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Dependence of variables construed as an atomic formula☆

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ABSTRACT

We define a logic $\mathfrak{D}$ capable of expressing dependence of a variable on designated variables only. Thus $\mathfrak{D}$ has similar goals to the Henkin quantifiers of [4] and the independence friendly logic of [6] that it much resembles. The logic $\mathfrak{D}$ achieves these goals by realizing the desired dependence declarations of variables on the level of atomic formulas. By [3] and [17], ability to limit dependence relations between variables leads to existential second order expressive power. Our $\mathfrak{D}$ avoids some difficulties arising in the original independence friendly logic from coupling the dependence declarations with existential quantifiers. As is the case with independence friendly logic, truth of $\mathfrak{D}$ is definable inside $\mathfrak{D}$. We give such a definition for $\mathfrak{D}$ in the spirit of [11,2] and [1].

1. Introduction

This paper is an attempt to make the independence friendly logic of [6] and the related compositional semantics in [8] more transparent.

Consider the sentence

$$\forall x_1 \ldots \forall x_n (\exists y/x_2)\ \psi(x_1, \ldots, x_n, y). \tag{1}$$

This sentence says that if you choose elements $x_1, \ldots, x_n$, then on the basis of just x_1 and $x_3, \ldots, x_n$ I can choose an element y so that $\psi(x_1, \ldots, x_n, y)$. The formal semantics says that y can be chosen as a function of $x_1, x_3, \ldots, x_n$. Now arguably the notation $(\exists y/x_2)$ mentions exactly the wrong thing. It expresses that I *don't need* x_2 for choosing y; what it should be saying is rather that I *can* choose y using the *other* chosen elements. In other words, the hidden fact behind the quantifier is that there is a *y determined by* $x_1, x_3, \ldots, x_n$, such that $\psi(x_1, \ldots, x_n, y)$.

For this and other reasons, one of us (Väänänen) proposed a different syntax which detaches the dependence from the quantifier and expresses it as a separate clause. In this paper we develop this idea. Our quantifiers will be just the classical $\forall$ and $\exists$, but we introduce new atomic formulas, called *dependence formulas*, as follows. When $n \geqslant 2$, the atomic formula

$$=\!(x_1, \ldots, x_{n-1}, x_n) \tag{2}$$

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will have the intuitive meaning

$$x_n \text{ is determined by } x_1, \ldots, x_{n-1}. \tag{3}$$

We will call the resulting formalism *dependence logic*. The semantics for the dependence formulas seems to us more intuitive than the semantics for slash quantifiers (1). One reason for this may be that natural languages have expressions that are like dependence formulas, but to the best of our knowledge they rarely or never have expressions that behave like $(\exists x/y)$ or $(\exists x\backslash y)$ (see (6)).

Most of this paper will recast facts about IF-logic within dependence logic. At the end we make some remarks about expressing dependence in English.

An elementary presentation of dependence logic has appear in [16].

2. At least three different syntaxes

Before we go to details, we need to mention a slight glitch in the definition of the semantics in [6]. Page 365 describes a semantics in terms of Skolem functions. In effect, the sentence (1) above is defined to be true if and only if a function F exists so that

$$\forall x_1 \ldots \forall x_n\ \psi(x_1, \ldots, x_n, F(x_1, x_3, \ldots, x_n)) \tag{4}$$

is true. Page 364 describes a game-theoretic semantics: Player $\forall$ chooses elements $a_1, \ldots, a_n$ to assign to the variables $x_1, \ldots, x_n$, and then player $\exists$ chooses an element b to assign to y. Player $\exists$ wins if the assigned elements make $\psi(x_1, \ldots, x_n, y)$ true. The sentence (1) is defined to be true if and only if player $\exists$ has a winning strategy that chooses b as a function of all previously chosen elements except the one chosen for x_2.

For the sentence (1) these two conditions are obviously equivalent; player $\exists$ can take the function F from the first definition as a winning strategy in the second, and vice versa. But they can come apart if the quantifier prefix contains two or more existential quantifiers. A Skolem function for the second existential quantifier will never have the variable from the first existential quantifier as one of its arguments, because skolemising removes all the existentially quantified variables. But in the game semantics there is no reason why player $\exists$ should not use her second choice as a function of her first choice (unless the quantifier explicitly forbids this).

The quantifier notation $(\exists y/\forall x)$ of [6] strongly suggests that the author intended the Skolem function semantics throughout. Hintikka has confirmed this orally and in several later writings, for example page 63f of [6] and page 413 of [7]. From the game-theoretic point of view it is more natural to allow strategies to depend on previous moves of either player; it certainly leads to a cleaner compositional semantics too. So following [8,15] we adopt the game-theoretic semantics for IF-logic, rather than the Skolem function semantics. One should bear this in mind when comparing our languages with Hintikka's.

Now consider the sentence

$$\forall x_0 \forall x_2 (x_0 \neq x_2 \vee (\exists x_1/x_0)(x_0 = x_1)) \tag{5}$$

which is one of a number of similar sentences considered in [10]. The sentence is always true. If the domain contains more than one element, then when player $\exists$ chooses a disjunct at $\vee$, she looks to see whether the elements a_0 and a_2 chosen for x_0 and x_2 are equal or unequal. If unequal, she chooses the left disjunct and wins. If equal, she chooses the right; then she chooses a_1 as a function of a_2, namely $a_1 = a_2$, and again she wins. (So she uses the $\vee$ to signal to herself at the $\exists$.) Translating into dependence formulas, we want to express that x_1 can be chosen as a function of x_2. But x_2 is not mentioned anywhere in $(\exists x_1/x_0)(x_0 = x_1)$. The moral is that we cannot expect a compositional translation from IF-logic to dependence-logic.

To compare the two logics it will be helpful to introduce a third kind of logic. The quantifier

$$(\exists y\backslash x_1, \ldots, x_n) \tag{6}$$

will mean that y can be chosen as a function of $x_1, \ldots, x_n$. The game-theoretic semantics is easy to write down: a strategy for player $\exists$ at this quantifier must express y as a function of $x_1, \ldots, x_n$. We can do the same for universal quantifiers. (Our notation follows Hintikka [7]. But be warned that he is talking Skolem functions, not games; so his formula means something a little different from ours.) Let us call the resulting logic *backslash-logic*.

Now sentences of IF-logic do translate into sentences of backslash-logic, and vice versa, though again the translations are not compositional. Given an IF-sentence ϕ, we look at each subformula occurrence of the form

$$(\exists y/x_1, \ldots, x_n)\psi, \tag{7}$$

and we list as $z_1, \ldots, z_k$ all the variables z such that (i) the subformula occurrence is within the scope of a quantifier $(\exists z \ldots)$ or $(\forall z \ldots)$, and (ii) z is not one of $y, x_1, \ldots, x_n$. Then we replace the subformula by

$$(\exists y\backslash z_1, \ldots, z_k)\psi. \tag{8}$$

Doing this with every quantifier occurrence in ϕ, the result is a backslash-logic sentence $\phi^\star$. One easily checks that the game conditions for ϕ to be true (resp. false) are exactly the same as those for $\phi^\star$ to be true (resp. false). The translation in the other direction, from backslash-logic to IF-logic, goes just the same way.

There are compositional translations between backslash-logic and dependence-logic—though we will have to check this in detail later (see Proposition 3). Namely:

$$=(t_1, \dots, t_{n-1}, t_n) \quad \equiv \quad \exists x_1 \dots \exists x_{n-1}(x_1 = t_1 \wedge \cdots \wedge x_{n-1} = t_{n-1} \wedge \exists x_n \backslash x_1 \dots x_{n-1}(x_n = t_n)). \tag{9}$$

And conversely:

$$(\exists x_n \backslash x_1 \dots x_{n-1})\phi \equiv \exists x_n(=(x_1, \dots, x_{n-1}, x_n) \wedge \phi). \tag{10}$$

To illustrate, we translate the sentence (5) first into backslash-logic:

$$\forall x_0 \forall x_2 (x_0 \neq x_2 \vee (\exists x_1 \backslash x_2)(x_0 = x_1)) \tag{11}$$

and then into dependence-logic:

$$\forall x_0 \forall x_2 (x_0 \neq x_2 \vee \exists x_1(=(x_2, x_1) \wedge (x_0 = x_1))). \tag{12}$$

If this sentence seems less paradoxical than (5), this suggests that the move to dependence-logic is going to be helpful for intuition.

Assume for the moment that our formal semantics will confirm the translation between backslash-logic and dependence-logic. Then it follows that dependence-logic has, on sentences, the same expressive power as IF-logic and hence the same expressive power of existential second order logic. (More precisely, for every sentence of dependence-logic there is an existential second order sentence that has exactly the same models, and vice versa.) In particular dependence-logic is in the strongest terms non-axiomatizable. Not only is there no arithmetically complete definable axiom-system, but to axiomatize this logic completely one would have to solve a whole range of questions undecidable in set theory, such as the Continuum Hypothesis. In other words, any axiomatization will be open ended, subject to endless future extensions, like the ZFC axioms for set theory. We will see that another characteristic feature of dependence-logic is that it does not have a negation in the classical sense. It is not only that some new sentences do not have a negation. The failure of negation is so complete that not a single sentence containing a non-eliminable occurrence of a dependence formula has a negation.

The reason for these consequences of giving the said meaning to $=(x_1, x_2)$ is that the ability to talk about arbitrary functional dependence between x_1 and x_2 allows us to refer to the surrounding mathematical world. For example, we can ask whether there is a one–one correspondence between two unary predicates. Being thus able to talk about cardinalities, we can talk about well-orderings and thereby about transitive models of set theory.

3. The interpretation of dependence formulas

The sentence

x_n is determined by $x_1, \dots, x_{n-1}$

is meaningless if the variables $x_1, \dots, x_n$ stand for particular things. For example is 5 dependent on 2 and ω? This way lies nonsense.

The proper setting for using this notion 'determined by' is where we have a range of assignments to the variables. Suppose for example that $n = 3$ and the relevant assignments (a_1, a_2, a_3) to (x_1, x_2, x_3) are listed as

$$(1, 2, 3), (1, 4, 3), (2, 2, 7), (4, 1, 6). \tag{13}$$

Within this range, x_3 is determined by x_1, because there is a function yielding x_3 in terms of x_1. But x_2 is not determined by x_1 and x_3, because the values $x_1 = 1$ and $x_3 = 3$ allow x_2 to be either 2 or 4.

So accordingly we define when a formula is satisfied, not by an assignment to its variables, but to a set X of assignments. More formally, if X is a set of assignments of elements of the structure $\mathfrak{M}$ to the variables in the terms $t_1, \dots, t_n$, then X satisfies the formula $=(t_1, \dots, t_n)$ in $\mathfrak{M}$ if and only if there is a function f such that for each assignment s in X,

$$t_n^{\mathfrak{M}}(s) = f(t_1^{\mathfrak{M}}(s), \dots, t_{n-1}^{\mathfrak{M}}(s)). \tag{14}$$

If $\phi(x_1, \dots, x_n)$ is a formula with no dependence formulas in it, then X satisfies ϕ in $\mathfrak{M}$ if and only if every assignment in X satisfies ϕ in $\mathfrak{M}$ in the usual sense.

4. Syntax and semantics

We allow arbitrary vocabularies, containing relation, constant and function symbols. Terms are built up from variable, constant and function symbols in the ordinary way.

Definition 1. A formula of $\mathfrak{D}$ has one of the following forms:

$t = t'$	$t \neq t'$
$R(t_1, \ldots, t_n)$	$\neg R(t_1, \ldots, t_n)$
$=(t_1, \ldots, t_{n-1}, t_n)$	$\neq(t_1, \ldots, t_{n-1}, t_n)$
$\phi \wedge \psi$	$\phi \vee \psi$
$\forall x\phi$	$\exists x\phi$

where $t, t', t_1, \ldots, t_n$ are terms, R is a relation symbol, ϕ and ψ are formulas of $\mathfrak{D}$, and x is a variable.

For the definition of semantics[1] we consider a structure $\mathfrak{M}$ and assignments s mapping some variables to elements of the universe M of $\mathfrak{M}$. The assignment with empty domain is denoted by $\emptyset$. The value $t^{\mathfrak{M}}\langle s\rangle$ of a term t in $\mathfrak{M}$ under the assignment s is defined as usual. We adopt the notation $s[a : x]$ for the modification of s obtained by changing (or adding) the value of s at x to a. If X is a set of assignments, let $X[M; x]$ be the set of all $s[a : x]$, where $a \in M$ and $s \in X$.

Definition 2. *Truth* $\mathfrak{M} \models_X \phi$ of a $\mathfrak{D}$-formula ϕ in the structure $\mathfrak{M}$ under an assignment set X is defined as follows:

$\mathfrak{M} \models_X t_1 = t_2$ iff $\forall s \in X(t_1^{\mathfrak{M}}\langle s\rangle = t_2^{\mathfrak{M}}\langle s\rangle)$.

$\mathfrak{M} \models_X t_1 \neq t_2$ iff $\forall s \in X(t_1^{\mathfrak{M}}\langle s\rangle \neq t_2^{\mathfrak{M}}\langle s\rangle)$.

$\mathfrak{M} \models_X R(t_1, \ldots, t_n)$ iff $\forall s \in X((t_1^{\mathfrak{M}}\langle s\rangle, \ldots, t_n^{\mathfrak{M}}\langle s\rangle) \in \mathrm{Val}_{\mathfrak{M}}(R))$

$\mathfrak{M} \models_X \neg R(t_1, \ldots, t_n)$ iff $\forall s \in X((t_1^{\mathfrak{M}}\langle s\rangle, \ldots, t_n^{\mathfrak{M}}\langle s\rangle) \notin \mathrm{Val}_{\mathfrak{M}}(R))$.

$\mathfrak{M} \models_X =(t_1, \ldots, t_{n-1}, t_n)$ iff $\forall s, s' \in X(t_1^{\mathfrak{M}}\langle s\rangle \neq t_1^{\mathfrak{M}}\langle s'\rangle$ or $\ldots$ or $t_{n-1}^{\mathfrak{M}}\langle s\rangle \neq t_{n-1}^{\mathfrak{M}}\langle s'\rangle$ or $t_n^{\mathfrak{M}}\langle s\rangle = t_n^{\mathfrak{M}}\langle s'\rangle)$.

$\mathfrak{M} \models_X \neq(t_1, \ldots, t_{n-1}, t_n)$ iff $X = \emptyset$.

$\mathfrak{M} \models_X \phi \wedge \psi$ iff $\mathfrak{M} \models_X \phi$ and $\mathfrak{M} \models_X \psi$.
$\mathfrak{M} \models_X \phi \vee \psi$ iff there are X_0 and X_1 such that $\mathfrak{M} \models_{X_0} \phi$, $\mathfrak{M} \models_{X_1} \psi$, and $X \subseteq X_0 \cup X_1$.

$\mathfrak{M} \models_X \forall x\phi$ iff there is Y such that $\mathfrak{M} \models_Y \phi$ and for every $s \in X$ we have $s[a : x] \in Y$ for every $a \in M$.

$\mathfrak{M} \models_X \exists x\phi$ iff there is Y such that $\mathfrak{M} \models_Y \phi$ and for every $s \in X$ we have $s[a : x] \in Y$ for some $a \in M$.

As special cases we define
$\mathfrak{M} \models_X =()$ iff $X = X$
$\mathfrak{M} \models_X \neq()$ iff $X = \emptyset$.

The formula $\neq(t_1, \ldots, t_{n-1}, t_n)$ is a kind of De Morgan dual of $=(t_1, \ldots, t_{n-1}, t_n)$, got by changing disjunctions to conjunctions and by interchanging identities and non-identities. So:
$\mathfrak{M} \models_X \neq(t_1, \ldots, t_{n-1}, t_n)$ iff $\forall s, s' \in X(t_1^{\mathfrak{M}}\langle s\rangle = t_1^{\mathfrak{M}}\langle s'\rangle$ and $\ldots$ and $t_{n-1}^{\mathfrak{M}}\langle s\rangle = t_{n-1}^{\mathfrak{M}}\langle s'\rangle$ and $t_n^{\mathfrak{M}}\langle s\rangle \neq t_n^{\mathfrak{M}}\langle s'\rangle)$.

This amounts to the condition $X = \emptyset$. One may wonder why introduce $\neq(t_1, \ldots, t_n, t)$ at all. The answer will emerge later. An immediate consequence of the definition is

If $\mathfrak{M} \models_X \phi$ and $Y \subseteq X$, then $\mathfrak{M} \models_Y \phi$. (15)
$\mathfrak{M} \models_\emptyset \phi$ for all $\phi \in \mathfrak{D}$. (16)

We illustrate with the formula $\exists y(=(x_1, \ldots, x_n, y) \wedge \phi)$ from (10):

$\mathfrak{M} \models_X \exists y(=(x_1, \ldots, x_n, y) \wedge \phi)$
$\Leftrightarrow$ *there is* Y *such that* $\mathfrak{M} \models_Y (=(x_1, \ldots, x_n, y) \wedge \phi)$ *and for every* $s \in X$ *we have* $s[a : y] \in Y$ *for some* $a \in M$
$\Leftrightarrow$ *there is* Y *such that* $\mathfrak{M} \models_Y =(x_1, \ldots, x_n, y)$ *and* $\mathfrak{M} \models_Y \phi$ *and for every* $s \in X$ *we have* $s[a : y] \in Y$ *for some* $a \in M$
$\Leftrightarrow$ *for some function* f, $\mathfrak{M} \models_{\{s[f(\underline{s}):y]:s\in X\}} \phi$

[1] Following [9].

where $\underline{s}$ means the restriction of s to (at most) the variables $x_1, \ldots, x_n$. For comparison, the trump semantics for IF-logic [8] yields the following semantics in backslash-logic:

$$\begin{array}{ll} & \mathfrak{M} \models_X (\exists y \backslash x_1, \ldots, x_n)\phi \\ \Leftrightarrow & \textit{for some } Y, \mathfrak{M} \models_Y \phi \textit{ and for every subset } W \\ & \textit{of } X \textit{ which is constant on } x_1, \ldots, x_n, \textit{ there is } a \in M \textit{ such that all} \\ & w[a : y] \textit{ with } w \in W \textit{ are in } Y. \end{array}$$

These are clearly the same condition on $\mathfrak{M}$, X and ϕ, so the equivalence (10) above holds. The equivalence (9) can be checked likewise:

Proposition 3. *The following conditions are equivalent:*

(a) $\mathfrak{M} \models_X =(t_1, \ldots, t_{n-1}, t_n)$
(b) $\mathfrak{M} \models_X \exists x_1 \ldots \exists x_{n-1}(x_1 = t_1 \wedge \cdots \wedge x_{n-1} = t_{n-1} \wedge \exists x_n \backslash x_1 \ldots x_{n-1}(x_n = t_n))$

Proof. Suppose $\mathfrak{M} \models_X =(t_1, \ldots, t_{n-1}, t_n)$. Let for each $s \in X$ an extension $\bar{s}$ be defined by

$$\begin{aligned} \bar{s}(x_1) &= t_1^{\mathfrak{M}}\langle s \rangle \\ &\vdots \\ \bar{s}(x_{n-1}) &= t_{n-1}^{\mathfrak{M}}\langle s \rangle. \end{aligned}$$

Let Y be the collection of all $\bar{s}$, where $s \in X$. Clearly,

$$\mathfrak{M} \models_Y (x_1 = t_1 \wedge \cdots \wedge x_{n-1} = t_{n-1}).$$

For $s \in X$ let $\check{s}$ extend $\bar{s}$ by $\check{s}(x_n) = t_n^{\mathfrak{M}}\langle s \rangle$. Let Z be the set of $\check{s}$, where $s \in X$. Now $\mathfrak{M} \models_Z (x_n = t_n)$ and the mapping $\bar{s} \mapsto \check{s}$ satisfies, by assumption (a), the uniformity condition that $\check{s}(x_n)$ is a function of $(\bar{s}(x_1), \ldots, \bar{s}(x_{n-1}))$, when $\bar{s} \in Y$. For the converse, suppose

$$\mathfrak{M} \models_Y (x_1 = t_1 \wedge \cdots \wedge x_{n-1} = t_{n-1} \wedge \exists x_n \backslash x_1 \ldots x_{n-1}(x_n = t_n))$$

so that each $s \in X$ has an extension $\bar{s} \in Y$ with $\{x_1, \ldots, x_{n-1}\} \subseteq \mathrm{dom}(\bar{s})$. Let for each $\bar{s} \in Y$ a further extension $\check{s}$ be defined, with $\check{s}(x_n)$ a function of $(\bar{s}(x_1), \ldots, \bar{s}(x_{n-1}))$, so that their family Z satisfies $\mathfrak{M} \models_Z (x_n = t_n)$. It follows that $t_n^{\mathfrak{M}}\langle s \rangle$ is a function of $(t_1^{\mathfrak{M}}\langle s \rangle, \ldots, t_{n-1}^{\mathfrak{M}}\langle s \rangle)$ for $s \in X$. $\square$

Hence as we anticipated in Section 2:

Every $\mathfrak{D}$-sentence is definable in existential second order logic, (17)

that is, it is possible to associate every $\mathfrak{D}$-sentence ϕ with a first order sentence $\Phi(R)$ with a new predicate symbol R such that the following are equivalent for all models $\mathfrak{M}$:

(a) $\mathfrak{M} \models_{\{\emptyset\}} \phi$
(b) $\mathfrak{M} \models \exists R \Phi(R)$.

On the other hand, it follows from [3,17] that the converse is true

Every existential second order sentence is definable in $\mathfrak{D}$, (18)

that is, we can associate every first order sentence $\Phi(R)$ with an $\mathfrak{D}$-sentence ϕ without R such that the above (a) and (b) are equivalent for all models $\mathfrak{M}$. The familiar properties of existential second order logic and of independence friendly logic then follow:

(a) $\mathfrak{D}$ satisfies the Compactness Theorem.
(b) $\mathfrak{D}$ satisfies the Downward and the Upward Löwenheim–Skolem theorems.
(c) $\mathfrak{D}$ satisfies the Separation Theorem: if $\phi \in \mathfrak{D}$ and $\psi \in \mathfrak{D}$ have no common models, there is a first order θ in the common vocabulary such that every model of ϕ is a model of θ, and θ and ψ have no common models. In particular, only first order definable formulas in $\mathfrak{D}$ have a negation.

It also follows from this that in infinite models $\mathfrak{D}$ is closed under infinite recursive conjunctions (see Corollary 10), indicating the subtle infinitary nature of $\mathfrak{D}$.

5. Game-theoretic semantics for $\mathfrak{D}$

An equivalent game-theoretic definition for the semantics of $\mathfrak{D}$ is given below. This game appears to be a perfect information game. However, we are only interested in the question whether the second player has what we call a *uniform* winning strategy. The effect of the uniformity constraint is that the game becomes actually a game of imperfect information. The only way player II can win with a uniform winning strategy is that she bases her strategy on information that is in game theory appropriately called *imperfect*.

Definition 4. The semantic game $G(\phi, s)$ of the logic $\mathfrak{D}$ in a model $\mathfrak{M}$ is the following game (of imperfect information): There are two players, I and II. In the beginning player II holds the pair (ϕ, s) consisting of the formula ϕ of $\mathfrak{D}$ and an assignment s of the free variables of ϕ.

(a) If ϕ is $t_1 = t_2$, $t_1 \neq t_2$, $R(t_1, \ldots, t_n)$, or $\neg R(t_1, \ldots, t_n)$, and s satisfies it in $\mathfrak{M}$, then II wins the game, otherwise player I wins.
(b) If ϕ is $=(t_1, \ldots, t_{n-1}, t_n)$, then II wins.
(c) If ϕ is $\neq(t_1, \ldots, t_{n-1}, t_n)$, then I wins.
(d) If $\phi = \psi \wedge \theta$, then II switches to hold (ψ, s) or (θ, s), and player I decides which.
(e) If $\phi = \psi \vee \theta$, then II switches to hold (ψ, s) or (θ, s), and can herself decide which.
(f) If $\phi = \forall x\psi$, then II switches to hold $(\psi, s[a : x])$ for some $a \in M$, and player I decides for which.
(g) If $\phi = \exists x\psi$, then II switches to hold $(\psi, s[a : x])$ for some $a \in M$, and can herself decide for which.

Thus $=(t_1, \ldots, t_{n-1}, t_n)$ is a safe haven for II. Respectively, $\neq(t_1, \ldots, t_{n-1}, t_n)$ is a safe haven for I. However, we are not so much interested in who has a winning strategy in this determined game, but who has a winning strategy with extra uniformity, as defined below:

Definition 5. We call a strategy τ of player I or II in the game $G(\phi, \emptyset)$ *uniform* if the following condition holds: Suppose s and s' are assignments arising from the game when II plays τ and the game ends in the same[2] dependence formula $=(t_1, \ldots, t_{n-1}, t_n)$. Suppose furthermore that s and s' agree about the values of $t_1, \ldots, t_{n-1}$. Then s and s' agree about the value of t_n.

Theorem 6. *Suppose ϕ is a sentence of $\mathfrak{D}$. Then $\mathfrak{M} \models_{\{\emptyset\}} \phi$ if and only if player II has a uniform winning strategy in the semantic game, starting with $(\phi, \emptyset)$.*

Proof. This is as in [8]. Consider the following strategy of player II: She keeps pointing to a set X (in the beginning she points to $X = \{\emptyset\}$) such that after every move of the game:

$(\star)$ If (ϕ, s) is the pair she holds and she points to X, then $s \in X$ and $\mathfrak{M} \models_X \phi$.

Let us check that II can actually follow this strategy and win. In the beginning $\mathfrak{M} \models_{\{\emptyset\}} \phi$, so $(\star)$ holds.

(a) Suppose ϕ is $t_1 = t_2$, $R(t_1, \ldots, t_n)$, $t_1 \neq t_2$, or $\neg R(t_1, \ldots, t_n)$. Since $\mathfrak{M} \models_X \phi$ and $s \in X$, we may conclude that s satisfies ϕ in $\mathfrak{M}$. So II wins.
(b) Suppose ϕ is $=(t_1, \ldots, t_{n-1}, t_n)$. II wins outright.
(c) Suppose ϕ is $\neq(t_1, \ldots, t_{n-1}, t_n)$. If the game comes to this point, II is holding (ϕ, s) and $(\star)$ holds, so $X \neq \emptyset$. Thus the game never comes to this point.
(d) Suppose ϕ is $\psi \wedge \theta$. It follows from $\mathfrak{M} \models_X \phi$ that $\mathfrak{M} \models_X \psi$ and $\mathfrak{M} \models_X \theta$. Whether II has to switch to (ψ, s) or to (θ, s), she knows where to point to maintain condition $(\star)$.
(e) Suppose ϕ is $\psi \vee \theta$. Since $\mathfrak{M} \models_X \phi$, $X \subseteq X_0 \cup X_1$ with $\mathfrak{M} \models_{X_0} \psi$ and $\mathfrak{M} \models_{X_1} \theta$. Now $s \in X_0$ or $s \in X_1$. In the first case II moves to point to X_0, in the second case she moves to point to X_1. Condition $(\star)$ remains valid.
(f) Suppose ϕ is $\forall x\psi$. Since $\mathfrak{M} \models_X \phi$, $\mathfrak{M} \models_Y \phi$ for some Y with $X[M; x] \subseteq Y$. Whichever a player I chooses, II can continue by pointing to Y, knowing that $s[a : x] \in Y$.
(g) Suppose ϕ is $\exists x\psi$. Since $\mathfrak{M} \models_X \phi$, there is Y such that $\mathfrak{M} \vdash_Y \psi$ and for every $s' \in X$ we have $s'[a : x] \in Y$ for some $a \in M$. In particular, since $s \in X$, there is $a \in M$ such that $s[a : x] \in Y$. Condition $(\star)$ remains valid, if II points to Y.

We claim that the strategy is uniform. Suppose s and s' are assignments arising from the game when II plays her winning strategy and the game ends in the same dependence formula $=(t_1, \ldots, t_{n-1}, t_n)$. It is easy to prove that the set X that II points to is uniquely determined by her strategy. Suppose s and s' agree about the values of $t_1, \ldots, t_{n-1}$. Since $\mathfrak{M} \models_X =(t_1, \ldots, t_{n-1}, t_n)$ and $s, s' \in X$, it follows that s and s' agree about the value of t_n. This strategy gives one direction of the theorem.

For the other direction, suppose player II has a uniform winning strategy τ in the semantic game starting from $(\phi, \emptyset)$. Let $X_{\phi'}$ be the set of s such that II has held (ϕ', s) in some play where she followed τ. We show by induction on the length of subformulas ϕ' of ϕ that $\mathfrak{M} \models_{X_{\phi'}} \phi'$.

(a) Suppose ϕ' is $t_1 = t_2$, $R(t_1, \ldots, t_n)$, $t_1 \neq t_2$, or $\neg R(t_1, \ldots, t_n)$. In this case $\mathfrak{M} \models_s \phi'$ for $s \in X_{\psi'}$ holds by definition.
(b) Suppose ϕ' is $=(t_1, \ldots, t_{n-1}, t_n)$. Suppose s and s' are in $X_{\phi'}$ and agree about the values of $t_1, \ldots, t_{n-1}$. Since τ is uniform, s and s' agree also about the value of t_n. Thus $\mathfrak{M} \models_{X_{\phi'}} =(t_1, \ldots, t_{n-1}, t_n)$.
(c) Suppose ϕ' is $\neq(t_1, \ldots, t_{n-1}, t_n)$. Now $X_{\phi'} = \emptyset$. Thus $\mathfrak{M} \models_{X_{\phi'}} \neq(t_1, \ldots, t_{n-1}, t_n)$ by (16).
(d) Suppose ϕ' is $\psi \wedge \theta$. By induction hypothesis $\mathfrak{M} \models_{X_\psi} \psi$ and $\mathfrak{M} \models_{X_\theta} \theta$. Now $X_{\phi'} \subseteq X_\psi \cap X_\theta$. Thus $\mathfrak{M} \models_{X_{\phi'}} \phi'$.
(e) Suppose ϕ' is $\psi \vee \theta$. By induction hypothesis $\mathfrak{M} \models_{X_\psi} \psi$ and $\mathfrak{M} \models_{X_\theta} \theta$. Clearly, $X_{\phi'} \subseteq X_\psi \cup X_\theta$. Thus $\mathfrak{M} \models_{X_{\phi'}} \phi'$.
(f) Suppose ϕ' is $\forall x\psi$. By induction hypothesis $\mathfrak{M} \models_{X_\psi} \psi$. For every $s \in X_{\phi'}$ and every $a \in M$ we have $s[a : x] \in X_\psi$. Therefore $\mathfrak{M} \models_{X_{\phi'}} \phi'$.
(g) Suppose ϕ' is $\exists x\psi$. By induction hypothesis $\mathfrak{M} \models_{X_\psi} \psi$. For every $s \in X_{\phi'}$ there is an $a \in M$ such that $s[a : x] \in X_\psi$. Therefore $\mathfrak{M} \models_{X_{\phi'}} \phi'$. □

[2] Two dependence formulas are here considered the same only if they are identical and in the same place in the original formula ϕ.

An equivalent game-theoretic definition using a determined game of perfect information can be easily devised along the lines of [15].

6. An algebraic truth definition

Alternatively, the truth definition can be given algebraically, emphasizing the compositionality of our semantics. Let us first recall the algebraic representation of the semantics of first order logic: If t is a term, let $\mathrm{var}(t)$ denote the set of n for which the variable x_n occurs in t. The *value* $t^{\mathfrak{M}}\langle s\rangle$ of a term t in a structure $\mathfrak{M}$ under the assignment $s : \mathrm{var}(t) \to M$ is defined inductively, as usual, by $x_n^{\mathfrak{M}}\langle s\rangle = s(n)$, $c^{\mathfrak{M}}\langle s\rangle = c^{\mathfrak{M}}$ and $f(t_1,\ldots,t_n)^{\mathfrak{M}}\langle s\rangle = f^{\mathfrak{M}}(t_1^{\mathfrak{M}}\langle s\rangle,\ldots,t_n^{\mathfrak{M}}\langle s\rangle)$. Let $\mathrm{Asg}(M)$ be the set of all assignments of variables into M. Let for any set Y of assignments in a structure $\mathfrak{M}$:

$$Y[a:n] = \{s : s[a:n] \in Y\}.$$

The *truth value* $[\phi]_{\mathfrak{M}}$ of a first order logic formula ϕ in $\mathfrak{M}$ is defined inductively, as usual, by:

$$\begin{array}{lcl}
[t_1 = t_2]_{\mathfrak{M}} & = & \{s : \mathrm{dom}(s) = \mathrm{var}(t_1) \cup \mathrm{var}(t_2) \text{ and } t_1^{\mathfrak{M}}\langle s\rangle = t_2^{\mathfrak{M}}\langle s\rangle\} \\
[t_1 \neq t_2]_{\mathfrak{M}} & = & \{s \in \mathrm{Asg}(M) : \mathrm{dom}(s) = \mathrm{var}(t_1) \cup \mathrm{var}(t_2) \text{ and } t_1^{\mathfrak{M}}\langle s\rangle \neq t_2^{\mathfrak{M}}\langle s\rangle\} \\
[R(t_1,\ldots,t_n)]_{\mathfrak{M}} & = & \{s \in \mathrm{Asg}(M) : \mathrm{dom}(s) = \bigcup_{1\le n\le n} \mathrm{var}(t_i) \text{ and} \\
 & & (t_1^{\mathfrak{M}}\langle s\rangle,\ldots,t_n^{\mathfrak{M}}\langle s\rangle) \in \mathrm{Val}_{\mathfrak{M}}(R))\} \\
[\neg R(t_1,\ldots,t_n)]_{\mathfrak{M}} & = & \{s \in \mathrm{Asg}(M) : \mathrm{dom}(s) = \bigcup_{1\le n\le n} \mathrm{var}(t_i) \text{ and} \\
 & & (t_1^{\mathfrak{M}}\langle s\rangle,\ldots,t_n^{\mathfrak{M}}\langle s\rangle) \notin \mathrm{Val}_{\mathfrak{M}}(R))\} \\
[\phi \wedge \psi]_{\mathfrak{M}} & = & [\phi]_{\mathfrak{M}} \cap [\psi]_{\mathfrak{M}} \\
[\phi \vee \psi]_{\mathfrak{M}} & = & [\phi]_{\mathfrak{M}} \cup [\psi]_{\mathfrak{M}} \\
[\forall x_n\phi]_{\mathfrak{M}} & = & \bigcap_{a\in M}[\phi]_{\mathfrak{M}}[a:n] \\
[\exists x_n\phi]_{\mathfrak{M}} & = & \bigcup_{a\in M}[\phi]_{\mathfrak{M}}[a:n]
\end{array}$$

and then for first order ϕ:

$$\mathfrak{M} \models_s \phi \iff s \in [\phi]_{\mathfrak{M}}.$$

For $\mathfrak{D}$ the algebraic definition is remarkably similar, but involves the power-set operation:

Definition 7. The *truth value* $[\![\phi]\!]_{\mathfrak{M}}$ of a formula ϕ in $\mathfrak{M}$ can be defined as follows:

$$\begin{array}{lcl}
[\![t_1 = t_2]\!]_{\mathfrak{M}} & = & \mathcal{P}([t_1 = t_2]_{\mathfrak{M}}) \\
[\![t_1 \neq t_2]\!]_{\mathfrak{M}} & = & \mathcal{P}([t_1 \neq t_2]_{\mathfrak{M}}) \\
[\![R(t_1,\ldots,t_n)]\!]_{\mathfrak{M}} & = & \mathcal{P}([R(t_1,\ldots,t_n]_{\mathfrak{M}}) \\
[\![\neg R(t_1,\ldots,t_n)]\!]_{\mathfrak{M}} & = & \mathcal{P}([\neg R(t_1,\ldots,t_n)]_{\mathfrak{M}}) \\
[\![=(t_1,\ldots,t_n)]\!]_{\mathfrak{M}} & = & \{X \subseteq \{s \in \mathrm{Asg}(M) : \mathrm{dom}(s) = \bigcup_{1\le n\le n} \mathrm{var}(t_i)\} : \\
 & & X \times X \subseteq \{(s,s') : t_1^{\mathfrak{M}}\langle s\rangle \neq t_1^{\mathfrak{M}}\langle s'\rangle \text{ or} \\
 & & \ldots \text{ or } t_{n-1}^{\mathfrak{M}}\langle s\rangle \neq t_{n-1}^{\mathfrak{M}}\langle s'\rangle \text{ or } t^{\mathfrak{M}}\langle s\rangle = t^{\mathfrak{M}}\langle s'\rangle\}\} \\
[\![\phi \wedge \psi]\!]_{\mathfrak{M}} & = & \bigcup_{X_0\in[\![\phi]\!]_{\mathfrak{M}},X_1\in[\![\psi]\!]_{\mathfrak{M}}} \mathcal{P}(X_0 \cap X_1) \\
[\![\phi \vee \psi]\!]_{\mathfrak{M}} & = & \bigcup_{X_0\in[\![\phi]\!]_{\mathfrak{M}},X_1\in[\![\psi]\!]_{\mathfrak{M}}} \mathcal{P}(X_0 \cup X_1) \\
[\![\forall x_n\phi]\!]_{\mathfrak{M}} & = & \bigcup_{Y\in[\![\phi]\!]_{\mathfrak{M}}} \mathcal{P}(\bigcap_{a\in M} Y[a:n]) \\
[\![\exists x_n\phi]\!]_{\mathfrak{M}} & = & \bigcup_{Y\in[\![\phi]\!]_{\mathfrak{M}}} \mathcal{P}(\bigcup_{a\in M} Y[a:n]).
\end{array}$$

It is obvious that

$$\mathfrak{M} \models_X \phi \iff X \in [\![\phi]\!]_{\mathfrak{M}}.$$

Example 8. Below we identify an element $(a_0,\ldots,a_n)$ of M^{n+1} with the function $(x_0,\ldots,x_n) \mapsto (a_0,\ldots,a_n)$. Moreover, we let

$$\mathrm{Id}_i = \{((a_0,\ldots,a_n),(b_0,\ldots,b_n)) \in M^{n+1} \times M^{n+1} : a_i = b_i\}.$$

- $[\![=()]\!]_{\mathfrak{M}} = \{\emptyset, \{\emptyset\}\} = \mathcal{P}(\{\emptyset\})$
- $[\![=(x_0)]\!]_{\mathfrak{M}} = [\![=(x_0,\ldots,x_0,x_0)]\!]_{\mathfrak{M}} = \mathcal{P}(M)$
- $[\![=(x_0,x_1)]\!]_{\mathfrak{M}} = \{X \subseteq M^2 : X \text{ is a function}\}$.
- $[\![x_0 = x_1]\!]_{\mathfrak{M}} = \{X \subseteq M^2 : X \text{ is an identity function}\}$.
- $[\![=(x_0,f(x_0))]\!]_{\mathfrak{M}} = \mathcal{P}(M)$.

- $[\![=(f(x_0), x_0)]\!]_{\mathfrak{M}} = \{X \subseteq M : f^{\mathfrak{M}} \text{ is one to one on } X\}$.
- $[\![=(x_0, \ldots, x_{n-1}, x_n)]\!]_{\mathfrak{M}} = \{X \subseteq M^{n+1} : X \times X \subseteq (M^{n+1} \setminus \mathrm{Id}_0) \cup, \ldots, \cup (M^{n+1} \setminus \mathrm{Id}_{n-1}) \cup \mathrm{Id}_n\}$.

There is a perfect symmetry between disjunction and conjunction on the one hand, and existential and universal quantifiers on the other hand, just as in the first order case. If ϕ happens to be first order (i.e. does not contain subformulas of the form $=(t_1, \ldots, t_n)$ or of the form $\neq(t_1, \ldots, t_n)$), then $[\![\phi]\!]_{\mathfrak{M}}$ has a largest element, namely $[\phi]_{\mathfrak{M}}$. If ϕ is in first order logic, then for all $\mathfrak{M}$ and X we have

$$\mathfrak{M} \models_X \phi \iff (\forall s \in X)(\mathfrak{M} \models_s \phi)$$

i.e.

$$X \in [\![\phi]\!]_{\mathfrak{M}} \iff X \subseteq [\phi]_{\mathfrak{M}}.$$

In this sense the semantics of $\mathfrak{D}$ is a conservative extension of the semantics of first order logic. The definition of $[\![\phi]\!]_{\mathfrak{M}}$ also demonstrates the compositionality of $\mathfrak{D}$.

7. Definability of truth

Some systems can define their own truth, see e.g. [13]. Since $\mathfrak{D}$ is able to express all existential second order sentences, it is in principle clear that it, too, can express its own truth definition, as is pointed out in [5] and proved in [6, Appendix]. On the basis of the inductive truth definition of $\mathfrak{D}$ it is possible to write down the truth definition explicitly in $\mathfrak{D}$. For this end, let us work in a model $\mathfrak{M}$ which has enough coding to code the basic syntactic concepts as well as all assignments by elements of the model. Let us denote the Gödel number of ϕ by $\ulcorner\phi\urcorner$. We denote the term denoting the natural number n in our model by $\underline{n}$. Let $\underline{\emptyset}$ be the term denoting the singleton set of the empty assignment. We construct a sentence $\theta(P)$ in $\mathfrak{D}$ with a new binary predicate symbol P, and a formula $\Theta(x)$ in $\mathfrak{D}$ such that:

Theorem 9. *The following conditions are equivalent for all sentences ϕ of $\mathfrak{D}$:*

(a) $\mathfrak{M} \models_{\{\emptyset\}} \phi$
(b) $\mathfrak{M} \models_{\{\emptyset\}} \exists P(\theta(P) \wedge P(\ulcorner\phi\urcorner, \underline{\emptyset}))$
(c) $\mathfrak{M} \models_{\{\emptyset\}} \Theta(\ulcorner\phi\urcorner)$.

As in [11], we get

Corollary 10. *$\mathfrak{D}$ is closed under recursive conjunctions in infinite models.*

Clearly, $\mathfrak{D}$ is not closed under recursive disjunctions: the disjunction of the sentences "P contains at most n elements", $n = 1, 2, \ldots$, would lead to a contradiction with the Compactness Theorem of $\mathfrak{D}$.

For the proof of Theorem 9, we introduce some notation, following [1]. Let $\mathrm{Vł}(t, x_1)$ denote the value of a term t in $\mathfrak{M}$ under the assignment coded by x_1. Let $\mathrm{E}(x_1, x_2, x_3)$ code the assignment $s[a : n]$, where s is the assignment coded by x_1, n is the value of x_2, and a is the value of x_3.

Definition 11. Let $\theta'(x_0, x_1, x_2)$ be the conjunction of the following formulas with free variables x_0, x_1, and x_2, and the binary predicate symbol P:

(a) Either x_0 is not the Gödel number of an identity of two terms t and t', or $\neg P(x_0, x_1)$ or the terms t and t' have the same value under the assignment x_1.
(b) Either x_0 is not the Gödel number of a negated identity of two terms t and t', or $\neg P(x_0, x_1)$ or the terms t and t' have different value under the assignment x_1.
(c) Either x_0 is not the Gödel number of an atomic formula $P(t_1, \ldots, t_n)$ of terms $t_1, \ldots, t_n$, or $\neg P(x_0, x_1)$ or the terms values of the terms $t_1, \ldots, t_n$ under the assignment x_1 satisfy the predicate P.
(d) Either x_0 is not the Gödel number of a negated atomic formula $\neg P(t_1, \ldots, t_n)$ of terms $t_1, \ldots, t_n$, or $\neg P(x_0, x_1)$ or the values of the terms $t_1, \ldots, t_n$ under the assignment x_1 fail to satisfy the predicate P.
(e) Either x_0 is not the Gödel number of a dependence formula $=(t_1, \ldots, t_n)$ of terms $t_1, \ldots, t_n$, or $\neg P(x_0, x_1)$ or $=(\mathrm{Vł}(t_1, x_1), \ldots, \mathrm{Vł}(t_n, x_1))$.
(f) Either x_0 is not the Gödel number $\ulcorner\phi \wedge \psi\urcorner$ of a conjunction, or $\neg P(x_0, x_1)$ or $(P(\ulcorner\phi\urcorner, x_1)$ and $P(\ulcorner\psi\urcorner, x_1))$.
(g) Either x_0 is not the Gödel number $\ulcorner\phi \vee \psi\urcorner$ of a disjunction, or $\neg P(x_0, x_1)$ or $P(\ulcorner\phi\urcorner, x_1)$ or $P(\ulcorner\psi\urcorner, x_1)$.
(h) Either x_2 is not a natural number, or x_0 is not $\ulcorner\forall z\phi\urcorner$, where z is the variable number x_2, or $\neg P(x_0, x_1)$, or $\forall x_3 P(\ulcorner\phi\urcorner, \mathrm{E}(x_1, x_2, x_3))$.
(i) Either x_2 is not a natural number, or x_0 is not $\ulcorner\exists z\phi\urcorner$, where z is the variable number x_2, or $\neg P(x_0, x_1)$, or $\exists x_3 P(\ulcorner\phi\urcorner, \mathrm{E}(x_1, x_2, x_3))$.

Let $\theta(P)$ be the formula $\forall x_0 \forall x_1 \forall x_2 \theta'(x_0, x_1, x_2)$.

Proposition 12. *There is a formula $\Theta(x_4, x_5)$ of $\mathfrak{D}$ such that the following conditions are equivalent for all natural numbers n and all X:*

(a) $(\mathfrak{M}, S) \models_{\{\emptyset\}} \theta(P) \wedge P(\underline{n}, \underline{\emptyset})$ *for some interpretation S of P in $\mathfrak{M}$.*
(b) $\mathfrak{M} \models_{\{\emptyset\}} \Theta(\underline{n}, \underline{\emptyset})$.

Proof. Since the proof is essentially in [3,17], we give just an outline in a special case. Suppose $\theta(P)$ were just

$$\forall x_0 \forall x_1 (\ (\neg P(x_0, x_1) \vee {=}(f(x_0, x_1), g(x_0, x_1))) \wedge \\ (\neg P(x_0, x_1) \vee \exists x_2 P(x_0, h(x_1, x_2)))).$$

We would then let $\Theta(x_4, x_5)$ be the formula

$$\begin{aligned}\forall x_0 \forall x_1 \forall y_0 \forall y_1 \exists u \exists x_2 \exists v (&{=}(x_0, x_1, u) \wedge \\ &{=}(y_0, y_1, v) \wedge \\ &{=}(x_0, x_1, x_2) \wedge \\ &(x_0 \neq y_0 \vee x_1 \neq y_1 \vee u = v) \wedge \\ &(u \neq \underline{1} \vee {=}(f(x_0, x_1), g(x_0, x_1))) \wedge \\ &(u \neq \underline{1} \vee x_0 \neq y_0 \vee y_1 \neq h(x_1, x_2) \vee v = \underline{1}) \wedge \\ &(x_0 \neq x_4 \vee x_1 \neq x_5 \vee u = \underline{1})).\end{aligned}$$

Suppose now $(\mathfrak{M}, S) \models_{s_0} \theta \wedge P(x_4, \underline{\emptyset})$, where $s_0(x_4) = n$. Let X be the set of all assignments s interpreting x_0 and x_1 in M. Thus $X \subseteq X_0 \cup X_1$ such that

$$s \in X_0 \text{ implies } \quad \neg S(s(x_0), s(x_1)) \tag{19}$$

$$s, s' \in X_1 \text{ implies } \quad f(s(x_0), s(x_1)) \neq f(s'(x_0), s'(x_1)) \text{ or} \tag{20}$$

$$g(s(x_0), s(x_1)) = g(s'(x_0), s'(x_1)). \tag{21}$$

Moreover, $X \subseteq X_2 \cup X_3$ such that

$$s \in X_2 \text{ implies } \neg S(s(x_0), s(x_1)) \tag{22}$$

$$s \in X_3 \text{ implies } \text{ there is an } a_s \in M \text{ such that } S(s(x_0), h(s(x_1), a_s)). \tag{23}$$

To prove $\mathfrak{M} \models_{\{\emptyset\}} \Theta(\underline{n})$ let X' be the set of all assignments s interpreting x_0, x_1, y_0, and y_1 in M. Let Y be obtained from X' by extending each $s \in X'$ by the assignments $s(u) = 1 \iff S(s(x_0), s(x_1))$, $s(v) = 1 \iff S(s(y_0), s(y_1))$, and $s(x_2) = a_{s \restriction \{x_0, x_1\}}$, if $s \restriction \{x_0, x_1\} \in X_3$ and otherwise $= 1$. We prove

$$\begin{aligned}\mathfrak{M} \models_Y &{=}(x_0, x_1, u) \wedge \\ &{=}(y_0, y_1, v) \wedge \\ &{=}(x_0, x_1, x_2) \wedge \\ &(x_0 \neq y_0 \vee x_1 \neq y_1 \vee u = v) \wedge \\ &(u \neq \underline{1} \vee {=}(f(x_0, x_1), g(x_0, x_1))) \wedge \\ &(u \neq \underline{1} \vee x_0 \neq y_0 \vee y_1 \neq h(x_1, x_2) \vee v = \underline{1}) \wedge \\ &(x_0 \neq x_4 \vee x_1 \neq \emptyset \vee u = \underline{1}).\end{aligned} \tag{24}$$

The first four conjuncts of (24) are trivially satisfied. Let $Y_0 = \{s \in Y : \neg S(s(x_0), s(x_1))\}$ and $Y_1 = \{s \in Y : S(s(x_0), s(x_1))\}$. Then $Y = Y_0 \cup Y_1$ and $\mathfrak{M} \models_{Y_0} u \neq \underline{1}$. To prove $\mathfrak{M} \models_{Y_1} {=}(f(x_0, x_1), g(x_0, x_1))$ let $s, s' \in Y_1$. Then by (20) and (21) $f(s(x_0), s(x_1)) \neq f(s'(x_0), s'(x_1))$ or $g(s(x_0), s(x_1)) = g(s'(x_0), s'(x_1))$. We have proved the fifth conjunct of (24). Next we prove

$$\mathfrak{M} \models_{Y_1} (x_0 \neq y_0 \vee y_1 \neq h(x_1, x_2) \vee v = \underline{1})). \tag{25}$$

Assume then $s \in Y_1$ is such that $s(x_0) = s(y_0)$ and $s(y_1) = h(s(x_1), s(x_2))$. By (23), $S(s(y_0), s(y_1))$ holds, whence $s(v) = 1$.

Finally, we prove

$$\mathfrak{M} \models_Y x_0 \neq x_4 \vee x_1 \neq \emptyset \vee u = \underline{1}. \tag{26}$$

Assume $s \in Y$ is such that $s(x_0) = n$ and $s(x_1) = \underline{\emptyset}$. By our assumption $(\mathfrak{M}, S) \models_{s_0} P(x_4, \underline{\emptyset})$. So (26) follows immediately from the definition of $s(u)$. For the converse, suppose $\mathfrak{M} \models_{s_0} \Theta(x_4)$. Let X' be the set of all possible assignments defined on $\{x_0, x_1, y_0, y_1\}$, and let Z consist of an extension $E(s)$ of each $s \in X'$ by a definition of $s(u), s(x_2), s(v) \in M$ so that

$$\begin{aligned}\mathfrak{M} \models_Z &{=}(x_0, x_1, u) \wedge \\ &{=}(y_0, y_1, v) \wedge \\ &{=}(x_0, x_1, x_2) \wedge \\ &(x_0 \neq y_0 \vee x_1 \neq y_1 \vee u = v) \wedge \\ &(u \neq \underline{1} \vee {=}(f(x_0, x_1), g(x_0, x_1))) \wedge \\ &(u \neq \underline{1} \vee x_0 \neq y_0 \vee y_1 \neq h(x_1, x_2) \vee v = \underline{1}) \wedge \\ &(x_0 \neq x_4 \vee x_1 \neq \underline{\emptyset} \vee u = \underline{1}).\end{aligned} \tag{27}$$

Let

$$S = \{(a_0, a_1) : \text{ there is an } s \in Z, \text{ with } s(x_0) = a_0, s(x_1) = a_1, s(u) = 1\}.$$

We show that $(\mathfrak{M}, S) \models_{\emptyset} \theta$. Let W the set of all assignments s interpreting x_0 and x_1 in M. Because of the first four conjuncts of (27) there are functions E_u and E_{x_2} on W such that if $s \in X'$, then

$$E(s)(u) = E_u(s \restriction \{x_0, x_1\}) \tag{28}$$

$$E(s)(v) = E_u(s \restriction \{y_0, y_1\}) \tag{29}$$

$$E(s)(x_2) = E_{x_2}(s \restriction \{x_0, x_1\}). \tag{30}$$

The fifth conjunct of (27) shows $Z \subseteq Z_0 \cup Z_1$, where $\mathfrak{M} \models_{Z_0} u \neq 1$ and $\mathfrak{M} \models_{Z_1} =\!(f(x_0, x_1), g(x_0, x_1))$. Let W_0 consist of all $s \in W$ with some extension $\bar{s} \in Z_0$, and $W_1 = W \setminus W_0$. If $s \in W_0$ with an extension $\bar{s} \in Z_0$, then $\bar{s}(u) \neq 1$, whence $\neg S(s(x_0), s(x_1))$. If $s, s' \in W_1$, and the extensions $\bar{s}, \bar{s'} \in Z$ are chosen arbitrarily (e.g. $\bar{s}(y_0) = \bar{s}(y_1) = \bar{s}(u) = \bar{s'}(y_0) = \bar{s'}(y_1) = \bar{s'}(u) = 1$), then $\bar{s}, \bar{s'} \in Z_1$. This implies $\mathfrak{M} \models_{W_1} =\!(f(x_0, x_1), g(x_0, x_1))$. We have proved that W satisfies the first conjunct of θ.

We next prove that W satisfies the second conjunct of θ. Let $Z \subseteq Z_2 \cup Z_3 \cup Z_4 \cup Z_5$ such that

$$\mathfrak{M} \models_{Z_2} u \neq \underline{1} \tag{31}$$

$$\mathfrak{M} \models_{Z_3} x_0 \neq y_0 \tag{32}$$

$$\mathfrak{M} \models_{Z_4} y_1 \neq h(x_1, x_2) \tag{33}$$

$$\mathfrak{M} \models_{Z_5} v = \underline{1}. \tag{34}$$

Let W_2 consist of all $s \in W$ with some extension $s' \in Z_2$, and $W_3 = W \setminus W_2$. If $s \in W_2$ with an extension $s' \in Z_2$, then $s(u) = s'(u) \neq 1$, whence $\neg S(s(x_0), s(x_1))$. If $s \in W_3$, let $G(s)$ extend s by the definition $G(s)(x_2) = E_{x_2}(s)$. Let V consist of all $G(s)$ with $s \in W_3$. We aim at proving $\mathfrak{M} \models_V P(x_0, h(x_1, x_2))$. Let $G(s) \in V$ with $s \in W_3$. Let $a_0 = s(x_0)$, $a_1 = s(x_1)$, and $a_2 = G(s)(x_2)$. Let us consider the following two assignments in X':

	s_1	s_2
x_0	a_0	a_0
x_1	a_1	$h(a_1, a_2)$
y_0	a_0	a_0
y_1	$h(a_1, a_2)$	$h(a_1, a_2)$

Then $E(s_1), E(s_2) \in Z$. As $s \in W_3$ and s_1 extends s,

$$E(s_1)(u) = 1. \tag{35}$$

As $s_1(x_0) = s_1(y_0)$, $E(s_1) \notin Z_3$. As

$$s_1(y_1) = h(a_1, a_2) = h(E(s_1)(x_1), E(s_1)(x_2)),$$

we have $E(s_1) \notin Z_4$. Thus $E(s_1) \in Z_5$, whence

$$E(s_1)(v) = 1. \tag{36}$$

As $s_2(x_0) = s_2(y_0)$ and $s_2(x_1) = s_2(y_1)$,

$$\begin{aligned} E(s_2)(u) &= E(s_2)(v) \text{ by (27)} \\ &= E_u(s_2 \restriction \{y_0, y_1\}) \text{ by (29)} \\ &= E_u(s_1 \restriction \{y_0, y_1\}) \text{ as } s_2 \text{ and } s_1 \text{ agree on } y_0 \text{ and } y_1 \\ &= E(s_1)(v) \text{ by (29)} \\ &= 1 \text{ by (36)}. \end{aligned}$$

We have found $s_2 \in Z$ such that $s_2(x_0) = a_0$, $s_2(x_1) = h(a_1, a_2)$, and $s_2(u) = 1$. This implies $S(a_0, a_1)$. We have proved that W satisfies the second conjunct of θ.

The last conjunct of (27) shows that $(\mathfrak{M}, S) \models_{\{\emptyset\}} P(\underline{n}, \underline{\emptyset})$. □

8. Dependence in English

There are a number of English phrases which allow definite noun phrases, but seem to yield nonsense if the definite noun phrases are read as names of individual objects. For example:

(a) x is determined by y.
(b) x is a function of y.
(c) x depends on y.
(d) x is independent of y.
(e) x varies with y.

All these five examples allow interpretation along the following lines. A family X of paired assignments (a, b) (with a assigned to x and b assigned to y) is given, and the phrase expresses some property of X which cannot be paraphrased as a property of any single pair (a, b) in X. We call an interpretation of this kind a *family* interpretation.

Three questions arise at once. First, what property of X is expressed? Second, how is X determined? Third, given X, how do these phrases express properties of X, given that a typical English sentence 'x does such-and-such to y' expresses a property of a single x and a single y?

8.1. What property of X?

There is an easy answer in case (a): x is determined by y if and only if for any two pairs (a_1, b_1) and (a_2, b_2) in X, if $b_1 = b_2$ then $a_1 = a_2$. In case (e) we nearly have the opposite answer: for any two pairs (a_1, b_1) and (a_2, b_2) in X, if $b_1 \neq b_2$ then $a_1 \neq a_2$. If this were exactly true then we would expect (e) to have the paraphrase

y is determined by x

and vice versa. In practice this is not how the phrases are used. There seems to be a causal implication; in case (a) the implication is that the choice of second coordinate acts so as to decide the first coordinate, and in case (e) the implication is that a change in the second coordinate causes a change in the first. Examples from google.com (the poor man's corpus) are

(For (a)) Hip bone density is determined by teenage exercise patterns.
(For (e)) Pet insurance varies with breed of dog.

To be sure of avoiding any causal implications, we probably need to take mathematical examples; but then we are in danger of leaving ordinary English usage behind.

The usage of (b) is harder to pin down. Mathematicians often assume that it means what we said (a) means, in other words that for some function f, X is the set of all pairs of the form $(f(b), b)$. But sometimes this is clearly not so. One we found on google.com some years ago was:

The quality of your essay is a function of the work you put into it.

The clear implication here is that you can improve your essay by putting in more work; this looks more like (e), and it would certainly be false if the quality was a constant function of the work! It's not entirely clear here whether the speaker means to rule out that other factors might be relevant too, because the class X is not spelt out. For example the implied class might be the class of all possible pairs

(the quality of your essay, how hard you try to improve your essay)

on the assumption that other factors are kept constant.

There is also a Gricean factor. Someone who tells you (b) would presumably have said so if x was determined by something less than y. The same applies to (a). This Gricean implication goes some way towards (e), but easy examples show that it is weaker than (e).

(c) is perhaps closer to the Gricean implication of (b). A typical use of (c) is where we have a bundle of factors, and y describes the values of some of them. In this case (c) expresses that x is not determined by any set of factors unless the set contains all the factors reported in y. So here we have an extra multiplicity: not just the family X, but a set of factors, some of which are involved in X. An example from google.com is

Your security depends on your workers' habits.

(d) seems to be a negation of (c); for example 'Your security is independent of your workers' habits' (in other words, it is entirely determined by other factors).

8.2. How is X given?

There is probably nothing useful that we can say here about the cases where X is given by the context of utterance. More important for us are the cases where no X seems to be given, even though one is needed for the interpretation.

Here is one of several similar examples in the books of Serge Lang [12]:

Let $0 < a \leq 1$, and m an integer with $|m| \geq 2$. Let $s = \sigma + iT_m$ with $-a \leq \sigma \leq 1 + a$ and T_m as above. Then

$$|\xi'/\xi(s)| \leq b(\log |m|)^2,$$

where b is a number depending on a but not on m and σ.

Never mind that this is heavily mathematical; students seem to have no trouble understanding the idiom without needing to have it explained to them. (One of us used this same example on page 545 of [8] to make the point that IF-like phenomena do occur quite naturally in mathematical writing. Here we make the further point that Lang's text is closer to the dependence-of-variables formulation of the present paper than it is to slashed quantifiers.)

The form of Lang's statement is roughly:

For all a, m and σ there is b such that $\psi(a, m, \sigma, b)$ and b depends on a but not on m and σ.

There is an obvious family X here, namely the family of 4-tuples (a, m, σ, b) such that $\psi(a, m, \sigma, b)$ holds. Three factors are relevant for finding b, namely a, m and σ. Lang's final clause tells us we do not need the second and third factors in order to determine b. But that leaves only the first factor, and Lang is certainly not telling us that b is determined by the first factor. He neither says nor implies anything about b being unique.

There are at least two ways to read this example. Perhaps more work and a wider range of examples are needed to decide which if either of them is right.

(A) The first reading is that Lang means that b is determined by a, but he does not intend this X as the relevant family. Rather his sentence means that *there is* a family X' such that certain things hold, and within this family b is determined by a. This reading will probably appeal to mathematicians who detect a function quantifier hidden under Lang's text.
(B) The second reading is that we got the right family X, but Lang is not saying anything about functional dependence. Rather he is saying first that for every choice of a, m and σ there is a b with (a, m, σ, b) in X (this is his first clause), and second that given a, m and σ we can choose the b without making any use of m and σ, but there is no b that works for all a.

If we try to build up a formal semantics for the relevant fragment of English, (B) will probably be easier to handle than (A), because (A) involves an existential quantification over sets.

8.3. How to shift from individuals to families?

There is a much-quoted paper of Barbara Partee [14] which helps us here. She describes some ways in which noun phrases can come to have interpretations not of type e (individuals) but of type $\langle e, t\rangle$ (functions from individuals to truth values, i.e. predicates), or even of type $\langle\langle e, t\rangle, t\rangle$ (predicates of predicates). For example the noun phrase 'an introvert' can stand for an individual introvert; but in the context

Mary considers John an introvert.

the verb 'considers' has the effect of raising the type of 'an introvert' from e to $\langle e, t\rangle$, so that it operates as a predicate. (This is an oversimplified version of one of her remarks.)

Our phrases seem to demand something similar, though not exactly any of the devices that Partee mentions. We need to say, for Lang's example above, that the word 'depending' has the effect of creating a semantic component of type $\langle e, \langle e, \langle e, \langle e, t\rangle\rangle\rangle\rangle$, i.e. a property of 4-tuples. Note that the first part of Lang's sentence, up to but excluding the word 'where', has a conventional semantics that does not require any higher-type object like the component we have just proposed. Nevertheless this first part contains all the raw materials used to construct the component. It seems that this type raising occurs after the first processing of the first part of the sentence. Partee makes the point that facts at a low type level can be translated into facts at a higher level, and in this case there is no problem about recasting Lang's first clause so that it involves the higher type component.

Dependence logic does something similar to what we have just described. Sentences of dependence logic with no occurrences of $=(\ldots)$ or $\neq(\ldots)$ can be interpreted exactly as ordinary first-order sentences; but there is a uniform paraphrase of the first-order interpretation in terms of sets of assignments. When a $=(\ldots)$ appears, it forces the interpretation into the higher-type form, using sets of assignments.

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