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On Definability in Dependence Logic

Juha Kontinen · Jouko Väänänen

Abstract We study the expressive power of open formulas of dependence logic introduced in Väänänen [Dependence logic (Vol. 70 of London Mathematical Society Student Texts), 2007]. In particular, we answer a question raised by Wilfrid Hodges: how to characterize the sets of teams definable by means of identity only in dependence logic, or equivalently in independence friendly logic.

Keywords Dependence logic · Independence friendly logic · Team

1 Introduction

We can associate any first order formula $\phi(x_1, \ldots, x_n)$ and any structure $\mathfrak{A}$ of the same vocabulary with the set

$$\{s : \mathfrak{A} \models_s \phi(x_1, \ldots, x_n)\} \tag{1}$$

of assignments $s : \{x_1, \ldots, x_n\} \to A$ that satisfy $\phi(x_1, \ldots, x_n)$ in $\mathfrak{A}$. The sets (1) form the well-studied Boolean algebra of first order definable relations on $\mathfrak{A}$. If $A = \{\texttt{Rose}, \texttt{Gill}, \texttt{Leon}, \texttt{1961}, \texttt{1950}\}$, $R^{\mathfrak{A}} = \{(\texttt{Gill}, \texttt{1961}), (\texttt{Leon}, \texttt{1950})\}$, $S^{\mathfrak{A}} =$

J. Kontinen (✉)
Department of Mathematics and Statistics, University of Helsinki,
P.O. Box 68, 00014 University of Helsinki, Finland
e-mail: juha.kontinen@helsinki.fi

J. Väänänen
ILLC, University of Amsterdam, Amsterdam, The Netherlands

J. Väänänen
Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland

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{(Rose, 1950)} and ϕ is $R(x_0, x_1) \vee S(x_0, x_1)$, then the set (1) would be $\{s_0, s_1, s_2\}$, where

	x_0	x_1
s_0	Gill	1961
s_1	Leon	1950
s_2	Rose	1950

(2)

On the other hand, we can turn the table around, considering (2) as a *database*, and ask what kind of properties does the set $\{s_0, s_1, s_2\}$ as a *whole* have, apart from the property that each $s \in \{s_0, s_1, s_2\}$ *individually* satisfies $\phi(x_0, x_1)$ in $\mathfrak{A}$. This is the approach of dependence logic, introduced in Väänänen (2007), basing on Hodges (1997a).

The basic idea of dependence logic is that certain properties of sets such as $\{s_0, s_1, s_2\}$ cannot be expressed merely in terms of what each individual $s \in \{s_0, s_1, s_2\}$ satisfies. An example of such a property of the database (2) is the fact that the value of x_0 completely determines the value of x_1 but not vice versa. That is, if we know x_0 we know x_1 but not vice versa.

Dependencies are at the heart of numerous questions related to interaction of agents. Typically, when we describe a strategy of a player in a game, we declare on which previous moves any particular move of the player is allowed to depend, and in a deterministic strategy the move is supposed to be completely determined by the moves that it is allowed to depend on. Insider trading is an example of a move which depends on disallowed moves. A border police officer may decide whether we can enter the country depending, not on what kind of food or music we like, but only on whether we have a certain kind of visa or not. A hereditary disease manifests itself as a dependence of a health condition on certain genes in a health record database of a large population. Finally, we may recognize important dependencies between our actions and future states of affairs on Earth. Modern society is full of examples of reliance on recognizing whether a certain kind of dependence is manifested or not. It seems therefore natural to subject the logic of dependence to the scrutiny of exact study.

We give below the exact definition of dependence logic. In harmony with the above discussion on manifestation of dependence in a *set* of assignments rather than in individual assignments, formulas of dependence logic express properties of sets of assignments, not properties of individual assignments. There is an obvious way to consider definability of properties of sets of assignments in second order logic. We establish an exact relationship between those properties of sets of assignments that are definable in dependence logic and those that are definable in a certain specific way in second order logic.

Our main result answers a question of Wilfrid Hodges in Hodges (1997b). The background of this question is the following: The independence friendly (IF) logic, incorporating explicit dependence of quantifiers on each other, was introduced in Hintikka and Sandu (1989) and Hintikka (1996). By the method of Enderton (1970) and Walkoe (1970) it can be seen that every sentence of IF logic has a definition in Σ^1_1, and vice versa. Hodges gave in Hodges (1997a) a compositional semantics for IF logic in terms of what he calls trumps, i.e., sets of assignments to a fixed finite set of variables. He showed in Hodges (1997b) that every formula of IF logic can be represented in an equivalent form in Σ_1^1 with an extra predicate interpreting the trump. Hodges went on to ask about the converse: what sets of subsets of an infinite domain M are expressible as the set of trumps of a formula of the logic IF by means of identity only. We show in this paper that the answer is: exactly those that can be defined in Σ_1^1 with an extra predicate, occurring only negatively, for the trump.

We use the framework of Väänänen (2007) and accordingly refer to dependence logic rather than IF logic. At the end of the paper we state our results also for IF logic.

2 Preliminaries

In this section we define dependence logic ($\mathcal{D}$) and recall some of its properties.

Definition 2.1 (Väänänen 2007) The syntax of $\mathcal{D}$ extends the syntax of FO, defined in terms of $\vee$, $\wedge$, $\neg$, $\exists$ and $\forall$, by new atomic (dependence) formulas of the form

$$=(t_1, \ldots, t_n), \tag{3}$$

where $t_1, \ldots, t_n$ are terms. If L is a vocabulary, we use $\mathcal{D}[L]$ to denote the set of formulas of $\mathcal{D}$ based on L.

The intuitive meaning of the dependence formula (3) is that the value of the term t_n is determined by the values of the terms $t_1, \ldots, t_{n-1}$. As singular cases we have

$$=(),$$

which we take to be universally true, and

$$=(t),$$

which declares that the value of the term t depends on nothing, i.e., is constant.

The set $\mathrm{Fr}(\phi)$ of free variables of a formula $\phi \in \mathcal{D}$ is defined as for first-order logic, except that we have the new case

$$\mathrm{Fr}(=(t_1, \ldots, t_n)) = \mathrm{Var}(t_1) \cup \cdots \cup \mathrm{Var}(t_n),$$

where $\mathrm{Var}(t_i)$ is the set of variables occurring in term t_i. If $\mathrm{Fr}(\phi) = \emptyset$, we call ϕ a sentence.

In order to define the semantics of $\mathcal{D}$, we first need to define the concept of a *team*. Let $\mathfrak{A}$ be a model with domain A. *Assignments* of $\mathfrak{A}$ are finite mappings from variables to A. The value of a term t in an assignment s is denoted by $t^{\mathfrak{A}}\langle s\rangle$. If s is an assignment, x a variable, and $a \in A$, then $s(a/x)$ denotes the assignment which agrees with s everywhere except that it maps x to a.

Let A be a set and $\{x_1, \ldots, x_k\}$ a finite (possibly empty) set of variables. A *team* X of A with domain $\{x_1, \ldots, x_k\}$ is any set of assignments from the variables $\{x_1, \ldots, x_k\}$ into the set A. We denote by $rel(X)$ the k-ary relation of A corresponding to X

$$rel(X) = \{(s(x_1), \ldots, s(x_k)) : s \in X\}.$$

If X is a team of A, and $F\colon X \to A$, we use $X(F/x_n)$ to denote the team $\{s(F(s)/x_n) : s \in X\}$ and $X(A/x_n)$ the team $\{s(a/x_n) : s \in X \text{ and } a \in A\}$.

We are now ready to define the semantics of $\mathcal{D}$. We restrict attention to formulas in negation normal form, i.e., negation is assumed to appear only in front of atomic formulas.

Definition 2.2 (Väänänen 2007) Let $\mathfrak{A}$ be a model and X a team of A. The satisfaction relation $\mathfrak{A} \models_X \varphi$ is defined as follows:

1. $\mathfrak{A} \models_X t_1 = t_2$ iff for all $s \in X$ we have $t_1^{\mathfrak{A}}\langle s\rangle = t_2^{\mathfrak{A}}\langle s\rangle$.
2. $\mathfrak{A} \models_X \neg t_1 = t_2$ iff for all $s \in X$ we have $t_1^{\mathfrak{A}}\langle s\rangle \neq t_2^{\mathfrak{A}}\langle s\rangle$.
3. $\mathfrak{A} \models_X {=}(t_1, \ldots, t_n)$ iff for all $s, s' \in X$ such that $t_1^{\mathfrak{A}}\langle s\rangle = t_1^{\mathfrak{A}}\langle s'\rangle, \ldots, t_{n-1}^{\mathfrak{A}}\langle s\rangle = t_{n-1}^{\mathfrak{A}}\langle s'\rangle$, we have $t_n^{\mathfrak{A}}\langle s\rangle = t_n^{\mathfrak{A}}\langle s'\rangle$.
4. $\mathfrak{A} \models_X \neg {=}(t_1, \ldots, t_n)$ iff $X = \emptyset$.
5. $\mathfrak{A} \models_X R(t_1, \ldots, t_n)$ iff for all $s \in X$ we have $(t_1^{\mathfrak{A}}\langle s\rangle, \ldots, t_n^{\mathfrak{A}}\langle s\rangle) \in R^{\mathfrak{A}}$.
6. $\mathfrak{A} \models_X \neg R(t_1, \ldots, t_n)$ iff for all $s \in X$ we have $(t_1^{\mathfrak{A}}\langle s\rangle, \ldots, t_n^{\mathfrak{A}}\langle s\rangle) \notin R^{\mathfrak{A}}$.
7. $\mathfrak{A} \models_X \psi \wedge \phi$ iff $\mathfrak{A} \models_X \psi$ and $\mathfrak{A} \models_X \phi$.
8. $\mathfrak{A} \models_X \psi \vee \phi$ iff $X = Y \cup Z$ such that $\mathfrak{A} \models_Y \psi$ and $\mathfrak{A} \models_Z \phi$.
9. $\mathfrak{A} \models_X \exists x_n \psi$ iff $\mathfrak{A} \models_{X(F/x_n)} \models \psi$ for some $F\colon X \to A$.
10. $\mathfrak{A} \models_X \forall x_n \psi$ iff $\mathfrak{A} \models_{X(A/x_n)} \psi$.

Above, we assume that the domain of X contains the variables free in φ. Finally, a sentence φ is true in a model $\mathfrak{A}$ if $\mathfrak{A} \models_{\{\emptyset\}} \varphi$.

From Definition 2.2 it follows that many familiar propositional equivalences of connectives do not hold in dependence logic. Example 2.3 below shows that the idempotence of disjunction fails in dependence logic. This can be used to show that the distributivity laws of disjunction and conjunction do not hold in dependence logic either. We refer to Sect. 3.3 of Väänänen (2007) for a detailed exposition on propositional equivalences of connectives in dependence logic.

Example 2.3 Let $\mathfrak{A}$ be a model with $A = \{0, 1, 2\}$. Consider the following team X of A:

	x_0	x_1	x_2
s_0	1	2	2
s_1	2	1	2
s_2	3	1	2

(4)

By Definition 2.2 part 3, $\mathfrak{A} \models_X {=}(x_0, x_1)$. Intuitively this means that the value of x_0 determines in this team the value of x_1: for each value of x_0 there is exactly one value of x_1. On the other hand, we have $\mathfrak{A} \not\models_X {=}(x_1, x_0)$, as the value of x_1 does not determine the value of x_0: for the value 1 of x_1 there are two different values of x_0. Note that by Definition 2.2 part 3, $\mathfrak{A} \models_X {=}(x_2)$. Note also that although $\mathfrak{A} \not\models_X {=}(x_1, x_0)$, we still have by Definition 2.2 part 8, $\mathfrak{A} \models_X {=}(x_1, x_0) \vee {=}(x_1, x_0)$, as we can take $Y = \{s_0, s_1\}$ and $Z = \{s_2\}$.

Example 2.4 Let $\mathfrak{A} = (A, \{(0,0), (1,0), (1,2), (2,2)\})$, where $A = \{0, 1, 2\}$. Consider team $X = \{\emptyset\}$ consisting of the empty assignment only. In this case $Y = X(A/x_0)$ is the team:

	x_0
s_0	0
s_1	1
s_2	2

(5)

Let $F : Y \to A$ be the mapping $F(s_0) = 0$, $F(s_1) = 0$, $F(s_2) = 2$. So $Z = Y(F/x_1)$ is

	x_0	x_1
s_0	0	0
s_1	1	0
s_2	2	2

(6)

By Definition 2.2 part 5, $\mathfrak{A} \models_Z R(x_0, x_1)$. By Definition 2.2 part 9, $\mathfrak{A} \models_Y \exists x_1 R(x_0, x_1)$. By Definition 2.2 part 10, $\mathfrak{A} \models_X \forall x_0 \exists x_1 R(x_0, x_1)$.

Let X be a team with domain $\{x_1, \ldots, x_k\}$ and $V \subseteq \{x_1, \ldots, x_k\}$. Denote by $X \upharpoonright V$ the team $\{s \upharpoonright V : s \in X\}$ with domain V. The following lemma shows that the truth of a formula depends only on the interpretations of the variables occurring free in the formula.

Lemma 2.5 *Suppose* $V \supseteq \mathrm{Fr}(\phi)$. *Then* $\mathfrak{A} \models_X \phi$ *if and only if* $\mathfrak{A} \models_{X \upharpoonright V} \phi$.

Proof See Lemma 3.27 in Väänänen (2007). □

Our goal in this paper is to characterize *definable sets of teams*, i.e., sets of the form

$$\{X : \mathfrak{A} \models_X \phi\}, \tag{7}$$

where $\mathfrak{A}$ is a fixed model, $\phi \in \mathcal{D}$, and X is a team over some fixed domain $\{x_1, \ldots, x_k\}$. In the case of dependence logic we could drop the assumption of teams having a fixed domain in (7) by Lemma 2.5. On the other hand, for IF logic the analogue of Lemma 2.5 does not hold, i.e., the truth of a formula may depend on the interpretations of variables that do not occur in the formula (see, e.g., formula (13)).

For reasons that we discuss in the next section we first attempt to characterize the set (7) in the special case when the vocabulary of $\mathfrak{A}$ is empty. Note that this case is still non-trivial. For example, if the domain of $\mathfrak{A}$ is infinite, the set of ϕ such that $\mathfrak{A} \models \phi$ is non-recursive even if the vocabulary of $\mathfrak{A}$ is empty. It turns out that the characterization can be then directly generalized to the case of $\mathfrak{A}$ having a non-empty vocabulary. The following fact [Fact 11.1 in Hodges (1997a), see Proposition 3.10 in Väänänen (2007)] is very basic:

Proposition 2.6 (Downward closure) *Suppose* $Y \subseteq X$. *Then* $\mathfrak{A} \models_X \varphi$ *implies* $\mathfrak{A} \models_Y \varphi$.

Another basic fact is the result that the expressive power of sentences of $\mathcal{D}$ coincides with that of existential second-order sentences (Σ^1_1):

Theorem 2.7 *For every sentence ϕ of $\mathcal{D}$ there is a sentence Φ of Σ^1_1 such that*

$$\text{For all models } \mathfrak{A}\text{: } \mathfrak{A} \models_{\{\emptyset\}} \phi \iff \mathfrak{A} \models \Phi. \tag{8}$$

Conversely, for every sentence Φ of Σ^1_1 there is a sentence ϕ of $\mathcal{D}$ such that (8) holds.

Proof Using the method of Walkoe (1970) and Enderton (1970) [see Theorems 6.2 and 6.15 in Väänänen (2007)]. □

However, Theorem 2.7 does not—a priori—tell us anything about definable sets of teams. In our main result below (Theorem 4.9) we generalize Theorem 2.7 from sentences to formulas. Since formulas of $\mathcal{D}$ define sets of teams and formulas of Σ^1_1 define sets of assignments, the two concepts cannot be directly compared. To remedy this we compare definability by a formula of $\mathcal{D}$ to definability by a sentence of Σ^1_1 with an extra predicate.

3 Two Examples

The two examples of this section demonstrate the difficulties in characterizing all definable properties of teams. The first example is the analogue of Theorem 3.2 in Cameron and Hodges (2001) [see also Example 7.1 in Hodges (1997a)].

Example 3.1 Let $L = \{R\}$ where R is an $n+k$-ary predicate symbol. Suppose A is a finite set such that $|A|^k \geq 2^{|A|^n}$ and S is a set of teams of A with domain $\{x_1, \ldots, x_n\}$ such that S is closed under subsets. Then there is an L-structure $\mathfrak{A}$ with domain A and a formula $\phi(x_1, \ldots, x_n)$ of dependence logic such that a team X with domain $\{x_1, \ldots, x_n\}$ satisfies ϕ in $\mathfrak{A}$ if and only if $X \in S$. As emphasized in Hodges (1997a) and elaborated further in Cameron and Hodges (2001), this shows that it is very difficult to say anything more about definable properties of teams on arbitrary structures except that they are closed downwards.

The previous example used in an essential way the predicate R. In the next example, we construct formulas defining certain downward closed properties of teams over the empty vocabulary.

Proposition 3.2 *Let $k \in \mathbb{N}$ and let $P(x)$ be a polynomial with positive integer coefficients. Then there is a formula $\varphi(\overline{x}) \in \mathcal{D}[\emptyset]$ such that for all finite sets A and teams X over $\{x_1, \ldots, x_k\}$*

$$A \models_X \varphi \Leftrightarrow |X| \leq P(|A|).$$

Proof Suppose first that $P(x) = c \in \mathbb{N}$. Note that $|X| \leq 1$ can be defined by the formula ψ:

$$=(x_1) \wedge \cdots \wedge =(x_k).$$

Therefore, $|X| \leq c$ can be expressed as

$$\psi \vee \psi \cdots \vee \psi,$$

where the disjunction is taken c times. Suppose then that $P(x) = x^c$. Now the following formula can be used

$$\exists y_1 \cdots \exists y_c \left(\bigwedge_{1 \leq i \leq k} =(y_1, \ldots, y_c, x_i) \right).$$

This formula declares that there is a function from the set X to the set A^c which is one-to-one. Finally, note that $|X| \leq (P_1 + P_2)(|A|)$ can be expressed as $\psi_1 \vee \psi_2$ assuming that ψ_i defines the property $|X| \leq P_i(|A|)$. □

4 Characterizing Definable Properties of Teams

In this section we first characterize the properties of teams definable in dependence logic over the empty vocabulary. We show that, over the empty vocabulary, definable team properties correspond exactly to the downwards monotone classes of Σ^1_1. Then we extend the characterization to non-empty vocabularies.

Definition 4.1 Let $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$ and R a k-ary predicate. We denote by Q_φ the following class of $\{R\}$-structures

$$Q_\varphi = \{(A, rel(X)) \mid A \models_X \varphi\}.$$

Lemma 4.2 *For every formula $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$, the class Q_φ is closed under isomorphisms.*

Since satisfiability is preserved in subteams, the class Q_φ is always monotone downwards. The question we are studying can be formulated as follows.

Question For which downwards monotone classes Q can we find a formula $\varphi \in \mathcal{D}[\emptyset]$ such that $Q = Q_\varphi$.

Denote by $\Sigma^1_1[\{R\}]$ existential second-order sentences of vocabulary $\{R\}$. It is easy to see that Σ^1_1-definability is an upper bound for the solution.

Proposition 4.3 *For every $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$ the class Q_φ is definable in $\Sigma^1_1[\{R\}]$.*

Proof By an analogous translation as in Sect. 4 of Hodges (1997b) [see Theorem 6.2 in Väänänen (2007)], for every $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$, there is a sentence $\psi \in \Sigma^1_1[\{R\}]$ such that for all sets A and teams X over $\{y_1, \ldots, y_k\}$ it holds that

$$A \models_X \varphi \Leftrightarrow (A, rel(X)) \models \psi.$$

□

Corollary 4.4 *Let $k \in \mathbb{N}$. There is no formula $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$ such that for all A and teams X*

$$A \models_X \varphi \Leftrightarrow |X| \text{ is finite.}$$

Proof This follows by Proposition 4.3 and the Compactness Theorem of Σ^1_1. □

Since, e.g., transitivity is not a downward monotone property, the family of classes we are looking for will be a proper subfamily of $\Sigma^1_1[\{R\}]$. We shall next show that there is a syntactic criterion for a $\Sigma^1_1[\{R\}]$ sentence to be monotone downwards.

Definition 4.5 Let R be a k-ary relation symbol and $\varphi \in \Sigma^1_1[\{R\}]$ a sentence. We say that φ is downwards monotone with respect to R if for all A and $B' \subseteq B \subseteq A^n$

$$(A, B) \models \varphi \Rightarrow (A, B') \models \varphi.$$

Definition 4.6 An occurrence of a relation symbol R in a formula φ is called positive (negative) if it is in the scope of an even (odd) number of nested negation symbols.

Proposition 4.7 *A sentence $\varphi \in \Sigma^1_1[\{R\}]$ is downwards monotone with respect to R iff there is $\psi \in \Sigma^1_1[\{R\}]$ such that*

$$\models \varphi \leftrightarrow \psi,$$

and R appears only negatively in ψ.

Proof Assume that $\varphi \in \Sigma^1_1[\{R\}]$ is downwards monotone with respect to R. Let φ^* be the formula acquired by replacing all the occurrences of R in φ by a new predicate variable R'. Using the downwards monotonicity of φ, it is straightforward to verify that

$$\models \varphi \leftrightarrow \exists R'(\varphi^* \wedge \forall \overline{x}(R(\overline{x}) \rightarrow R'(\overline{x}))).$$

Note that, on the right hand side, the predicate R appears only negatively.

For the other direction, we may assume that negation appears in φ only in front of atomic formulas. Now the claim follows by induction on the construction of φ (case $\varphi = \neg R(\bar{t})$ being the only non-trivial one). □

In the following, we shall be using the fact that Σ^1_1 formulas can be transformed to the so-called Skolem Normal Form (Skolem 1920; see Skolem 1970).

Theorem 4.8 (Skolem Normal Form Theorem) *Every Σ^1_1 formula is equivalent to a formula of the form*

$$\exists f_1 \cdots \exists f_n \forall x_1 \cdots \forall x_m \psi,$$

where ψ is a quantifier-free formula.

We are now ready to prove the main result of this paper.

Theorem 4.9 *Let Q be a downwards monotone class of $\{R\}$-models. Then there is a formula $\chi \in \mathcal{D}[\emptyset]$ such that $Q = Q_\chi$ if and only if Q is $\Sigma^1_1[\{R\}]$-definable.*

Proof Note that Proposition 4.3 already gives the other half of the claim. Assume that Q is a downwards monotone and $\Sigma^1_1[\{R\}]$-definable. We need to find a formula $\chi \in \mathcal{D}[\emptyset]$ such that $Q = Q_\chi$. By Theorem 4.8, there is a sentence $\lambda \in \Sigma^1_1[\{R\}]$ of the form

$$\exists f_1 \cdots \exists f_n \forall x_1 \cdots \forall x_m \psi \tag{9}$$

defining Q. We may assume that ψ is in conjunctive normal form and that for all the function symbols appearing in ψ there are unique pairwise distinct variables $z_1, \ldots, z_s$, where $(z_1, \ldots, z_s)$ is a subsequence of $(x_1, \ldots, x_m)$, such that all occurrences of f are of the form $f(z_1, \ldots, z_s)$ [see the proof of Theorem 6.15 in Väänänen (2007) for details]. As in the proof of Proposition 4.7, we then pass on to the equivalent formula

$$\exists R'(\lambda^* \wedge \forall \overline{x}(R(\overline{x}) \rightarrow R'(\overline{x})))$$

and translate it again to Skolem normal form

$$\exists f_1 \cdots \exists f_n \exists f_{n+1} \exists f_{n+2} \forall x_1 \cdots \forall x_{m'} (\psi' \wedge (\neg R(\overline{x}) \vee f_{n+1}(\overline{x}) = f_{n+2}(\overline{x}))),$$

i.e., we replace all subformulas of the form $R'(t_1, \ldots, t_k)$ by the formula $f_{n+1}(t_1, \ldots, t_k) = f_{n+2}(t_1, \ldots, t_k)$ and move the universal quantifiers in front by changing bound variables if necessary. We still need to make sure that all the occurrences of the new function symbols f_{n+1} and f_{n+2} are of the form $f(z_1, \ldots, z_s)$ for some pairwise distinct variables $z_1, \ldots, z_s$ ($(z_1, \ldots, z_s)$ a subsequence of $(x_1, \ldots, x_m)$). This can be achieved by the introduction of new universally quantified first-order variables and existentially quantified function variables. These reductions also impose changes to the quantifier-free part of the formula. More precisely, they can add a new conjunct (a disjunction of identities) to

$$(\psi' \wedge (\neg R(\overline{x}) \vee f_{n+1}(\overline{x}) = f_{n+2}(\overline{x})))$$

or new disjuncts (identity atoms) to all the conjuncts via the equivalence $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$. However, after these reductions, the quantifier-free part of the formula is still in conjunctive normal form and only one of the conjuncts has a literal of the form $\neg R(\overline{x})$. In other words, the predicate R has in total only one occurrence in the formula and it is negative.

Let us now assume that the sentence λ in (9) defining Q satisfies all the conditions required above. The formula $\chi(y_1, \ldots, y_k) \in \mathcal{D}[\emptyset]$ defining Q is now defined as

$$\forall x_1 \cdots \forall x_m \exists x_{m+1} \cdots \exists x_{m+n} (\theta_1 \wedge \theta_2),$$

where θ_1 is the formula

$$\bigwedge_{1\le i\le n} =(z^i_1,\ldots,z^i_{s_i},x_{m+i}),$$

and $(z^i_1,\ldots,z^i_{s_i})$ is the unique tuple of variables to which f_i is applied in ψ. The formula θ_2 is acquired from ψ by first replacing the terms $f_i(z^i_1,\ldots,z^i_{s_i})$ by the corresponding variables x_{m+i} in ψ. Note that our assumptions on the way function terms can occur guarantee that the variable x_{m+i} always denotes the same element as the term $f_i(z^i_1,\ldots,z^i_{s_i})$ in the translation. Finally, we replace the subformula $\neg R(x_1,\ldots,x_k)$ in ψ by the formula

$$\bigvee_{1\le i\le k} y_i \neq x_i. \tag{10}$$

Let $\mathfrak{A}$ be a structure, $R^{\mathfrak{A}}\subseteq A^k$, and $(a_1,\ldots,a_k)\in A^k$. The idea of (10) is that the condition $(a_1,\ldots,a_k)\notin R^{\mathfrak{A}}$ is equivalent with the condition that $R^{\mathfrak{A}}$ can be partitioned into k parts $B_1,\ldots,B_k$ so that for all $(b_1,\ldots,b_k)\in B_i$, it holds that $a_i\neq b_i$.

We shall next show that the translation works as intended, i.e., that for all A and teams X over $\{y_1,\ldots,y_k\}$

$$A\models_X \chi(y_1,\ldots,y_k)\Leftrightarrow (A, rel(X))\models\lambda.$$

Clearly, it suffices to show that for all functions $\overline{f}$ of the appropriate arity

$$A\models_{X^*}\theta_2\Leftrightarrow (A, rel(X),\overline{f})\models \forall x_1\cdots\forall x_m\psi,$$

where

$$X^*=\{s\overline{a}f_1(\overline{a})\cdots f_n(\overline{a})\mid s\in X \text{ and } \overline{a}\in A^m\}$$

and $f_i(\overline{a})$ denotes the result of applying function f_i to the appropriate subsequence of $\overline{a}$ determined by the way $z^i_1,\ldots,z^i_{s_i}$ reside in $x_1,\ldots,x_m$. Recall that ψ is assumed to be in conjunctive normal form

$$\psi=\bigwedge_{1\le j\le e}\bigvee_{1\le i\le r_j}\alpha_{ji}.$$

Hence, the formula θ_2 can be written as

$$\theta_2=\bigwedge_{1\le j\le e}\bigvee_{1\le i\le r_j}\alpha^*_{ji},$$

where α^*_{ji} arises from α_{ji} by replacing the terms $f_i(z^i_1,\ldots,z^i_{s_i})$ by the variables x_{m+i} and $\neg R(x_1,\ldots,x_k)$ by $\bigvee_{1\le i\le k} y_i\neq x_i$.

Let us assume first that the claim holds for all the conjuncts of ψ. Suppose that

$$(A, rel(X), \overline{f}) \models \forall x_1 \cdots \forall x_m \bigwedge_{1 \le j \le e} \bigvee_{1 \le i \le r_j} \alpha_{j_i}.$$

Then, for all j we have that

$$(A, rel(X), \overline{f}) \models \forall x_1 \cdots \forall x_m \bigvee_{1 \le i \le r_j} \alpha_{j_i}.$$

By the assumption, it holds that

$$A \models_{X^*} \bigvee_{1 \le i \le r_j} \alpha^*_{j_i}$$

for all j, and thus

$$A \models_{X^*} \bigwedge_{1 \le j \le e} \bigvee_{1 \le i \le r_j} \alpha^*_{j_i}.$$

The other direction is analogous. Therefore, it suffices to show the claim for disjunctions of atomic formulas. Suppose that $\vee_{1 \le i \le r} \alpha_i$ is a disjunction of atomic formulas in which R appears only negatively. Assume that

$$(A, rel(X), \overline{f}) \models \forall x_1 \cdots \forall x_m \bigvee_{1 \le i \le r} \alpha_i.$$

Then, for each $\overline{a} \in A^m$, some α_i is satisfied. Define a partition $Y_1, \ldots Y_r$ of X^* as follows: $s\overline{a} f_1(\overline{a}) \cdots f_n(\overline{a})$ is put to Y_v iff v is the least index j for which

$$(A, rel(X), \overline{f}) \models \alpha_j(\overline{a}).$$

It is easy to verify that $X^* = \cup_{1 \le i \le r} Y_i$ and that

$$A \models_{Y_i} \alpha^*_i.$$

For the other direction (here we need the assumption that at most one α_j is of the form $\neg R(x_1, \ldots, x_k)$), suppose that

$$A \models_{X^*} \bigvee_{1 \le i \le r} \alpha^*_i. \tag{11}$$

By definition, there is a partition of X^* into sets $Y_1, \ldots, Y_r$ such that

$$A \models_{Y_i} \alpha^*_i.$$

We may assume that α_1 is the formula $\neg R(x_1, \ldots, x_k)$. We next define a new partition of X^* in the following way. For $i = 2, \ldots, r$ we do the following. Starting with the case $i = 2$, we inflate Y_2 to the maximal $W_2 \subseteq X^*$ satisfying

$$A \models_{W_2} \alpha_2^*.$$

In the case $i = l$ we define W_l as the maximal $W' \subseteq X^* \backslash (W_2 \cup W_3 \cdots \cup W_{l-1})$ satisfying

$$A \models_{W'} \alpha_l^*.$$

Finally, we define $W_1 = Y_1 \backslash (W_2 \cup \cdots \cup W_r)$. Since $W_1 \subseteq Y_1$, this new partition also witnesses (11) by the downward closure. If, in the new partition, some tuple $s\overline{a} f_1(\overline{a}) \cdots f_n(\overline{a}) \in W_1$, then we must have

$$s'\overline{a} f_1(\overline{a}) \cdots f_n(\overline{a}) \in W_1$$

for all $s' \in X$. This follows from the maximality of the sets $W_2, \ldots, W_r$ and the fact that the variables $y_1, \ldots, y_k$ do not appear in any of the formulas α_i^* for $i > 1$. Therefore,

$$A \models_{W_1} \bigvee_{1 \le i \le k} y_i \neq x_i$$

implies that

$$(A, rel(X), \overline{f}) \models \neg R(\overline{a})$$

for all $\overline{a} \in A^m$ such that, for some s, we have $s\overline{a} f_1(\overline{a}) \cdots f_n(\overline{a}) \in W_1$.

We may conclude that

$$(A, rel(X), \overline{f}) \models \forall x_1 \cdots \forall x_m \bigvee_{1 \le i \le r} \alpha_i.$$

□

Let us then consider definability over a non-empty vocabulary. Suppose that $\varphi(y_1, \ldots, y_k) \in \mathcal{D}[L]$ and L is non-empty. In this case the formula $\varphi(y_1, \ldots, y_k)$ gives rise to a mapping Q_φ assigning to each L-structure $\mathfrak{A}$ a set of k-ary relations on A. The generalization of Theorem 4.9 to arbitrary vocabularies L can be formulated in terms of global mappings as described above. Instead, we formulate the general theorem below localized to a fixed model:

Theorem 4.10 *Let L be a vocabulary, $\mathfrak{A}$ a L-model and F a family of sets of k-tuples of A which is closed under subsets. Then the following are equivalent:*

1. $F = \{rel(X) : \mathfrak{A} \models_X \psi(y_1, \ldots, y_k)\}$ *for some formula* $\psi(y_1, \ldots, y_k) \in \mathcal{D}[L]$.

2. $F = \{Y : (\mathfrak{A}, Y) \models \phi(R)\}$ *for some sentence* $\phi \in \Sigma^1_1[L \cup \{R\}]$, *in which* R *occurs only negatively.*

Proof The direction from 1 to 2 follows by an analogous translation as in Sect. 4 of Hodges (1997b) [see Theorem 6.2 in Väänänen (2007)] implying that for every $\psi(y_1, \ldots, y_k) \in \mathcal{D}[L]$, there is a sentence $\phi \in \Sigma^1_1[L \cup \{R\}]$ such that for all models $\mathfrak{A}$ and teams X over $\{y_1, \ldots, y_k\}$ it holds that

$$\mathfrak{A} \models_X \psi \Leftrightarrow (\mathfrak{A}, rel(X)) \models \phi.$$

The proof from 2 to 1 is also analogous to the proof in the case $L = \emptyset$. Proposition 4.7 generalizes directly to the case where $\phi \in \Sigma^1_1[L \cup \{R\}]$. By the same arguments as in the proof of Theorem 4.9, we may assume that ϕ is in the special Skolem normal form as before, and that the predicate symbol R has only one negative occurrence in ϕ. The critical part in the proof is the case of a disjunction of atomic formulas of which only one disjunct is allowed to be of the form $\neg R(x_1, \ldots, x_k)$. In the case $L = \emptyset$, the other disjuncts are just identities or their negations. In the general case, the other disjuncts can be arbitrary atomic formulas or their negations from the vocabulary L, and they may involve new terms arising from the function symbols and constants in L. However, the variables $y_1, \ldots, y_k$ do not appear in these formulas. Therefore, exactly the same argument as in the case $L = \emptyset$ still applies. □

5 The Case of IF Logic

In this section we state our results for IF logic. We begin by briefly recalling the syntax and semantics of IF logic.

The syntax of IF logic extends FO by slashed quantifiers of the form $(\exists x/W)$ and $(\forall x/W)$, where W is a finite set of variables. The intuitive meaning, e.g., of a formula $(\exists x/\{y\})\phi$ is that "there exists x, independently of y, such that ϕ". Hodges gave a compositional semantics for IF logic in terms of, what he calls trumps (Hodges 1997a). A trump for a formula $\phi(x_1, \ldots, x_n)$, with $x_1, \ldots, x_n$ free, corresponds to a team with domain $\{x_1, \ldots, x_n\}$. A truth definition similar to Definition 2.2 for IF formulas can be found, e.g., in the Appendix of Cameron and Hodges (2001). Instead of giving the definition here, we discuss its similarities and differences to Definition 2.2.

In IF logic, atomic formulas and connectives $\wedge$, $\vee$, and $\neg$ are treated just like in Definition 2.2. With respect to trumps with a fixed domain $\{x_1, \ldots, x_n\}$, the meaning of a formula of the form $(\exists x/W)\phi$, where $W \subseteq \{x_1, \ldots, x_n\}$, is that "there is an x, depending only on variables other than in W, such that ϕ". This can be expressed in dependence logic as

$$\exists x(=(x_{j_1}, \ldots, x_{j_r}, x) \wedge \phi), \tag{12}$$

where $\{x_{j_1}, \ldots, x_{j_r}\} = \{x_1, \ldots, x_n\} \backslash W$. Note that if we consider trumps over variables $\{x_1, \ldots, x_{n+m}\}$, the variables $x_{n+1}, \ldots, x_{n+m}$ need to be added to the dependence formula in (12). This simple observation actually marks a difference between IF logic and $\mathcal{D}$, since, unlike with $\mathcal{D}$, the truth of an IF-formula may depend on the interpretations of variables that do not occur in the formula. For example, the truth of the formula φ

$$\varphi = \exists x/\{y\}(x = y) \tag{13}$$

in a trump X with domain $\{x, y, z\}$ depends on the values of z in X, although z does not occur in φ. The relationship of IF logic and dependence logic is discussed further in Väänänen (2007).

Suppose that $\varphi(y_1, \ldots, y_k) \in \mathrm{IF}[\emptyset]$ and R is a k-ary predicate. Just like in the case of dependence logic, we denote by Q_φ the class of $\{R\}$-structures $(A, rel(X))$ such that X is a trump with domain $\{y_1, \ldots, y_k\}$ for $\varphi(y_1, \ldots, y_k)$ in A. We are now ready to formulate the analogues of Theorems 4.9 and 4.10 for IF logic.

Theorem 5.1 *Let Q be a downwards monotone class of $\{R\}$-models. Then there is a formula $\varphi(y_1, \ldots, y_k) \in \mathrm{IF}[\emptyset]$ such that $Q = Q_\varphi$ if and only if Q is $\Sigma^1_1[\{R\}]$-definable.*

Proof The implication from left to right is analogous to the case of dependence logic using the downward closure of IF logic observed in Hodges (1997a) and the translation of IF logic into Σ^1_1 defined in Sect. 4 of Hodges (1997b). For the converse, it suffices to show that the defining formula $\chi(y_1, \ldots, y_k)$ in the proof of Theorem 4.9 can be translated into an equivalent formula of IF logic (equivalent with respect to trumps with fixed domain $\{y_1, \ldots, y_k\}$). It is easy to verify that the following formula is as wanted

$$\forall x_1 \cdots \forall x_m (\exists x_{m+1}/W_1) \cdots (\exists x_{m+n}/W_n)\theta_2, \tag{14}$$

where W_i is the set

$$(\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_k\} \cup \{x_{m+j} : 1 \leq j \leq i-1\}) \setminus \{z_{i_1}, \ldots, z_{i_s}\}.$$

The semantics of the (unslashed) universal quantifier of IF-logic in (14) coincides with that of dependence logic. Also, by (12) it is clear that the block of slashed existential quantifiers in (14) is equivalent with the construction

$$\exists x_{m+1} \cdots \exists x_{m+n} (\theta_1 \wedge \cdots).$$

Finally, since the semantics of quantifier-free formulas in IF-logic is defined just like in dependence logic, we are done. □

The IF version of Theorem 4.10 can be proved analogously.

Theorem 5.2 *Let L be a vocabulary, $\mathfrak{A}$ a L-model and F a family of sets of k-tuples of A which is closed under subsets. Then the following are equivalent:*

1. $F = \{rel(X) : X$ *is a trump for* $\psi(y_1, \ldots, y_k)$ *in* $\mathfrak{A}\}$ *for some formula* $\psi(y_1, \ldots, y_k) \in \mathrm{IF}[L]$.
2. $F = \{Y : (\mathfrak{A}, Y) \models \phi(R)\}$ *for some sentence* $\phi \in \Sigma^1_1[L \cup \{R\}]$, *in which R occurs only negatively.*

6 Uniform Definability of a Quantifier

In this section we discuss some consequences of Theorem 4.10 and present an open problem.

Recall that the existential quantifier of $\mathcal{D}$ is defined by

$$\mathfrak{A} \models_X \exists x_n \psi \text{ iff } \mathfrak{A} \models_{X(F/x_n)} \models \psi \quad \text{for some } F \colon X \to A.$$

Denote by $\exists^1$ the following variant of the existential quantifier

$$\mathfrak{A} \models_X \exists^1 x_n \psi \text{ iff there is an } a \in A \text{ such that } \mathfrak{A} \models_{X(a/x_n)} \models \psi.$$

It is easy to see that $\exists^1 x \psi$ can be expressed in a "uniform" way as

$$\exists x (=(x) \wedge \psi).$$

The analogue of $\exists^1$ for the universal quantifier is

$$\mathfrak{A} \models_X \forall^1 x_n \psi \text{ iff for all } a \in A \text{ holds that } \mathfrak{A} \models_{X(a/x_n)} \models \psi.$$

It is an open question whether the quantifier $\forall^1$ can be given a uniform definition in the logic $\mathcal{D}$. It is easy to verify that extending the syntax of $\mathcal{D}$ by $\forall^1$ does not increase the expressive power of $\mathcal{D}$. This follows from the fact that Theorem 6.2 in Väänänen (2007) generalizes to cover also the case of $\forall^1$. More interestingly, Theorem 4.10, and the fact that Proposition 2.6 can be also shown to hold for the extension of $\mathcal{D}$ in terms of $\forall^1$, shows that the quantifier $\forall^1$ does not increase the expressive power of $\mathcal{D}$ with respect to open formulas either. It remains open whether the quantifier $\forall^1$ is "uniformly" definable in the logic $\mathcal{D}$.

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