On the semantics of informational independence

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Abstract

The semantics of the independence friendly logic of Hintikka and Sandu is usually defined via a game of imperfect information. We give a definition in terms of a game of perfect information. We also give an Ehrenfeucht-Fraïssé game adequate for this logic and use it to define a Distributive Normal Form for independence friendly logic.

Keywords: Independence friendly logic, Ehrenfeucht-Fraïssé game

1 Introduction

We consider a logic $\mathcal{IF}$ which is the closure of atomic and negated atomic formulas of first order logic under the operations

$$\begin{array}{ll} (1) & \wedge, \\ (2) & \vee/v_{i_1}\dots v_{i_k}, \\ (3) & \forall, \\ (4) & \exists v_n/v_{i_1}\dots v_{i_k}, \end{array}$$

where $n \notin \{i_1, \dots i_k\}$. This is, mutatis mutandis, the independence friendly logic of Hintikka and Sandu [4]. The intuitive interpretation of $\phi \vee /v_{i_1}\dots v_{i_k}\psi$ is "ϕ or ψ independently of $v_{i_1}, ..., v_{i_k}$." Likewise, the interpretation of $\exists v_n/v_{i_1}\dots v_{i_k}$ is "there exists v_n independently of $v_{i_1}, ..., v_{i_k}$." The *extended* independence friendly logic $\mathcal{EIF}$ is the Boolean closure of $\mathcal{IF}$, that is, the closure of the set of sentences of $\mathcal{IF}$ under the operations

$$\begin{array}{ll} (1) & \wedge, \\ (5) & \neg, \end{array}$$

which are understood classically.

Hintikka and Sandu define the semantics of $\mathcal{IF}$ via a non-determined game of imperfect information. Hodges [5] gives for his independence friendly logic an inductive truth definition a'la Tarski, which actually makes the semantics compositional. Caicedo and Krynicki [1] use a variation of Hodges' truth-definition and prove a Prenex Normal Form Theorem.

For first order logic it is customary to define truth in terms of the concept of a sequence satisfying a formula in a structure. However, here we have to deal with sequences in which some elements are independent of some others. But it does not make sense to speak about independence of elements in a single sequence. For example, we may have the sequence

$$12 \qquad 32 \qquad 51$$

for interpreting the variables v_0, v_1 and v_2. It does not make sense to say that 51 is independent of 12. But if we have a set of sequences, such as

12	32	51
17	32	51
3	32	51
10	46	58
2	46	58
17	46	58
19	46	58

there is a clear sense in which the third elements 51 and 58 are independent of the first element of each triple. This interpretation of independence is an essential feature of the semantics given by Hodges ([5, 6]). The treatment of Caicedo and Krynicki is also based on sets of sequences.

Thus we define the concept of a set X of sequences satisfying a formula ϕ. For ordinary first order formulas this is equivalent to saying that every individual sequence in the set X satisfies ϕ. We show that by defining the basic concepts of semantics systematically in this higher order setting, we get for $\mathcal{IF}$ not only a compositional Tarski-style semantics as in Hodges [6], but also a game-theoretic semantics of perfect information, and an Ehrenfeucht-Fraïssé game. The Ehrenfeucht-Fraïssé game is likewise a game of perfect information. The new games are all determined.

The Ehrenfeucht-Fraïssé game is used in Section 5 to analyse $\mathcal{IF}$ in terms of the so called *Distributive Normal Form*, introduced for first order logic by Hintikka [3]. This is a normal form obtained by pushing quantifiers of a formula in a sense as deep as possible. So it is a kind of opposite for the Prenex Normal Form.

2 Notation

The variables $v_{i_1}, ..., v_{i_k}$ in $\exists/v_{i_1} \ldots v_{i_k}$ and $\vee/v_{i_1} \ldots v_{i_k}$ are considered free occurrences of variables in the formula. We treat $v_{i_1}, ..., v_{i_k}$ in $/v_{i_1} \ldots v_{i_k}$ as a set. So $/v_{i_1} \ldots v_{i_k}$ is in fact a shorthand for $/\{v_{i_1} ... v_{i_k}\}$. We further shorten $\vee/\emptyset$ to $\vee$ and $\exists/\emptyset$ to $\exists$. If $v_{i_1}, ..., v_{i_k}$ covers all the free variables of ϕ and ψ, then we write

$$\phi \oplus \psi$$

for $\phi \vee /v_{i_1}, ..., v_{i_k} \psi$. This is the classical totally independent disjunction.

Suppose $\mathcal{A}$ is a structure with universe A. If $W \subseteq \omega$, let A^W be the set of functions $f : W \to A$. We use $S(A^W)$ to denote the power set of A^W. Let $X \in S(A^W), F : X \to B$ for some set B, and $V \subseteq W$. We say that F is V*-uniform* ([5]) if for all $f, g \in X$ we have

$$(\forall n \in W \setminus V)(f(n) = g(n)) \Rightarrow F(f) = F(g).$$

If $W \subseteq \omega, X \in S(A^W), F : X \to A, f \in X, a \in A$ and $n \in \omega$, we define $f[a, n] \in A^{W \cup \{n\}}$ by:

$$f[a,n](m) = \begin{cases} a & \text{, if } m = n \\ f(m) & \text{, otherwise} \end{cases}$$

and

$$X[F,n] = \{f[F(f),n] : f \in X\}.$$

Furthermore, let

$$X[A;n] = \{f[a,n] : a \in A, f \in X\}.$$

3 Semantics

We define the satisfaction relation for the logic $\mathcal{IF}$. Our definition is like [6]. For sentences this gives the same semantics as [4].

Definition 3.1 ([6]) Let $\mathcal{A}$ be a structure. The notion

$$\mathcal{A} \models_X \phi$$

is defined by the following inductive definition for all $W \subseteq \omega$, $X \in S(A^W)$, $V \subseteq \{v_n : n \in W\}$ and all formulas ϕ of $\mathcal{IF}$ with free variables among v_n, $n \in W$.

(1) $\mathcal{A} \models_X \phi \iff (\forall f \in X)(\mathcal{A} \models \phi(f(i_1),...,f(i_m))$, when $\phi(v_{i_1},...,v_{i_m})$ is an atomic or negated atomic formula.

(2) $\mathcal{A} \models_X \phi \wedge \psi \iff (\mathcal{A} \models_X \phi$ and $\mathcal{A} \models_X \phi)$

(3) $\mathcal{A} \models_X \phi \vee /v_{i_1} \ldots v_{i_k}\psi \iff (\exists F : X \to 2)(\mathcal{A} \models_{F^{-1}(\{0\})} \phi$ and $\mathcal{A} \models_{F^{-1}(\{1\})} \psi$ and F is $\{i_1,...,i_k\}$-uniform)

(4) $\mathcal{A} \models_X \forall v_n \phi \iff \mathcal{A} \models_{X[A;n]} \phi$

(5) $\mathcal{A} \models_X \exists v_n/v_{i_1} \ldots v_{i_k}\phi \iff (\exists F : X \to A)(\mathcal{A} \models_{X[F,n]} \phi$ and F is $\{i_1,...,i_k\}$-uniform)

For sentences ϕ we define

$$\mathcal{A} \models \phi \iff \mathcal{A} \models_{\{\emptyset\}} \phi \ (\iff \mathcal{A} \models_{A^n} \phi).$$

The logical consequence $\phi \models \psi$ is defined to mean

$$\forall \mathcal{A} \forall X(\mathcal{A} \models_X \phi \Rightarrow \mathcal{A} \models_X \psi)$$

and *logical equivalence* of ϕ and ψ is defined to mean that both $\phi \models \psi$ and $\psi \models \phi$.

We consider the variables $v_{i_1} \ldots v_{i_k}$ as occurring free in $\exists v_n/v_{i_1} \ldots v_{i_k}\phi$.

Example 3.2 Let ϕ be the formula $\exists v_1/v_0(v_0 = v_1)$. Then $\phi \vee \phi \not\models \phi$, but $\phi \oplus \phi \models \phi$. Thus $\phi \vee \phi$ and $\phi \oplus \phi$ are not logically equivalent.

The following lemma is well-known, but we give a proof to illustrate the truth-definition:

Lemma 3.3 Suppose A is a set and $R \subseteq A^4$. We prove that the following conditions are equivalent:

(1) $(A,R) \models \forall v_0 \forall v_1 \exists v_2/v_1 \exists v_3/v_0 v_2 R(v_0,v_1,v_2,v_3)$

(2) $(A,R) \models \begin{pmatrix} \forall v_0 & \exists v_2 \\ \forall v_1 & \exists v_3 \end{pmatrix} R(v_0,v_1,v_2,v_3)$.

Proof. Let $\mathcal{A} = (A, R)$. Suppose first (1). Thus

$$\mathcal{A} \models_{A^2} \exists v_2/v_1 \exists v_3/v_0 v_2 R(v_0, v_1, v_2, v_3).$$

Therefore there is a $\{1\}$-uniform $F : A^2 \to A$ such that

$$\mathcal{A} \models_X \exists v_3/v_0 v_2 R(v_0, v_1, v_2, v_3),$$

where $X = \{h \in A^3 : h(2) = F(h(0), h(1))\}$. By the uniformity of F, we can write $F(a, b)$ as $f(a)$ for some $f : A \to A$. Thus $X = \{h \in A^3 : h(2) = f(0)\}$. Let $G : X \to A$ be $\{0, 2\}$-uniform such that

$$\mathcal{A} \models_{X[G,3]} R(v_0, v_1, v_2, v_3).$$

Since $G(a, b, f(a))$ only depends on b, we can write it as $g(b)$ for some $g : A \to A$. Now $X[G, 3] = \{h \in A^4 : h(2) = f(0), h(3) = g(1)\}$ and (2) follows.

Conversely, suppose (2) holds, that is, there are functions $f : A \to A$ and $g : A \to A$ such that for all $a, b \in A$ we have $(a, b, f(a), g(b)) \in R$. If we define G by $G(a, b, f(a)) = g(b)$ and let $X = \{(a, b, f(a)) : a, b \in A\}$, we have

$$\mathcal{A} \models_{X[G,3]} R(v_0, v_1, v_2, v_3),$$

whence

$$\mathcal{A} \models_X \exists v_3/v_0 v_2 R(v_0, v_1, v_2, v_3).$$

If we now define $F(a, b) = f(a)$, we have

$$\mathcal{A} \models_{(A^2)[F,2]} \exists v_3/v_0 v_2 R(v_0, v_1, v_2, v_3),$$

whence

$$\mathcal{A} \models_{A^2} \exists v_2/v_1 \exists v_3/v_0 v_2 R(v_0, v_1, v_2, v_3),$$

and (1) follows. □

Corollary 3.4 Let $\phi(P, Q)$ be the sentence $\forall v_0 \forall v_1 \exists v_2/v_1 \exists v_3/v_0, v_2((\neg P(v_0) \vee \neg v_0 = v_1 \vee (v_2 = v_3 \wedge Q(v_2))) \wedge (\neg Q(v_2) \vee \neg v_2 = v_3 \vee (v_0 = v_1 \wedge P(v_0)))$ Then $\mathcal{A} \models \phi(P, Q)$ if and only if $P^{\mathcal{A}}$ and $Q^{\mathcal{A}}$ are equinumerous.

We shall next give the above semantics in terms of a semantic game. This game seems to be new. There are two players called I and II. A *position* in the game is a pair $p = (\phi, X)$, where ϕ is a formula, called the formula *of* the position, $X \in S(A^W)$, and the free variables of ϕ are among v_n, $n \in W$.

Definition 3.5 Let $\mathcal{A}$ be a structure. The game

$$G(\mathcal{A}, \phi)$$

is defined by the following inductive definition for all sentences ϕ of $\mathcal{IF}$. The type of the move of each player is determined by the position as follows:

(1) The position is (ϕ, X) and $\phi = \phi(v_{i_1}, ..., v_{i_k})$ is atomic or negated atomic. Then the game ends. Player II wins if

$$(\forall f \in X)(\mathcal{A} \models \phi(f(i_1), ..., f(i_k))).$$

Otherwise player I wins.

(2) The position is $(\phi \wedge \psi, X)$. Now player I chooses whether the game continues from position (ϕ, X) or (ψ, X).

(3) The position is $(\phi \vee /v_{i_1} \ldots v_{i_k} \psi, X)$. Now player II chooses $F : X \to 2$ such that F is $\{i_1, ..., i_k\}$-uniform and then player I chooses whether the game continues from position $(\phi, F^{-1}(\{0\}))$ or $(\psi, F^{-1}(\{1\}))$.

(4) The position is $(\forall v_n \phi, X)$. Now the game continues from the position $(\phi, X[A; n])$.

(5) The position is $(\exists/v_{i_1} \ldots v_{i_k} \phi, X)$. Now player II chooses $F : X \to A$ such that F is $\{i_1, ..., i_k\}$-uniform and then the game continues from the position $(\phi, X[F, n])$.

In the beginning, the position is $(\phi, \{\emptyset\})$.

The above game is a game of perfect information: the strategies of both players are allowed to depend on the whole sequence of previous positions. Hence it is *determined* by the Gale-Stewart theorem [2], as a zero-sum game of perfect information. For example, if $\mathcal{A}$ has at least 2 elements, player I has a winning strategy in $G(\mathcal{A}, \forall v_0 \exists v_1/v_0(v_0 = v_1))$. The winning strategy consists of player I doing nothing, player II simply has only losing moves.

Theorem 3.6 $\mathcal{A} \models \phi$ if and only if player II has a winning strategy in $G(\mathcal{A}, \phi)$.

Proof. We use induction to prove the following more general claim. Suppose $X \in S(A^W)$. Then the following conditions are equivalent for every formula ϕ of $\mathcal{IF}$ with free variables among v_n, $n \in W$:

(1) $\mathcal{A} \models_X \phi$

(2) Player II has a winning strategy $G(\mathcal{A}, \phi)$ in position (ϕ, X)

If ϕ is atomic or negated atomic, the claim follows from definitions. Also the case $\phi = \theta \wedge \psi$ is trivial.

Suppose $\phi = \theta \vee /v_{i_1} \ldots v_{i_k} \psi$ and (1) holds. Let $F : X \to 2$ be $\{i_1 \ldots i_n\}$-uniform such that $\mathcal{A} \models_{F^{-1}(\{0\})} \theta$ and $\mathcal{A} \models_{F^{-1}(\{1\})} \psi$. Player II starts $G(\mathcal{A}, \phi)$ in position (ϕ, X) by playing F. No matter how player I wants to continue, player II has a winning strategy by induction hypothesis. Conversely, if (2) holds and the winning strategy of II gives F such that II wins from position $(\phi, F^{-1}(\{0\}))$ and from position $(\phi, F^{-1}(\{1\}))$, and then again the induction hypothesis implies (1).

Suppose $\phi = \exists v_n/v_{i_1} \ldots v_{i_k} \psi$ and (1) holds. Let $F : X \to A$ be $\{i_1 \ldots i_n\}$-uniform such that $\mathcal{A} \models_{X[F,n]} \psi$. By induction hypothesis, player II wins in position $(\psi, X[F, n])$. Conversely, if (2) holds, the winning strategy of II gives a $\{i_1 \ldots i_n\}$-uniform $F : X \to A$ and the induction hypothesis gives from this (1). □

Every sentence of $\mathcal{IF}$ is equivalent to a Σ^1_1-sentence of second order logic, that is, a formula of the form

$$\exists R_1 ... \exists R_n \phi,$$

where $R_1, ..., R_n$ are new predicate symbols and ϕ is first order. Conversely, every Σ_1^1-sentence of second order logic is equivalent to a sentence of $\mathcal{IF}$. A sentence of $\mathcal{IF}$ has a negation if and only if it is first order definable (For IF-logic this was pointed out in [7]).

4 Characterizing definability and elementary equivalence

Let

$$\mathcal{A} \equiv\!>_{\mathcal{IF}} \mathcal{B} \iff \forall \phi \in \mathcal{IF}(\mathcal{A} \models \phi \Rightarrow \mathcal{B} \models \phi)$$

and

$$\mathcal{A} \equiv_{\mathcal{IF}} \mathcal{B} \iff \forall \phi \in \mathcal{IF}(\mathcal{A} \models \phi \Leftrightarrow \mathcal{B} \models \phi).$$

Since $\mathcal{IF}$ contains first order logic and first order logic is closed under negation, $\mathcal{A} \equiv\!>_{\mathcal{IF}} \mathcal{B}$ implies the ordinary elementary equivalence $\mathcal{A} \equiv \mathcal{B}$ of first order logic. Therefore $\mathcal{A} \equiv\!>_{\mathcal{IF}} \mathcal{B}$ implies $\mathcal{A} \cong \mathcal{B}$ and hence $\mathcal{A} \equiv_{\mathcal{IF}} \mathcal{B}$ in the domain of finite models. We point out later that this is not so for infinite models. Note that $\mathcal{A} \equiv_{\mathcal{IF}} \mathcal{B}$ is equivalent to elementary equivalence of $\mathcal{A}$ and $\mathcal{B}$ in the *extended* independence friendly logic $\mathcal{EIF}$. So to make a difference between the ordinary $\mathcal{IF}$ and the extended $\mathcal{EIF}$ on the level of elementary equivalence it is necessary to study the relation $\mathcal{A} \equiv\!>_{\mathcal{IF}} \mathcal{B}$ rather than the stronger $\mathcal{A} \equiv_{\mathcal{IF}} \mathcal{B}$. Corollary 4.9 and Proposition 4.10 below further show that $\equiv\!>_{\mathcal{IF}}$ is the right concept of elementary equivalence for $\mathcal{IF}$.

We now introduce an Ehrenfeucht-Fraïssé game adequate for $\mathcal{IF}$ and use this game to characterize $\equiv\!>_{\mathcal{IF}}$.

Definition 4.1 Let $\mathcal{A}$ and $\mathcal{B}$ be two structures of the same vocabulary. The game EF_n has two players and n moves. The position after move m is a pair (X, Y), where $X \subseteq A^{i_m}$ and $Y \subseteq B^{i_m}$, $i_m \leq m$. In the beginning the position is $(\{\emptyset\}, \{\emptyset\})$. After move $m-1$ the position is (X_{m-1}, Y_{m-1}) and there are the following possibilities for the continuation of the game:

Case 1: Player I chooses a function $F : X_{m-1} \to 2$ and $v_{i_1}, ..., v_{i_k} \subseteq \{v_0, ..., v_{m-2}\}$ such that F is $\{i_1, \ldots, i_k\}$-uniform. Then player II chooses a function $G : Y_{m-1} \to 2$ such that G is $i_1, \ldots, i_k$-uniform. Now player I chooses $i < 2$ and the game continues from position $(F^{-1}(\{i\}), G^{-1}(\{i\}))$.

Case 2: Player I decides that the game should continue from the new position

$$(X_{m-1}[A; m], Y_{m-1}[B; m]).$$

Case 3: Player I chooses a function $F : X_{m-1} \to A$ and $v_{i_1} \ldots v_{i_k} \subseteq \{0, ..., m-2\}$ such that F is $V = \{i_1, \ldots, i_k\}$-uniform. Then player II chooses a function $G : Y_{m-1} \to B$ such that G is V-uniform. Then the game continues from the position $(X_{m-1}[F, m], Y_{m-1}[G, m])$.

After n moves the position (X_n, Y_n) is reached and the game ends. Player II is the winner, if

$$\mathcal{A} \models_{X_n} \phi \Rightarrow \mathcal{B} \models_{Y_n} \phi$$

holds for all atomic and negated atomic formulas ϕ. Otherwise player I wins.

This is also a game of perfect information and the concept of winning strategy is defined as usual. The game is determined by the Gale-Stewart theorem ([2]).

Lemma 4.2 Suppose $\mathcal{A}$ is an infinite model and $\mathcal{B}$ a finite model of the empty vocabulary. Then player I has a winning strategy in EF_4.

Proof. The first move of player I uses Case 3 with $V = \emptyset$: He chooses an element $a \in A$ and the function $F : \{\emptyset\} \to A$ defined by $F(\emptyset) = a$. Suppose II answers with $G : \{\emptyset\} \to B$ mapping $G(\emptyset) = b$. The game continues in position $(\{a\}^1, \{b\}^1)$. Next player I demands that the game continues in position $(\{a\}^1[1], \{b\}^1[1])$. Let f be a one to one mapping of A into $A \setminus \{a\}$. We let player I play now $V = \emptyset$ and $F'(a, x) = f(x)$ as in Case 3. Player II answers with G'. Let $X = \{(a, x, f(x)) : x \in A\}$ and $Y = \{(b, y, G'(b, y)) : y \in B\}$. Let $g(x) = G'(b, x)$. Then g cannot be one to one from B into $B \setminus \{b\}$. Let us first assume $g(c) = b$ for some $c \in B$. Thus $(b, c, b) \in Y$ does not satisfy the negated atomic sentence $\neg(v_0 = v_2)$, while every sequence in X satisfies it. Thus I has won. Let us then assume $g(c) = g(d)$ for some $c \neq d \in B$. In this case player I chooses $V = \{1\}$ and $F''(a, x, f(x)) = x$. (This is like in [1, Example 1.4]). Since f is one to one, F'' is V-uniform. Suppose player II responds with G''. It is clear that I has won. □

Definition 4.3 The set $\mathrm{Fml}^m_{v_{i_1} \ldots v_{i_k}}$ of $\mathcal{IF}$-formulas of rank $\leq m$ in free variables $v_{i_1}, ..., v_{i_k}$ is defined as follows:

- $\mathrm{Fml}^0_{v_{i_1} \ldots v_{i_k}}$ is the set of atomic and negated atomic formulas in variables $v_{i_1}, ..., v_{i_k}$.
- If ϕ and ψ are in $\mathrm{Fml}^m_{v_{i_1} \ldots v_{i_k}}$, so then is $\phi \wedge \psi$.
- If $\phi_0, \phi_1 \in \mathrm{Fml}^m_{v_{i_1} \ldots v_{i_k}}$, then $\phi_0 \oplus \phi_1$ is in $\mathrm{Fml}^m_{v_{i_1} \ldots v_{i_k}}$.
- If $\phi_0, \phi_1 \in \mathrm{Fml}^m_{v_{i_1} \ldots v_{i_k}}$, then $\phi_0 \vee /v_{j_1} \ldots v_{j_l} \phi_1$ is in $\mathrm{Fml}^{m+1}_{v_{i_1} \ldots v_{i_k} v_{j_1} \ldots v_{j_l}}$.
- If ϕ is in $\mathrm{Fml}^m_{v_{i_1} \ldots v_{i_k}}$, then $\forall v_n \phi$ is in $\mathrm{Fml}^{m+1}_{\{v_{i_1}, \ldots, v_{i_k}\} \setminus \{v_n\}}$.
- If ϕ is in $\mathrm{Fml}^m_{v_{i_1} \ldots v_{i_k}}$, then $\exists v_n / v_{j_1} \ldots v_{j_l} \phi$ is in $\mathrm{Fml}^{m+1}_{(\{v_{i_1}, \ldots, v_{i_k}\} \cup \{v_{j_1}, \ldots, v_{j_l}\}) \setminus \{v_n\}}$.

Let us write $\mathcal{A} \equiv\!>^n_{\mathcal{IF}} \mathcal{B}$, if $\mathcal{A} \models \phi$ implies $\mathcal{B} \models \phi$ for all sentences ϕ in $\mathrm{Fml}^n_\emptyset$, and $\mathcal{A} \equiv^n_{\mathcal{IF}} \mathcal{B}$, if $\mathcal{A} \models \phi$ is equivalent to $\mathcal{B} \models \phi$ for all sentences ϕ in $\mathrm{Fml}^n_\emptyset$.

Theorem 4.4 Suppose $\mathcal{A}$ and $\mathcal{B}$ are models of the same vocabulary. Then the following conditions are equivalent:

(1) Player II has a winning strategy in the game $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$.

(2) $\mathcal{A} \equiv\!>^n_{\mathcal{IF}} \mathcal{B}$.

Proof. If $n \in \omega, W \subseteq \omega, f \in A^n$ and $\pi : n \to W$ is a bijection, let $\pi(f) \in A^W$ such that $\pi(f)(i) = f(\pi(i))$ for $i = 0, ..., n-1$. If $X \subseteq A^n$, let πX be the set of $\pi(f)$ with $f \in X$. We prove the equivalence of the following two statements:

$(3)_m$ Player II has a winning strategy in the game $\mathrm{EF}_m(\mathcal{A}, \mathcal{B})$ in position (X, Y), where $X \subseteq A^n$ and $Y \subseteq B^n$.

$(4)_m$ If $\phi = \phi(v_{i_0}, ..., v_{i_{n-1}})$ is a formula in $\mathrm{Fml}^m_{v_{i_0}, \ldots, v_{i_{n-1}}}$, then $\mathcal{A} \models_{\pi X} \phi \Rightarrow \mathcal{B} \models_{\pi Y} \phi$, where $\pi(j) = i_j$ for $j = 0, ..., n-1$.

The proof is by induction on m. The case $m = 0$ is true by construction. Let us then assume $(3)_m \iff (4)_m$ as an induction hypothesis. Assume now $(3)_{m+1}$ and let $\phi = \phi(v_{i_0}, ..., v_{i_{n-1}})$ be a formula in $\mathrm{Fml}^{m+1}_{v_{i_0},...,v_{i_{n-1}}}$ such that

$$\mathcal{A} \models_{\pi X} \phi.$$

Case 1: $\phi = \psi_0 \vee /v_{i_1} \ldots v_{i_k} \psi_1$, where $\psi_0, \psi_1 \in \mathrm{Fml}^m_{v_{j_1},...,v_{j_n}}$. By assumption there is a $\{i_1, \ldots, i_k\}$-uniform $F : \pi X \to 2$ such that for all $i < 2$ $\mathcal{A} \models_{F^{-1}(\{1\})} \psi_i$. We let I play F' and $V = \{i_1, \ldots, i_k\}$, where $F' : X \to 2$ such that $F'(f) = F(\pi(f))$ for $f \in X$. Then II plays a V-uniform $G' : Y \to 2$ and the game continues in some position $(F'^{-1}(\{i\}), G'^{-1}(\{i\}))$ according to the choice of I. We claim that for all $i < 2$

$$\mathcal{B} \models_{\pi G^{-1}(\{i\})} \psi_i.$$

Let us consider $\mathcal{B} \models_{\pi G^{-1}(\{i\})} \psi_i$. We let I demand that the game continues in position $(F'^{-1}(\{i\}), G'^{-1}(\{i\}))$. Let $G : \pi Y \to 2$ such that $G'(g) = G(\pi(g))$ for $g \in Y$. The induction hypothesis and the equation $(\pi X)F^{-1}(\{0\}) = \pi'(F'^{-1}(\{i\}))$ give $\mathcal{B} \models_{(\pi' Y)G^{-1}(\{0\})} \psi_i$. Now $\mathcal{B} \models_{\pi Y} \phi$ follows.

Case 2: $\phi = \forall v_{i_n} \psi$, where $\psi \in \mathrm{Fml}^m_{v_{i_0},...,v_{i_n}}$. By assumption, $\mathcal{A} \models_{(\pi X)[A;i_n]} \psi$. We let now I demand that the game continues in position $(X[A;n], Y[B;n])$. The induction hypothesis and the equation $(\pi X)[A;i_n] = \pi'(X[A;n])$, where π' extends π by $\pi'(n) = i_n$, give $\mathcal{B} \models_{\pi'(Y[B;n])} \psi$. Now $\mathcal{B} \models_{\pi Y} \phi$ follows trivially.

Case 3: $\phi = \exists v_{i_n} / v_{i_1} \ldots v_{i_k} \psi$, where $\psi \in \mathrm{Fml}^{m+1}_{\{v_{j_1},...,v_{j_l}\} \cup \{v_{i_1},...,v_{i_k}\}}$. By assumption there is a $\{i_1, \ldots, i_k\}$-uniform $F : \pi X \to A$ such that $\mathcal{A} \models_{(\pi X)[F,i_n]} \psi$. We let I play F' and $i_1, \ldots, i_k$, where $F' : X \to A$ such that $F'(f) = F(\pi(f))$ for $f \in X$. Then II plays a $\{i_1, \ldots, i_k\}$-uniform $G' : Y \to B$ and the game continues in position $(X[F', n], Y[G', n])$. Let $G : Y \to B$ such that $G'(g) = G(\pi(g))$ for $g \in Y$. The induction hypothesis and the equations $(\pi X)[F, i_n] = \pi'(X[F', n])$ and $(\pi Y)[G, i_n] = \pi'(Y[G', n])$, where π' extends π by $\pi'(n) = i_n$, give $\mathcal{B} \models_{(\pi' Y)[G,i_n]} \psi$. Now $\mathcal{B} \models_{\pi Y} \phi$ follows.

To prove the converse implication, assume $(4)_{m+1}$. To prove $(3)_{m+1}$ we consider the possible moves that player I can make in the position (X, Y).

Case 1: Player I chooses a function $F : X \to 2$ and a set $V \subseteq \{0, ..., n-1\}$ such that F is V-uniform. Let $\phi_j, j < M$ be a complete list (up to equivalence) of formulas in $\mathrm{Fml}^m_{v_{i_0},...,v_{n-1}}$. Let $\{j_1, ..., j_s\} = \{i_t : t \in V\}$. Let $\pi(j) = i_j$ for $j = 0, ..., n-1$ and

$$M^i = \{j < M : \mathcal{A} \models_{\pi(F^{-1}(\{i\}))} \phi_j\}.$$

Let $F' : \pi X \to n$ such that $F'(\pi(f)) = F(f)$ for $f \in X$. Now

$$M^i = \{j < M : \mathcal{A} \models_{F'^{-1}(\{i\})} \phi_j\},$$

whence

$$\mathcal{A} \models_{\pi X} \bigwedge_{j \in M^0} \phi_j \vee \bigwedge_{j \in M^1} \phi_j$$

and hence

$$\mathcal{B} \models_{\pi Y} \bigwedge_{j \in M^0} \phi_j \vee \bigwedge_{j \in M^1} \phi_j.$$

Thus there is a $\{j_1, ..., j_s\}$-uniform $G' : \pi Y \to 2$ such that for all $i < n$

$$\mathcal{B} \models_{G'^{-1}(\{i\})} \bigwedge_{j \in M^0} \phi_j.$$

Now we let player II play a $G : Y \to 2$ such that $G'(\pi f) = G(f)$. The game continues from position $(F^{-1}(\{i\}), G^{-1}(\{i\}))$ for some $i < 2$. Suppose player I wants to continue in position $(F^{-1}(\{i\}), G^{-1}(\{i\}))$. Given that now

$$\mathcal{A} \models_{\pi(F^{-1}(\{i\}))} \phi \Rightarrow \mathcal{B} \models_{\pi(G^{-1}(\{i\}))} \phi$$

for all $\phi \in \mathrm{Fml}^m_{v_{i_1} \dots v_{i_n}}$, the induction hypothesis implies that II has a winning strategy in position $(F^{-1}(\{i\}), G^{-1}(\{i\}))$.

Case 2: Player I decides that the game should continue from the new position $(X[m], Y[m])$. We claim that

$$\mathcal{A} \models_{\pi(X[m])} \phi \Rightarrow \mathcal{B} \models_{\pi(Y[m])} \phi$$

for all $\phi(v_{i_0}, ..., v_{i_n}) \in \mathrm{Fml}^m_{v_{i_0}, \dots, v_{i_n}}$, where $\pi(j) = i_j$ for $j = 0, ..., n$. From this the induction hypothesis would imply that II has a winning strategy in position $(X[m], Y[m])$. So let us assume $\mathcal{A} \models_{\pi(X[m])} \phi$, where $\phi \in \mathrm{Fml}^m$. By definition,

$$\mathcal{A} \models_{\pi' X} \forall v_{i_n} \phi,$$

where π' is the restriction of π to the set $\{0, ..., n-1\}$. Since $\forall v_{i_n} \phi \in \mathrm{Fml}^{m+1}$, $(4)_{m+1}$ gives

$$\mathcal{B} \models_{\pi' Y} \forall v_{i_n} \phi$$

and $\mathcal{B} \models_{\pi(Y[m])} \phi$ follows.

Case 3: Player I chooses a function $F : X \to A$ and a set $V \subseteq \{0, ..., n-1\}$ such that F is V-uniform. Let $\phi_i, i < M$ be a complete list (up to equivalence) of formulas in $\mathrm{Fml}^m_{v_{i_0}, \dots, v_{i_n}}$. Let $\{j_1, ..., j_s\} = \{i_t : t \in V\}$. Let $\pi(j) = i_j$ for $j = 0, ..., n$ and

$$M^0 = \{i < M : \mathcal{A} \models_{\pi(X[F,n])} \phi_i\}.$$

Let $F' : \pi X \to A$ such that $F'(\pi(f)) = F(f)$ for $f \in X$. Now

$$M^0 = \{i < M : \mathcal{A} \models_{(\pi X)[F', i_n]} \phi_i\},$$

whence

$$\mathcal{A} \models_{\pi X} \exists v_{i_n} / v_{j_1} \dots v_{j_s} \bigwedge_{i \in M^0} \phi_i$$

and hence

$$\mathcal{B} \models_{\pi Y} \exists v_{i_n} / v_{j_1} \dots v_{j_s} \bigwedge_{i \in M^0} \phi_i.$$

Thus there is a $\{j_1, ..., j_s\}$-uniform $G' : \pi Y \to B$ such that

$$\mathcal{B} \models_{(\pi Y)[G', i_n]} \bigwedge_{i \in M^0} \phi_i.$$

Now we let player II play a $G : Y \to B$ such that $(\pi Y)[G, i_n] = \pi'(Y[G', n])$. The game continues from position $(X[F, n], Y[G, n])$. Given that now

$$\mathcal{A} \models_{\pi(X[F,n])} \phi \Rightarrow \mathcal{B} \models_{\pi(Y[G,n])} \phi$$

for all $\phi \in \mathrm{Fml}^m_{v_{i_1} \dots v_{i_n}}$, the induction hypothesis implies that II has a winning strategy in position $(X[F, n], Y[G, n])$. $\square$

Corollary 4.5 Suppose $\mathcal{A}$ and $\mathcal{B}$ are models of the same vocabulary. Then the following conditions are equivalent:

(1) $\mathcal{A} \equiv>^n_{\mathcal{IF}} \mathcal{B}$.

(2) For all natural numbers n, player II has a winning strategy in the game $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$.

Corollary 4.6 Suppose $\mathcal{A}$ and $\mathcal{B}$ are models of the same vocabulary. Then the following conditions are equivalent:

(1) $\mathcal{A} \equiv_{\mathcal{IF}} \mathcal{B}$.

(2) For all natural numbers n, player II has a winning strategy both in the game $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$ and in the game $\mathrm{EF}_n(\mathcal{B}, \mathcal{A})$.

The two games $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$ and $\mathrm{EF}_n(\mathcal{B}, \mathcal{A})$ can be put together into one game by simply making the moves of the former symmetric with respect to $\mathcal{A}$ and $\mathcal{B}$. Then player II has a winning strategy in this new game if and only if $\mathcal{A} \equiv^n_{\mathcal{IF}} \mathcal{B}$. Instead of a game we could have used a notion of a back-and-forth sequence.

The Ehrenfeucht-Fraïssé game can be used to prove non-expressibility results for $\mathcal{IF}$, but we do not yet have examples where a more direct proof using compactness, interpolation and Löwenheim-Skolem theorems would not be simpler.

Proposition 4.7 There are countable models $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \equiv>_{\mathcal{IF}} \mathcal{B}$, but $\mathcal{B} \not\equiv_{\mathcal{IF}} \mathcal{A}$.

Proof. Let $\mathcal{A}$ be the standard model of arithmetic. Let $\Phi_n, n \in \omega$ be the list of all Σ^1_1-sentences true in $\mathcal{A}$. Suppose

$$\Phi_n = \exists R^n_1 ... \exists R^n_{k_n} \phi_n.$$

Let $\mathcal{A}^*$ be an expansion of $\mathcal{A}$ in which each ϕ_n is true. Let $\mathcal{B}^*$ be a countable non-standard elementary extension of $\mathcal{A}^*$. Let $\mathcal{B}$ be the reduct of $\mathcal{B}^*$ to the language of arithmetic. By construction, $\mathcal{A} \equiv>_{\mathcal{IF}} \mathcal{B}$. On the other hand, $\mathcal{B} \not\equiv>_{\mathcal{IF}} \mathcal{A}$ as non-wellfoundedness of the integers in $\mathcal{B}$ can be expressed by a Σ^1_1-sentence and hence by a sentence of $\mathcal{IF}$. $\square$

Let us say that a model class K is *closed under the relation* $\equiv>^n_{\mathcal{IF}}$, if $\mathcal{A} \in K$ and $\mathcal{A} \equiv>^n_{\mathcal{IF}} \mathcal{B}$ imply $\mathcal{B} \in K$.

Proposition 4.8 Suppose K is a model class and n is a natural number. Then the following conditions are equivalent:

(1) K is definable in $\mathcal{IF}$ by a sentence in $\mathrm{Fml}_\emptyset^n$.

(2) K is closed under the relation $\equiv\!>^n_{\mathcal{IF}}$.

Proof. Suppose K is the class of models of $\phi \in \mathrm{Fml}_\emptyset^n$. If $\mathcal{A} \models \phi$ and $\mathcal{A} \equiv\!>^n_{\mathcal{IF}} \mathcal{B}$, then by definition, $\mathcal{B} \models \phi$. Conversely, suppose K is closed under $\equiv\!>^n_{\mathcal{IF}}$. Let

$$\phi_{\mathcal{A}} = \bigwedge\{\phi \in \mathrm{Fml}_\emptyset^n : \mathcal{A} \models \phi\},$$

where the conjunction is taken over a finite set which covers all such ϕ up to logical equivalence. Let θ be the disjunction of all $\phi_{\mathcal{A}}$, where $\mathcal{A} \in K$. Again we take the disjunction over a finite set up to logical equivalence. We show that K is the class of models of θ. If $\mathcal{A} \in K$, then $\mathcal{A} \models \phi_{\mathcal{A}}$, whence $\mathcal{A} \models \theta$. On the other hand, suppose $\mathcal{A} \models \phi_{\mathcal{B}}$ for some $\mathcal{B} \in K$. Now $\mathcal{B} \equiv\!>^n_{\mathcal{IF}} \mathcal{A}$, for if $\mathcal{B} \models \phi$ and $\phi \in \mathrm{Fml}_\emptyset^n$, then ϕ is logically equivalent with one of the conjuncts of $\phi_{\mathcal{B}}$, whence $\mathcal{A} \models \phi$. As K is closed under $\equiv\!>^n_{\mathcal{IF}}$, we have $\mathcal{A} \in K$. □

Corollary 4.9 Suppose K is a model class. Then the following conditions are equivalent:

(1) K is definable in $\mathcal{IF}$.

(2) There is a natural number n such that K is closed under the relation $\equiv\!>^n_{\mathcal{IF}}$.

The above corollary gives also a characterization of Σ^1_1-definability in second order logic. No assumptions about cardinalities are involved, so if we restrict to finite models we get a characterization of NP-definability.

Proposition 4.10 Suppose K is a model class. Then the following conditions are equivalent:

(1) K is definable in $\mathcal{EIF}$.

(2) There is a natural number n such that K is closed under the relation $\equiv^n_{\mathcal{IF}}$.

Proof. Suppose (2) holds. Let

$$\phi_{\mathcal{A}} = \bigwedge\{\phi \in \mathrm{Fml}_\emptyset^n : \mathcal{A} \models \phi\} \wedge \bigwedge\{\neg\phi \in \mathrm{Fml}_\emptyset^n : \mathcal{A} \not\models \phi\},$$

where the conjunction is made finite. Let θ be the (finite) disjunction of all $\phi_{\mathcal{A}}$, where $\mathcal{A} \in K$. We show that K is the class of models of θ. If $\mathcal{A} \in K$, then $\mathcal{A} \models \phi_{\mathcal{A}}$, whence $\mathcal{A} \models \theta$. On the other hand, suppose $\mathcal{A} \models \phi_{\mathcal{B}}$ for some $\mathcal{B} \in K$. Now $\mathcal{B} \equiv^n \mathcal{A}$, for if $\mathcal{B} \models \phi$ and $\phi \in \mathrm{Fml}_\emptyset^n$, then ϕ is one of the conjuncts of $\phi_{\mathcal{B}}$, whence $\mathcal{A} \models \phi$. On the other hand, if $\mathcal{B} \not\models \phi$, then $\neg\phi$ is one of the conjuncts of $\phi_{\mathcal{B}}$, whence $\mathcal{A} \not\models \phi$. As K is closed under $\equiv^n$, we have $\mathcal{A} \in K$. □

5 Distributive normal forms

Definition 5.1 Suppose $\mathcal{A}$ is a model, $X \subseteq A^m$ and n is a natural number. We define the *constituent* $\phi^n_{\mathcal{A},X}$ as follows:

$$\begin{array}{rcl}
\phi^0_{\mathcal{A},X} & = & \bigwedge_{\psi} \psi(v_0,...,v_{m-1}) \\
\phi^{n+1}_{\mathcal{A},X} & = & \bigwedge_{F_1,V_1} (\phi^n_{\mathcal{A},F^{-1}(\{0\})} \vee /V_1 \ \ \phi^n_{\mathcal{A},F^{-1}(\{1\})}) \wedge \\
 & & \forall v_m \phi^n_{\mathcal{A},X[A,m]} \ \wedge \\
 & & \bigwedge_{F_2,V_2} \exists v_m/V_2 \ \ \phi^n_{\mathcal{A},X[F_2,m]},
\end{array}$$

where ψ ranges over all atomic and negated atomic formulas $\psi(v_0,...,v_{m-1})$ which every $(a_0,...,a_{n-1}) \in X$ satisfies in $\mathcal{A}$, F_1 and V_1 range over all $F_1 : X \to 2$ and $V_1 \subseteq \{0,...,n-1\}$ for which F_1 is V_1-uniform, and F_2 and V_2 range over all $F_2 : X \to A$ and $V_2 \subseteq \{0,...,n-1\}$ for which F_2 is V_2-uniform. Let

$$C_{n,m}$$

be the (finite) set of all constituents $\phi^n_{\mathcal{A},X}$, where $\mathcal{A}$ is a model and $X \subseteq A^m$.

Note, that $\phi^n_{A,X} \in \mathrm{Fml}^n_{v_0,...v_{m-1}}$ and always $\mathcal{A} \models_X \phi^n_{\mathcal{A},X}$. The proof is an easy induction on n.

Proposition 5.2 Suppose $\mathcal{A}, \mathcal{B}$ are models, $X \subseteq A^m, Y \subseteq B^m$ and n is a natural number. Then the following conditions are equivalent:

(1) $\mathcal{B} \models_Y \phi^n_{\mathcal{A},X}$
(2) Player II has a winning strategy in the game $\mathrm{EF}_n(\mathcal{A},\mathcal{B})$ in position (X, Y)
(3) If $\phi \in \mathrm{Fml}^n_{v_0,...,v_{m-1}}$, then $\mathcal{A} \models_X \phi$ implies $\mathcal{B} \models_Y \phi$.

Proof. The equivalence is essentially contained in the proof of Theorem 4.4.□

Theorem 5.3 (Distributive Normal Form) If $\phi \in \mathrm{Fml}^n_{v_0...v_{m-1}}$, then there are $\phi_0, ..., \phi_k \in C_{n,m}$ such that ϕ and $\phi_0 \oplus ... \oplus \phi_k$ are logically equivalent.

Proof. Let $\{\phi_0, ..., \phi_k\}$ be the finite list of all $\phi^n_{\mathcal{A},X}$ for which $\mathcal{A} \models_X \phi$. To prove the logical equivalence of ϕ and $\phi_0 \oplus ... \oplus \phi_k$, suppose first $\mathcal{B} \models_Y \phi$. Then $\phi^n_{\mathcal{B},Y}$ is one of the ϕ_i, so $\mathcal{B} \models_Y \phi_0 \oplus ... \oplus \phi_k$. Conversely, suppose $\mathcal{B} \models_Y \phi^n_{\mathcal{A},X}$ for some $\mathcal{A}$ and X with $\mathcal{A} \models_X \phi$. By Proposition 5.2, $\mathcal{B} \models_Y \phi$. □

The original constituents of Hintikka for first order logic have a strong maximality property: any two mutually consistent constituents of the same level are logically equivalent. This implies the uniqueness of the Distributive Normal Form. We may ask to what extent our Distributive Normal Form is unique. The following observations seems to suggest that the uniqueness of the Distributive Normal Form is characteristic to first order logic.

Let us call a constituent $\phi^n_{\mathcal{A},X}$ m-**maximal**, if

$$\phi^m_{\mathcal{B},Y} \models \phi^n_{\mathcal{A},X}$$

holds for all $\phi^m_{\mathcal{B},Y}$ which are consistent with $\phi^n_{\mathcal{A},X}$.

Proposition 5.4 Suppose $\mathcal{A}$ is a model, $X \subseteq A^l$ and n is a natural number. Then the following are equivalent:

(1) $\phi^n_{\mathcal{A},X}$ is first order definable.
(2) $\phi^n_{\mathcal{A},X}$ is m-maximal for some m.

Proof. (1)→(2):
Let m be a natural number such that $\neg\phi^n_{\mathcal{A},X}$ is definable in $\mathrm{Fml}^m_{v_{i_1},\ldots,v_{i_l}}$. We show that $\phi^n_{\mathcal{A},X}$ is m-maximal. Suppose therefore, that $\mathcal{D} \models_U \phi^m_{\mathcal{B},Y} \wedge \phi^n_{\mathcal{A},X}$. To prove that $\phi^m_{\mathcal{B},Y} \rightarrow \phi^n_{\mathcal{A},X}$, suppose $\mathcal{C} \models_Z \phi^m_{\mathcal{B},Y}$. If $\mathcal{C} \models_Z \neg\phi^n_{\mathcal{A},X}$, then $\mathcal{C} \models_Z \phi^m_{\mathcal{B},Y}$ and Proposition 5.2 imply $\mathcal{B} \models_Y \neg\phi^n_{\mathcal{A},X}$. Now $\mathcal{D} \models_U \phi^m_{\mathcal{B},Y}$ and Proposition 5.2 imply $\mathcal{D} \models_U \neg\phi^n_{\mathcal{A},X}$, a contradiction.

(2)→(1): We show

$$\mathcal{C} \not\models_Z \phi^n_{\mathcal{A},X} \iff \mathcal{C} \models_Z \bigoplus\{\phi^m_{\mathcal{B},Y} \ : \ \{\phi^m_{\mathcal{B},Y}, \phi^n_{\mathcal{A},X}\} \text{ inconsistent}\}.$$

To prove the implication from right to left, suppose $\mathcal{C} \models_Z \phi^m_{\mathcal{B},Y}$ with $\{\phi^m_{\mathcal{B},Y}, \phi^n_{\mathcal{A},X}\}$ inconsistent. Then, of course, $\mathcal{C} \not\models_Z \phi^n_{\mathcal{A},X}$. For the converse implication, suppose $\mathcal{C} \not\models_Z \phi^n_{\mathcal{A},X}$. Then $\mathcal{C} \models_Z \phi^m_{\mathcal{C},Z}$, so all we have to show is that $\{\phi^m_{\mathcal{C},Z}, \phi^n_{\mathcal{A},X}\}$ is inconsistent. But if $\mathcal{D} \models_U \phi^m_{\mathcal{C},Z} \wedge \phi^n_{\mathcal{A},X}$, then m-maximality implies $\phi^m_{\mathcal{C},Z} \models \phi^n_{\mathcal{A},X}$, which immediately gives $\mathcal{C} \models_Z \phi^n_{\mathcal{A},X}$, a contradiction. □

The following result shows that maximality fails for non-first order constituents in a particularly strong way.

Proposition 5.5 There are models $\mathcal{A}$ and $\mathcal{B}$ such that for all natural numbers n:

(1) $\{\phi^n_{\mathcal{A}}, \phi^n_{\mathcal{B}}\}$ is consistent
(2) $\phi^n_{\mathcal{A}} \not\models \phi^5_{\mathcal{B}}$
(3) $\phi^n_{\mathcal{B}} \not\models \phi^5_{\mathcal{A}}$

Proof. We consider a vocabulary with just one unary predicate symbol P. Let three models for this vocabulary be defined as follows:

$$\begin{array}{rcl} \mathcal{A} & = & (\mathbf{R}, \mathbf{Q}) \\ \mathcal{B} & = & (\mathbf{R}, \mathbf{R} \setminus \mathbf{Q}) \\ \mathcal{C} & = & (\mathbf{Q}, \mathbf{N}), \end{array}$$

Where $\mathbf{R}$, $\mathbf{Q}$ and $\mathbf{N}$ are the sets of real, rational and natural numbers, respectively. A simple downward Löwenheim-Skolem argument shows that $\mathcal{A} \equiv\!>_{\mathcal{IF}} \mathcal{C}$ and $\mathcal{B} \equiv\!>_{\mathcal{IF}} \mathcal{C}$, whence (1) holds. It is easy to write a sentence ψ in $\mathrm{Fml}^5_\emptyset$ which expresses the existence of a one-one function from the whole universe into P. Thus ψ is true in $\mathcal{B}$ but not in $\mathcal{A}$. Hence $\mathcal{B} \not\equiv >^5_{\mathcal{IF}} \mathcal{A}$ and (2) holds. (3) follows by symmetry. $\Box$

In first order logic the distributive normal form has a uniqueness feature: if two disjunctions of non-trivial constituents $\phi_1 \vee ... \vee \phi_n$ and $\phi'_1 \vee ... \vee \phi'_m$ are equivalent, then each ϕ_i is equivalent with some ϕ'_j and vice versa. This is based on the fact that constituents that are mutually consistent are actually equivalent. With the notation of the above proof we have in $\mathcal{IF}$ mutually consistent non-equivalent constituents $\phi^5_{\mathcal{A}}, \phi^5_{\mathcal{B}}$ and $\phi^5_{\mathcal{C}}$ and the non-uniqueness of the Distributive Normal Form of $\phi^5_{\mathcal{C}}$: The sentences $\phi^5_{\mathcal{C}}$, $\phi^5_{\mathcal{C}} \oplus \phi^5_{\mathcal{A}}$ and $\phi^5_{\mathcal{C}} \oplus \phi^5_{\mathcal{B}}$ are all logically equivalent.

It should be noted that first order definability of an $\mathcal{IF}$-formula does not mean that the formula behaves otherwise like a first order formula. A well-known example is

$$\forall v_0 \exists v_1/v_0 (v_0 = v_1).$$

This sentence is logically equivalent to a first order sentence and hence has a negation in $\mathcal{IF}$, but this negation is not the result of a syntactic operation on the original formula. Rather, the negation is the result of a semantic consideration. A more dramatic example is the following: It is easy to write down an $\mathcal{IF}$-sentence $\phi \in \mathrm{Fml}^{200}_{\emptyset}$ of the empty vocabulary which says that the universe has size at most 2^{100}. So the relatively short sentence ϕ is first order definable, but neither ϕ nor its negation has a short first order definition, in fact none of quantifier-rank $\leq 2^{100}$.

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