Second order logic or set theory?

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Second order logic

Structuralism

- Some results

Anti-foundationalism

- Some results

Categoricity

- Some remarks

Set theory

Realism

Formalism

Foundationalism

Non-standard models

Summary

- **Part One:**
  - Second order logic and set theory capture mathematical concepts to the same extent of categoricity.
  - Non-standard and countable models have the same role in second order logic and set theory.
- **Part Two:**
  - Second order characterizable structures have a canonical hierarchy.
  - Second order truth cannot be expressed as truth in a particular structure.
  - Understanding second order logic seems to be essentially beyond second order logic itself.

Part One

- Second order view
- Set theory view
- Catogoricity

The second order logic view

- Mathematical propositions are of the form,

$$M \models \phi \tag{1}$$

where *M* is a specific mathematical structure, like the reals, Euclidean space, etc, and *φ* is a second order sentence. Or of the from

$$\models \phi \tag{2}$$

What are the specific structures?

- Specific structures are structures *M* that have arisen from mathematical practice:

$$\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n \ldots$$

What are the specific structures?

- Specific structures are structures M that have a second (or higher) order characterization ϑ_M.

$$M \models \theta_M$$

$$\forall M', M''((M' \models \theta_M \wedge M'' \models \theta_M) \rightarrow M' \cong M'')$$

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$$M \models \phi$$ Case of (1)

iff

$$\models \vartheta_M \rightarrow \phi$$ Case of (2)

Judgements in second order logic

- What counts as evidence for the assertion that

$$\models \vartheta_M \rightarrow \phi$$

holds?

Evidence

Evidence for $\vDash \vartheta_M \rightarrow \varphi$ is a proof of φ from ϑ_M (and comprehension et al. axioms).

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The proof tells us more than just $\vDash \vartheta_M \rightarrow \varphi$.
If we study a formal system in which the proof is given, then φ holds in the entire ``cloud" of models of ϑ_M around *M*. Such models are often called non-standard.

The set theory view

- Mathematical propositions are of the form,

$$\Phi(a_1, ..., a_n)$$

  where $\Phi(x_1, ..., x_n)$ is a formula of set theory with quantifiers ranging over all sets and $a_1,\ldots,a_n$ are some specific definable mathematical objects.

- No (1)/(2) distinction.

What are the specific objects of set theory?

- Definable objects.
- Anything one might need in mathematics:

$$\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n \ldots$$
$$\sin(x), \zeta(x), \Gamma(x)$$
$$\sqrt{2}, \pi, e, \log 5, \zeta(5)$$

- Not every real is definable.
- A well-order of the reals need not be definable.

The set theory view modified

- Mathematical propositions are of the form,

$$V_\alpha \models \Phi(a_1, ..., a_n)$$

  where α is some rather large ordinal, although anything bigger than ω+5 (or ω_1) is rarely needed outside set theory itself.
- First order variables actually range over α^{th} order objects over the integers.

Judgements in set theory

- What counts as evidence for the assertion that

$$\Phi(a_1, ..., a_n)$$

holds?

Evidence

- We can use the evidence that

$$ZFC \vdash \Phi(a_1, ..., a_n)$$

- Of course, this tells more than the mere assertion that $\Phi(a_1, ..., a_n)$ holds in the universe of sets.

Proofs and categoricity

- Categoricity is provable from Comprehension Axioms (CA) for the classical specific structures.
  - Peano(S,0,S',0') proves isomorphism of {S,0} and {S',0'}.
  - Peano(S,0) and Peano(S',0') have non-isomorphic models.
- Non-standard models of CA tell us about the nature of the evidence, not about (lack of) categoricity.
- It is the same in set theory.
  - ZFC(∈,∈') proves isomorphism of {∈} and {∈'}.
  - ZFC(∈) and ZFC(∈') have non-isomorphic models.

Part Two

- Second order characterizable structures
- Their global structure
- Their existence

Cardinality matters

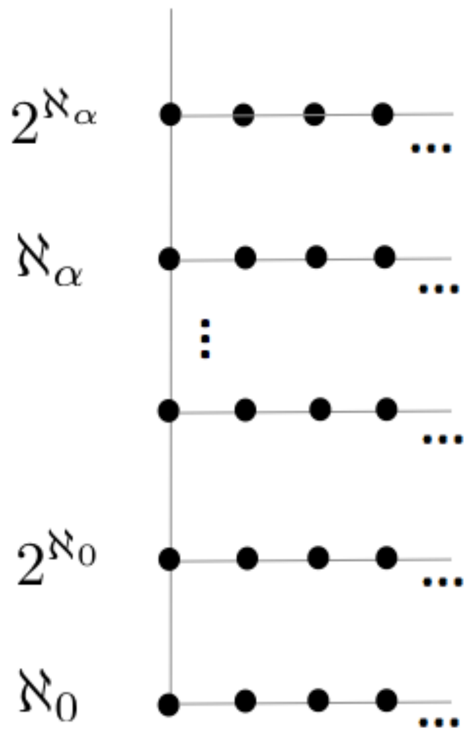
Recap:

$$M \models \theta_M$$

$$\forall M', M''((M' \models \theta_M \wedge M'' \models \theta_M) \rightarrow M' \cong M'')$$

- *M* is second order characterizable → |*M*| is second order characterizable.
- If κ is second order characterizable, then so are κ^+ and 2^κ.
- The second order theory of 2^κ is not Turing reducible to the second order theory of κ.

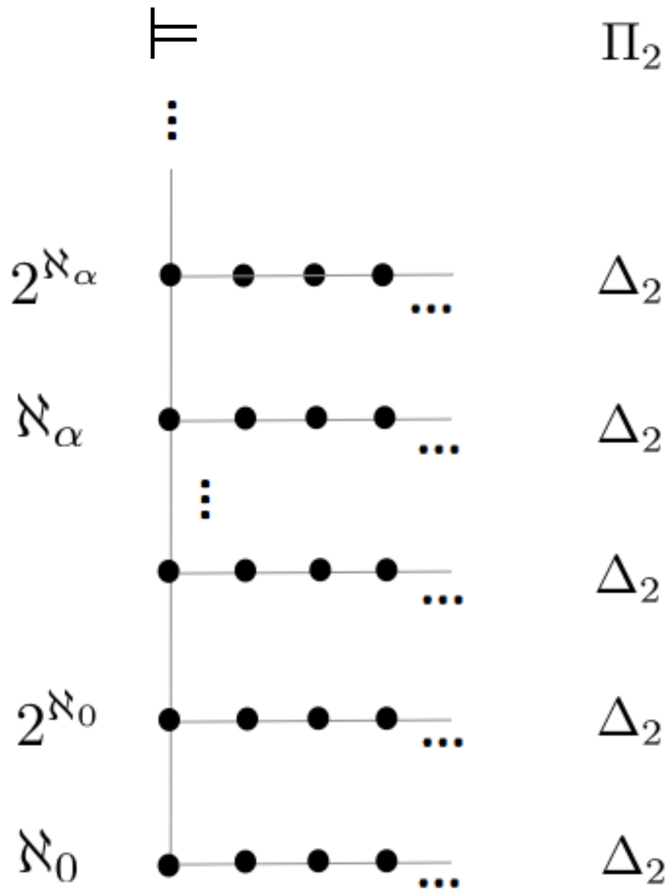
Second order characterizable structures



Definability matters

- If *M* is second order characterizable, the second order theory of *M* is Δ_2.
- The second order theory of all structures is Π_2-complete, hence not (Turing-reducible to) the second order theory of any particular (s. o. c.) structure.

Second order characterizable structures



Second order truth

- Conclusion: In second order logic truth in all structures cannot be reduced to truth in any particular specific structure.

The existence of second order characterizable structures

- The set of second order sentences that charcterize some structure is not Π_2.
- Second order characterizations depend on the propositions

$$``\varphi \text{ has a model}". \qquad (3)$$

- This is a new form of proposition. But what counts as evidence for such propositions? A proof? Of what?
- Likely choice: ZFC $\vdash$ ``φ has a model"; leaves second order logic behind.

Complete formulas

- A second order sentence is complete if it has a model and for any second order sentence logically implies the sentence or its negation.
- Categorical sentence are complete.
- Ajtai: Axiom of Constructibility implies that complete sentences are categorical.
- Ajtai, Solovay: Consistently, there are complete sentences that are non-categorical.
- Again, ``φ is complete" is not Π_2-definable.

Summary

- **Part One:**
  - Propositions of second order logic and set theory are of a different form but both refer to real mathematical objects and use proofs as evidence.
  - Second order logic and set theory capture mathematical concepts such as natural and real numbers to the same extent of categoricity.
  - Second order logic and set theory both have non-standard and countable models if evidence is formalized.
- **Part Two:**
  - Second order characterizable structures have a canonical hierarchy based on cardinality.
  - Second order truth cannot be expressed as truth in a particular structure.
  - Obtaining second order characterizable structures seems to go beyond second order logic.

Theses

- Second order logic is the $\sum_2$-part of set theory. Mathematics outside set theory resides there.
- As a weaker form of set theory, second order logic is an important milestone. One can develop second order model theory.
- Set theory provides a foundation for second order logic.

Thank you!