt^2 + g_{xx}(z)dx^2 + g_{zz}(z)dz^2$.



$$\sigma^1 = t, \quad \sigma^2 = x, \qquad X^\mu = (t, x, 0, 0, z(t,x) \to z(x))$$

$$h_{ab} = \begin{pmatrix} g_{tt} + g_{zz}\dot{z}^2 & g_{zz}\dot{z}z' \\ g_{zz}z'\dot{z} & g_{xx} + g_{zz}z'^2 \end{pmatrix} \qquad \sqrt{-h} = \sqrt{-g_{tt}g_{xx} - g_{tt}g_{zz}z'^2 - g_{xx}g_{zz}\dot{z}^2}$$

Extremize ($T = 1/(2\pi\alpha')$ = tension, $\dot{z} = 0$, $g(z) = 1 - z^4/z_0^4$ specialising to the 5d AdS BH)

$$S = T\Delta t \int_{-L/2}^{L/2} dx \sqrt{-h} = T\Delta t \, 2 \int_\epsilon^{z_*} dz \, x'(z)\sqrt{-h} = \Delta t \frac{\mathcal{L}^2}{\pi\alpha'} \int_\epsilon^{z_*} \frac{dz}{z^2}\sqrt{1 + g(z)x'(z)^2}$$

Key technical point: $dx\, L = dx\, L(z(x), z'(x)) = dz\, x'(z) L = dz\, L_{\text{new}}(x'(z))$

[22] Rey-Yee, hep-th/9803001; Sonnenschein, hep-th/9910089

Equation of motion is

$$\frac{\partial L}{\partial x(z)} - \frac{d}{dz}\frac{\partial L}{\partial x'(z)} = 0 \Rightarrow \frac{\partial L}{\partial x'(z)} = \frac{-g_{tt}g_{xx}x'}{\sqrt{-g_{tt}g_{zz} - g_{tt}g_{xx}x'^2}} = \text{constant}. \tag{5}$$

The constant is fixed neatly so that the maximum value of z is z_*, $z(x=0) = z_*$, $z'(z_*) = 0$. This gives $z'^2 = (dz/dx)^2$ from which by integration (Exercise)

$$L = 2\int_{\epsilon\to 0}^{z_*} \frac{dz}{\sqrt{\frac{g_{xx}}{g_{zz}}\left(\frac{g_{tt}g_{xx}}{g_{tt}^* g_{xx}^*} - 1\right)}} = 2\int_0^{z_*} \frac{dz}{\sqrt{g(z)\left[\frac{g(z)}{z^4}\frac{z_*^4}{g(z_*)} - 1\right]}} \qquad g_{tt}^* \equiv g_{tt}(z_*), \text{etc.}$$

Inserting this z'^2 to the action gives the extremal action (Exercise)

$$S = T\Delta t\, 2\int_\epsilon^{z_*} dz \sqrt{\frac{-g_{tt}g_{zz}}{1 - \frac{g_{tt}^* g_{xx}^*}{g_{tt}g_{xx}}}} = T\Delta t\, 2\int_\epsilon^{z_*} dz \frac{\mathcal{L}^2}{z^2}\frac{1}{\sqrt{1 - \frac{z^4}{g(z)}\frac{g(z_*)}{z_*^4}}}$$

Separate divergence of the 5d AdS BH at $z \to 0$:

$$\int_\epsilon^{z_*} dz\, \frac{1}{z^2} f(z) = \int_\epsilon^{z_*} dz\, \frac{1}{z^2}\left[f(z) - 1 + 1\right] = \frac{1}{\epsilon} + \int_0^{z_*} dz\, \frac{1}{z^2}\left[f(z) - 1\right] - \frac{1}{z_*};$$

throw away $1/\epsilon$. For Euclidian Wilson loop $\langle \Delta t \times R - \text{loop}\rangle \sim \exp[-\Delta t V(R)]$ so write here $V(L, z_0) = S/\Delta t$:

$$V(L,z_0) = 2T\mathcal{L}^2\left[\int_0^{z_*}\frac{dz}{z^2}\left(\frac{1}{\sqrt{1-\frac{z^4}{g(z)}\frac{g(z_*)}{z_*^4}}}-1\right)-\frac{1}{z_*}\right]$$

Scaling $z = yz_0$, $z_* = y_m z_0$: [23]

$$L = 2z_0\int_0^{y_m}\frac{dy}{\sqrt{(1-y^4)[q(y_m)/q(y)-1]}} \qquad q(y)\equiv y^4/(1-y^4) \tag{6}$$

$$V(L,z_0) = \frac{\mathcal{L}^2}{\alpha'\pi z_0}\left\{\int_0^{y_m}\frac{dy}{y^2}\left[\frac{1}{\sqrt{1-q(y)/q(y_m)}}-1\right]-\frac{1}{y_m}\right\} \tag{7}$$

Units of length and energy given by

$$\pi z_0 = \frac{1}{T_H}, \qquad -V_{Q\bar{Q}} \equiv \frac{\mathcal{L}^2}{\alpha'\pi z_0} = \sqrt{g^2N_c}\,T_H.$$

Small L (and $y_m\to 0$, $q(y)\approx y^4$):

$$V(L) = -\frac{0.2285\sqrt{g^2N_c}}{L}$$

Intermediate $0 < L < L_{\text{max}}$: can fit to

$$V(L) = -\frac{4}{3}\frac{\alpha_s}{L}+\sigma L$$

Small L again for $y_m\to 1$:

$$V(L,z_0)\to V_{Q\bar{Q}} = \frac{\mathcal{L}^2}{\pi\alpha'}\int_\epsilon^{z_0}\frac{dz}{z^2}\Rightarrow -\frac{\mathcal{L}^2}{\alpha'\pi z_0}$$

After some y_m, for $L >$ some value (see Fig) the dominant config is that with separate $Q\bar{Q}$. This is also a solution of (5) with $x' = 0$, $z_* = z_0$.

[23] Plotted with Mathematica using ParametricPlot[{L[ym], V[ym]}, {ym, 0.15, 0.994}] in Fig.1

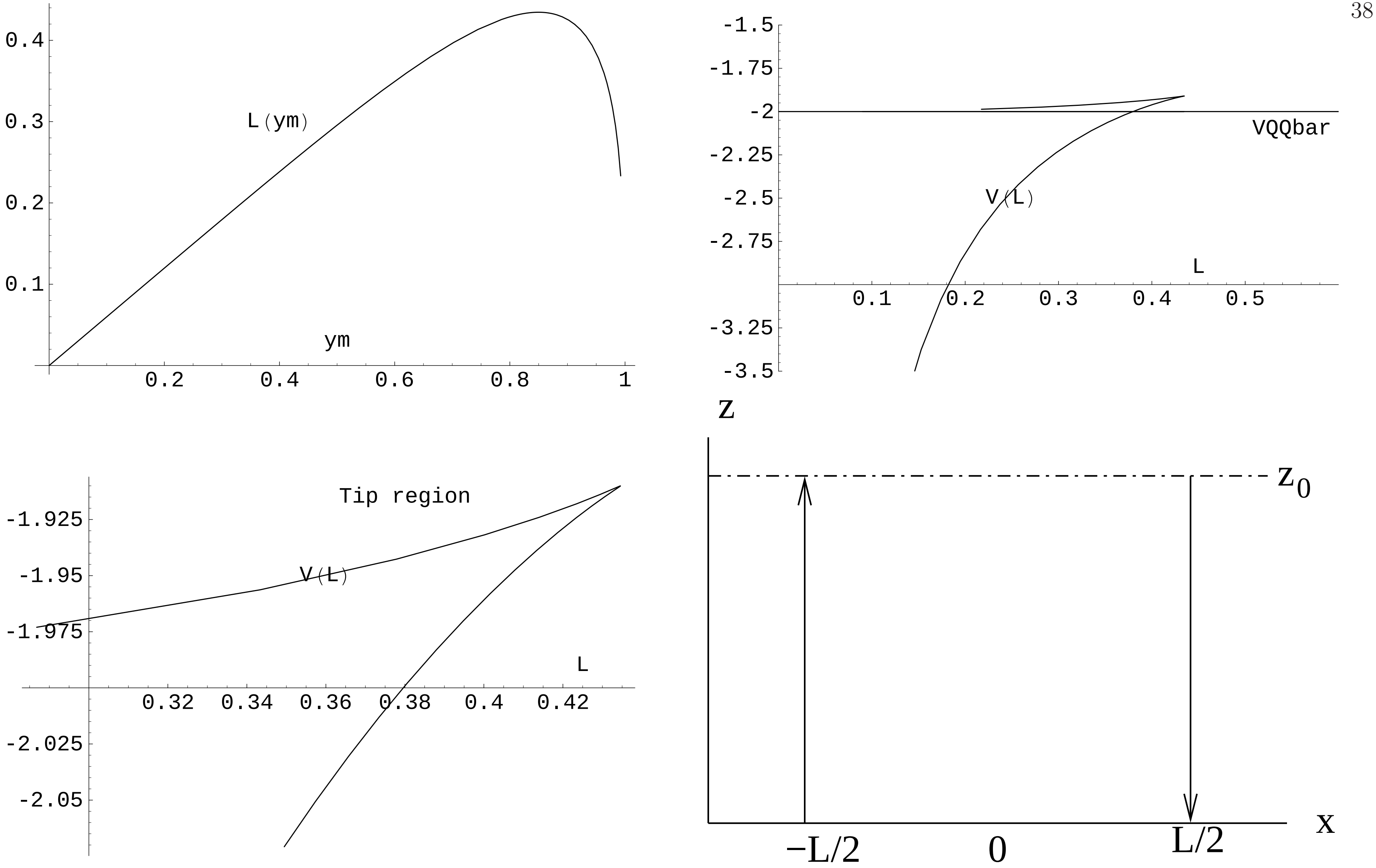


Figure 1: The distance L as function of y_m evaluated from (6), the extremal action and its tip region from (7) (all scaled by the factors outside the integral) and the $Q\bar{Q}$ configuration.

Picture from AdS/CFT:

- at small distances conformally invariant form $V \sim 1/L$, $\sqrt{..}$ dependence on g^2N_c is typical of strong coupling.
- at some distance interaction is screened and $Q\bar{Q}$ separate.
- one can put in numbers [24]
- mathematics of the curves is pretty: the independent $Q\bar{Q}$ solution is obtained also from the string-connected solution when $z_* \to z_0$ and the two branches approach each other

This was just an example of numerous applications of AdS/CFT to Wilson loop computations. Works even for gluonic scattering amplitudes!! [25]

[24] Large number of papers, mine is Kajantie-Tahkokallio-Yee, hep-ph/0609254
[25] Alday-Maldacena, arXiv:0705.0303

Addendum

There is no exact solution with the symmetry

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-e^{a(\tau,z)}d\tau^2 + \tau^2 e^{b(\tau,z)}d\eta^2 + e^{c(\tau,z)}(dx_2^2 + dx_3^2) + dz^2\right]$$

but there **is** [26] an exact solution with the symmetry

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-a(t,z)dt^2 + b(t,z)d\mathbf{x}^2 + dz^2\right]$$

[26] Binetruy et al, hep-th/9910219, Kajantie-Tahkokallio, hep-th/0612226

Solution in 1+3 dimensions: $r = r(t)$

$$ds^2 = [-a(t,z)dt^2 + b(t,z)d\mathbf{x}^2 + dz^2]/z^2, \quad R_{MN} + 4g_{MN} = 0$$

$$a(t,z) = \frac{\left[\left(1 - \frac{r''}{4r}z^2\right)^2 - \left(\frac{r''}{4r} - \frac{r'^2}{4r^2}\right)^2 z^4 - \frac{1}{4r^4 z_0^4} z^4\right]^2}{\left[\left(1 - \frac{r'^2}{4r^2}\, z^2\right)^2 + \frac{1}{4r^4 z_0^4} z^4\right]} \overrightarrow{r(t) = 1} \frac{(1 - z^4/(4z_0^4))^2}{1 + z^4/(4z_0^4)}$$

$$b(t,z) = r^2 \left[\left(1 - \frac{r'^2}{4r^2}\, z^2\right)^2 + \frac{1}{4r^4 z_0^4} z^4\right] \overrightarrow{r(t) = 1}\ 1 + \frac{z^4}{4z_0^4}$$

Again a time dependent solution with a "horizon" at $a(t,z) = 0$:

$$z_{H\pm}^2 = \frac{4r^2}{rr'' \pm \sqrt{4/z_0^4 + (r'^2 - rr'')^2}} \overrightarrow{r(t) = 1}\ 2z_0^2, \quad \pi T_H = \frac{1}{z_0}$$

Is there a coordinate transformation transforming away the t-dependence?

Boundary metric now is RW:

$$g_{\mu\nu}(x,0) = (-1, r^2(t), r^2(t), r^2(t))$$

but $r(t)$ is any function of t. Brane gravity adds a brane and Einstein with G_4 to determine $r(t)$.

$T_{\mu\nu}$:

$$g_{\mu\nu}(t,z) = g^{(0)}_{\mu\nu}(t) + g^{(2)}_{\mu\nu}(t)z^2 + g^{(4)}_{\mu\nu}(t)z^4 + ...,$$

$$T_{\mu\nu} = \frac{\mathcal{L}^3}{4\pi G_5}\left[g^{(4)}_{\mu\nu} - \frac{1}{8}g^{(0)}_{\mu\nu}[(\mathrm{Tr}\, g_{(2)})^2 - \mathrm{Tr}\, g^2_{(2)}] - \frac{1}{2}(g_{(2)}g^{-1}_{(0)}g_{(2)})_{\mu\nu} + \frac{1}{4}\mathrm{Tr}\, g_{(2)}\cdot g_{(2)\mu\nu}\right]$$

$$T_{tt} = \epsilon(t) = \frac{3N_c^2}{8\pi^2}\left(\frac{1}{z_0^4 r^4} + \frac{r'^4}{4r^4}\right) \sim \frac{T_0^4}{r^4(t)} + \frac{1}{t^4} \sim \underbrace{T^4(t)}_{\text{radiation}} + \underbrace{t^{-4}}_{\text{curvature}},$$

$$T^1_1(t) = \frac{1}{3}\epsilon(t) - \underbrace{\frac{N_c^2}{8\pi^2}\frac{r'^2 r''}{r^3}}_{\text{trace anomaly}/3}$$

$\epsilon = 3p \sim T^4$ are known, $T(t) =?$, $s = p'(T) =?$, $r(t) =?$. Obvious that $r(t) = t/t_0$ works nicely:

$$\epsilon(t) = \frac{3\pi^2 N_c^2}{8}\frac{1}{t^4}\left(\frac{t_0^4}{\pi^4 z_0^4} + \frac{1}{4}\right) \Rightarrow T(t) = \frac{1}{\pi z_0}\frac{t_0}{t} \quad \text{if } t_0 \gg z_0$$

$T \sim 1/\tau^{1/3}$ follows if expansion is in 1d, thermalisation in 3d.

The JP argument for fixing $r \sim t^p$: $R = -20/\mathcal{L}^2$, $R^{\mu\nu}R_{\mu\nu} = 80/\mathcal{L}^4$,

$$R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = \frac{1}{\mathcal{L}^4}\left\{40 + 72\left[\frac{z^2}{z_0^2 b(t,z)}\right]^4\right\}$$

This is $112/\mathcal{L}^4$ at the horizon if $r(t) = t/t_0$. If $r(t) \sim t^{p>1}$, this is $\sim t^{8(p-1)}$ for $t \gg z_0$ (similarly $p < 1$) , thus

$$r(t) = \frac{t}{t_0}$$

We thus have gravity dual of matter in the center of spherical bang:

Thermalisation condition: temperature can be defined for

$$t_0/z_0 = \pi T t_0 > 1 = \hbar$$