

# Multiverse Set Theory and Absolutely Undecidable Propositions

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# 1 Introduction

After the incompleteness theorems of Gödel, and especially after Cohen proved the independence of the Continuum Hypothesis from the ZFC axioms, the idea offered itself that there are *absolutely undecidable* propositions in mathematics, propositions that cannot be solved at all, by any means. If that were the case, one could throw doubt on the idea that mathematical propositions have a determined truth-value and that there is a unique well-determined reality of mathematical objects where such propositions are true or false. In this paper we try to give this doubt rational content by formulating a position in the foundations of mathematics which allows for multiple realities, or “parallel universes”.

The phrase “multiple realities”, as well as “parallel universe”, may sound immediately self-contradictory and ill-defined. We try to make sense of it anyway. It helps perhaps to look forward: according to our concept of “multiple realities” a working set theorist will not be able to be sure whether there are multiple realities or just one<sup>1</sup>, and will certainly not be able to talk about individual realities.

The word “reality” is famously not unproblematic in foundations of mathematics, but we are only concerned in this paper with the question whether it makes sense to talk about multiple realities or not, assuming it makes sense to talk about reality at all. We are not concerned with the problem of what “reality” means, apart from the multiplicity question.

Let us perform a thought experiment<sup>2</sup> to the effect that there are two realities, or “parallel universes”, in mathematics,  $V_1$  and  $V_2$ . Suppose we have a sentence  $\varphi$ , perhaps the Continuum Hypothesis itself, that is true in  $V_1$  but false in  $V_2$ . Obviously we would not say that  $\varphi$  is *true*, because it is in fact false in  $V_2$ . Neither would we say that it is false either, because it is true in  $V_1$ . So it is neither true nor false. What about its negation  $\neg\varphi$ ? Since  $\varphi$  is not true, should we not declare  $\neg\varphi$  true? But if  $\neg\varphi$  is true, why is  $\varphi$  not declared false? If negation has lost its meaning, have we lost also faith in the Law of Excluded Middle  $\varphi \vee \neg\varphi$ ? What has happened to the laws of logic in general?

The above thought experiment shows that allowing a divided reality may call for a re-evaluation of the basic logical operations and laws of logic. However, we can keep all the familiar laws of (classical) logic if we decide to call “true” those propositions that hold *both* in  $V_1$  and  $V_2$ , and “false” those propositions that are false in *both*  $V_1$  and  $V_2$ . A disjunction is called “true” if one disjunct is true in  $V_1$  and the other in  $V_2$ . Thus  $\varphi \vee \neg\varphi$  is still true, whatever  $\varphi$ , despite the fact that  $\varphi$  itself is neither true nor false. By developing this approach to the interpretation of logical constants we can make sense of the situation that there are two realities. At the same time we make sense of the situation that some propositions are absolutely undecidable: they are absolutely undecidable because they are true in one reality and false in another.

The reader will undoubtedly ask, is this not just what Gödel proved in his Completeness Theorem: Undecidability of  $\varphi$  by given axioms ZFC means the

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<sup>1</sup>Unless he or she adopts the stronger language of Section 5.

<sup>2</sup>Familiar from the supervaluation theory of truth.

existence of two models  $M_1$  and  $M_2$  of ZFC, one for  $\varphi$  and another for  $\neg\varphi$ . This is indeed the “outside” view about a theory, such as ZFC, familiar already to Skolem [25] and von Neumann [32]. But we are trying to make sense of this from the “inside” of ZFC. A theory like ZFC is a theory of all mathematics; everything is “inside” and we cannot make sense of the “outside” of the universe inside the theory ZFC itself, except in a metamathematical approach. If we formulate  $V_1$  and  $V_2$  inside ZFC in any reasonable way, modeling the fact that they are two “parallel” versions of  $V$ , it is hard to avoid the conclusion that  $V_1 = V_2$ , simply because  $V$  is “everything”. This is why the working set theorist will not be able to recognize whether he or she has one or several universes.

Already von Neumann [32] introduced the concept of an inner model and Gödel made this explicit in his universe  $L$  of constructible sets. If we assume the existence of a  $\sigma$ -complete total measure on the reals, we must conclude  $V \neq L$  (Ulam). Do we not have in this case two universes,  $L$  and  $V$ ? The difference with the above situation with  $V_1$  and  $V_2$  is that in the former case we *know* that  $L$  is not the entire universe, but in the latter case we consider both  $V_1$  and  $V_2$  as being the entire universe, whatever this means.

Our problem is now obvious: we want two universes in order to account for absolute undecidability and at the same time we want to say that both universes are “everything”. We solve this problem by thinking of the domain of set theory as a *multiverse* of parallel universes, and letting variables of set theory range—intuitively—over each parallel universe simultaneously, *as if* the multiverse consisted of a Cartesian product of all of its parallel universes<sup>3</sup>. The axioms of the multiverse are just the usual ZFC axioms and everything that we can say about the multiverse is in harmony with the possibility that there is just one universe<sup>4</sup>. But at the same time the possibility of absolutely undecidable propositions keeps alive the possibility that, in fact, there are several universes. The intuition that this paper is trying to follow is that the parallel universes are more or less close to each other and differ only “at the edges”.

Our *multiverse* consists of a multitude of universes. Truth in the multiverse means truth in each universe separately. The same for falsity. Thus negation does not have the usual meaning of not-true. Still the Law of Excluded Middle, as well as other principles of classical logic, are valid. Absolutely undecidable propositions are true in some universes of the multiverse and false in some others. So an absolutely undecidable proposition is neither true nor false, i.e. it lacks a truth-value. The idea is *not* that every model that the axioms of set theory admit is a universe in the multiverse; that would mean that we could dispense with the multiverse entirely and only talk about the axioms.

We are not admitting<sup>5</sup> the possibility that mathematical propositions do have truth-values but for some of them mathematicians will never be able to figure out what the truth-value is. We are only concerned with the possibility that mathematicians are never able to find the truth-value of some proposition

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<sup>3</sup>But the Cartesian product is just a mental image. We cannot form the Cartesian product because we cannot even isolate the universes from each other.

<sup>4</sup>Until we start using the stronger methods of Section 5.

<sup>5</sup>Not only because the human race may be wiped out tomorrow.

because such a truth-value does not exist.

It is the purpose of this paper to present the multiverse approach to set theory in all detail. In Section 2 we give some background and a review of views on absolute undecidability, of Gödel and others. In Section 3 we present the multiverse. In Section 4 we present elements of first order logic in the multiverse setup. In Section 5 we introduce new methods, based on [27], to get a better understanding of the multiverse.

## 2 Background

John von Neumann wrote in 1925:

Das abzählbar Unendliche als solches ist unanfechtbar: es ist ja nichts weiter als der allgemeine Begriff der positive ganzen Zahl, auf dem die Mathematik beruht und von dem selbst Kronecker und Brouwer zugeben, daß er von “Gott geschaffen” sei. Aber seine Grenzen scheinen sehr verschwommen und ohne anschaulich-inhaltliche Bedeutung zu sein.<sup>6</sup>[32]

An extreme form of the multiverse idea is the claim that there is no more truth in set theory than what the axioms give. Von Neumann writes:

Unter “Menge” wird hier (im Sinne der axiomatischen Methode) nur ein Ding verstanden, von dem man nicht mehr weiß und nicht mehr wissen will, als aus den Postulaten über es folgt.<sup>7</sup>(ibid.)

Von Neumann refers to Skolem and Löwenheim [18] as sources of the non-categoricity of his, or any other set theory. It is worth noting that von Neumann puts so much weight on categoricity. Indeed, if set theory had a categorical axiomatization, the categoricity proof itself, carried out in set theory, would be meaningful. But with non-categoricity everything is lost.<sup>8</sup>

For a time Gödel contemplated the idea that there could be *absolutely undecidable* propositions in mathematics<sup>9</sup>. He wrote in [11, p. 155]:

The consistency of the proposition  $A$  (that every set is constructible) is also of interest in its own right, especially because it is very plausible that with  $A$  one is dealing with an absolutely undecidable proposition, on which set theory bifurcates into two different systems, similar to Euclidean and non-Euclidean geometry.

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<sup>6</sup>“The denumerable infinite as such is beyond dispute; indeed, it is nothing more than the general notion of the positive integer, on which mathematics rests and of which even Kronecker and Brouwer admit that it was ‘created by God’. But its boundaries seem to be quite blurred and to lack intuitive, substantive meaning.” (English translation from [31].)

<sup>7</sup>Here (in the spirit of the axiomatic method) one understands by “set” nothing but an object of which one knows no more and wants to know no more than what follows about it from the postulates.

<sup>8</sup>See however [28].

<sup>9</sup>For more on Gödel’s views on absolute undecidability see [30].

Later Gödel turned against this view:

For if the meanings of the primitive terms of set theory as explained on page 262 and in footnote 14 are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false. ([9, page 260])

[I]t has been suggested that, in case Cantor's continuum problem should turn out to be undecidable from the accepted axioms of set theory, the question of its truth would lose its meaning, exactly as the question of the truth of Euclid's fifth postulate by the proof of the consistency of non-Euclidean geometry became meaningless for the mathematician. I therefore would like to point out that the situation in set theory is very different from that in geometry, both from the mathematical and from the epistemological point of view.[9, page 267]

We can study geometries in set theory, but not the other way around. More importantly, there is no stronger theory in which we would study set theory<sup>10</sup>. Set theory is meant to be the ultimate foundation for all mathematics. If we imagined a mathematical theory of models of set theory  $T$ , we would need a background theory in which this would be possible. If that background theory is a set theory  $T^*$ , we again must ask, is  $T^*$  talking about one universe or a multiverse? A lot of the investigation of set theory since Cohen's result on the Continuum Hypothesis can be seen as a study of models of finite parts of ZFC, but no stronger theory ZFC\* is needed because ZFC can prove the existence of models for any of its finite parts. The goal, following Cohen's result, is not so much to show that reality has many facets but rather to show that the axioms leave many things undecided. Still the fact that so many things are left undecided lends credibility to the idea that this is not *only* because the axioms are too weak but also because they try to describe something which is not unique.

Another sense in which the independence of Euclid's Fifth Postulate is different from the independence of CH was pointed out by Kreisel [15, 1(b)]: The Fifth is undecided even from the second order axioms of geometry, while second order axioms in set theory fix the levels of the cumulative hierarchy (Zermelo [34]) and thereby fix CH. So the independence of CH is in this respect of a weaker kind than the independence of Euclid's Fifth.

Saharon Shelah has emphasized the interest in proving set theoretical results in ZFC alone and has demonstrated the possibilities with his pcf-theory [22]. On the universe of set theory Shelah writes:

I am in my heart a card-carrying Platonist seeing before my eyes the universe of sets, but I cannot discard the independence phenomena.  
[21]

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<sup>10</sup>Apart from class theories such as the Mostowski-Kelley-Morse impredicative class theory. But these do not change the basic questions.

...I do not agree with the pure Platonic view that the interesting problems in set theory can be decided, we just have to discover the additional axiom. My mental picture is that we have many possible set theories, all conforming to ZFC. [23]

### 3 The multiverse of sets

The informal description of the multiverse is very much like an informal description of the universe of sets. So we start with an overview of the one-universe view.

#### 3.1 The one universe case

The so-called *iterative* concept of set<sup>11</sup> became soon entrenched in set theory in the early 20th century. Let us recall the basic idea. Roughly speaking the universe of set theory is, according to the iterative set view, the closure of the urelements<sup>12</sup> (aka individuals) under iterations of the power-set operation and taking unions. The crucial factors are the power-set operation and the length of the iteration. It seems difficult to say what the power-set of an infinite set should be like, apart from being closed under rather obvious operations and containing subsets that are actually definable.

Satisfying the Axiom of Choice in the final universe requires us to add choice sets for sets of non-empty sets, and this is a potential source of variation. Different ways to choose the choice sets may lead to different universes. The one-universe view holds that the choice-functions can be chosen in a canonical way leading to a unique universe. Of course, no actually “selecting” takes place because the whole picture of iterative set is just a helpful image for understanding the axioms.

To make the iterative concept of set even more intuitive the concept of a *stage* was introduced<sup>13</sup>. The concept of stage takes from the concept of iterative set the aspect of iteration: elements of a set are thought to have been formed at stages *prior to* the stage where the set itself is formed. As Shoenfield (ibid.) explains, “prior to” is not meant in a temporal sense but rather in a logical sense, as when we say that one theorem must be proved before another.

The idea of first focusing on the stages suggests itself naturally. If the stages are thought to be (intuitively) well-ordered, one can rely on the strong rigidity of well-orders. When Gödel formed the inner model *HOD* of hereditarily ordinal definable sets he noted:

... in the ordinals there is certainly no element of randomness, and hence neither in sets defined in terms of them. This is particularly

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<sup>11</sup>The iterative concept of set was first suggested by Mirimanoff [19] and made explicit by von Neumann [32]. For a thorough discussion of this concept of set see [4] and [20].

<sup>12</sup>Quickly found unnecessary.

<sup>13</sup>Apparently this explanatory concept is folklore. It features in [24] and again [2, pp. 321-344].

clear if you consider von Neumann's definition of ordinals, because it is not based on any well-ordering relations of sets, which may very well involve some random element.<sup>14</sup>

Unlike ZFC set theory itself, the theory of well-order is decidable<sup>15</sup> and its complete extensions are well understood. This further emphasizes the advantage of taking the concept of a stage as a stepping stone in the understanding of the iterative concept of set. Indeed, Boolos [4] formalizes the concept of *stage* as his *stage theory* and derives the ZFC axioms except the Axiom of Choice from that theory.

Although the universe is, according to the iterative set view, the *minimal* universe closed under the said operations and iterations, there is a commonly held view that the universe should be at the same time *maximal*, for otherwise we face immediately the question, what else is there, outside the universe so to speak.

There is no general agreement about what would be the right criterion of maximality. According to the independence results of Cohen, we can make CH true either by restricting the reals of the universe (to Gödel's  $L$ ), or by adding reals to the universe (by e.g. starting with  $2^{\aleph_0} = \aleph_2$  and collapsing  $\aleph_1$  to  $\aleph_0$ ). This demonstrates the basic problem of finding criteria for maximality.

One avenue to maximality, already emphasized by Gödel<sup>16</sup> is to maximize the length of the iteration by means of Axioms of Infinity, such as the assumption of inaccessible (and larger) so-called large cardinals.

Gödel seems to have strongly favored the iterative concept of set:

As far as sets occur in mathematics (at least in the mathematics of today, including all of Cantor's set theory), they are sets of integers, or of rational numbers (i.e., of pairs of integers), or of real numbers (i.e., of sets of rational numbers), or of functions of real numbers (i.e., of sets of pairs of real numbers), etc. When theorems about all sets (or the existence of sets in general) are asserted, they can always be interpreted without any difficulty to mean that they hold for sets of integers as well as for sets of sets of integers, etc. (respectively, that there either exist sets of integers, or sets of sets of integers, or ... etc., which have the asserted property).<sup>17</sup>

As Gödel says, no contradictions have arisen from this concept:

has never led to any antinomy whatsoever: that is, the perfectly "naive" and uncritical working with this concept of set has so far proved completely self-consistent. (ibid 259)

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<sup>14</sup>See Gödel 1946 ibid.

<sup>15</sup>Mostowski-Tarski 1949. Proved in [6].

<sup>16</sup>His "Remarks before the Princeton Bicentennial Conference on Problems in Mathematics", 1946, pages 150-153 in [10].

<sup>17</sup>[10, p. 258]

Favoring the iterative concept of set does not mean that mathematicians really (should) think that all objects in mathematics are iterative sets. The representation of everything as iterative sets is just a way to find a common ground on which all of mathematics can be understood.

### 3.2 The multiverse

There are at least two sources of possible variation in the cumulative hierarchy. The first is the power-set operation and the second is the length of the iteration. The latter is less interesting from the multiverse point of view. If we adopt a new Axiom of Infinity, such as the existence of a strongly inaccessible cardinal  $\kappa$ , we have immediately two universes,  $V_\kappa$  and  $V$ . However, we should not think of them as “parallel” universes. In fact,  $V_\kappa$  is rather an initial segment of  $V$  and exists as a set in  $V$ . Moreover, truth in  $V_\kappa$  is definable in  $V$ . There is no reason to think of  $V_\kappa$  as a parallel universe to  $V$ , one which we cannot distinguish from  $V$  and one about which we do not know whether it is the same as  $V$  or not. It is rather the opposite. We know that  $V \neq V_\kappa$  (since  $V_\kappa$  is a set) and  $V_\kappa$  satisfies “there are no inaccessible cardinals” (if we chose  $\kappa$  to be the first inaccessible), unlike  $V$ . The fact that ZFC does not decide (if the existence of inaccessible cardinals is consistent) the truth of “there are no inaccessible cardinals” does not mean that we cannot assign a truth-value to this statement. We would simply say that the statement is false because its negation is a new axiom that we have adopted.

The other possible source of variability in the cumulative hierarchy is the power-set operation. Lindström [17] presents a detailed analysis of the problem of the power-set operation. He accepts the power-set of  $\mathbb{N}$ , because he can *visualize*  $\mathcal{P}(\mathbb{N})$  as the sets of infinite branches of the full binary tree  $2^{<\omega}$ . But, says Lindström, we have no way of visualizing the power-set of  $\mathcal{P}(\omega)$ , i.e. the set  $\mathcal{P}(\mathcal{P}(\omega))$ . In other words, Lindström sees no problem with the infinite sets  $\mathbb{N}$  and  $\mathbb{R}$  but the set of all sets of reals he finds problematic, because he does not know how to form a picture of it in his mind. As it happens, it is exactly the set of sets of reals that decides also the problematic  $CH$ <sup>18</sup>.

Lindström emphasizes the role of visualization in making sense of set theory, and arrives at criticizing the power-set operation. A different view of set theory acknowledges that there are unvisualizable sets, such as a well-ordering of the reals, but these “random” sets are necessary for a smooth development of set theory and therefore they are accepted. So sets are roughly divided into two categories: the “simple” sets, studied in descriptive set theory, and the “arbitrary” sets, studied in the more abstract areas of set theory.

Problems related to the power-set operation, and more generally to the “arbitrary” sets, is where the multiverse idea emerges. It may just be the nature of the power-set operation that it eludes uniqueness. In anticipation of this we leave the uniqueness untouched and allow different cumulative hierarchies to

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<sup>18</sup>If ZFC is consistent, there are two models of ZFC which have the same ordinals, cardinals, and reals, but which differ on  $CH$  (folklore, personal communication from Matt Foreman).



emerge, in “parallel”. Note that this does not mean that we abandon the Power-Set Axiom, which only says that whatever subsets of a given set we happen to have, they can be collected together.

If we allow different power-sets to emerge, we should not be able to talk about them explicitly. For example, if we had two power-sets  $\mathcal{P}_1(X)$  and  $\mathcal{P}_2(X)$  for a set  $X$ , we could immediately use the Axiom of Extensionality (which we will assume) to derive  $\mathcal{P}_1(X) = \mathcal{P}_2(X)$ . So the universes are completely hidden from each other.

We cannot name individual universes by any means. Some parallel universes may be distinguishable by their properties. For example, if there are parallel universes satisfying  $V = L$ , we can use the sentence  $V \neq L \vee \varphi$  to say that  $\varphi$  is true in those universes. However, we cannot name individual universes. If we could, we would be able to say what is in that universe and what is not, while at the same time the universes should be “everything”. What is the use of multiverse set theory if we cannot say anything about the individual universes? The point of the multiverse is that it makes it possible to think of the set theoretic reality as a definite well-defined structure and still doubt the uniqueness of the power-set operation, and keep open the possibility of absolute undecidability. Absolute undecidability has in the multiverse then a rational explanation, rather than being left as a sign of vague uncertainty. In Section 5 we extend our framework to allow more access to the internal structure of the multiverse without being able to name the universes.

Let us then finally build the cumulative multiverse. It looks very familiar:

$$\begin{cases} \mathbb{V}_0 & = \emptyset \\ \mathbb{V}_{\alpha+1} & = \mathcal{P}(\mathbb{V}_\alpha) \\ \mathbb{V}_\nu & = \bigcup_{\alpha < \nu} \mathbb{V}_\alpha, \text{ if } \nu \text{ limit.} \\ \mathbb{V} & = \bigcup_\alpha \mathbb{V}_\alpha \end{cases} \quad (1)$$

The ordinals used in the equations (1) are ordinals simultaneously in all the parallel universes. Also the empty set  $\emptyset$  is the empty set in each universe separately. This seems unnecessary because surely there is no variability or randomness in the empty set. Indeed, we have introduced so far no way of seeing whether the empty sets of different universes are all equal or not. Intuitively there is every reason to believe that they are equal. When we come to higher and higher levels  $V_\alpha$  there is less and less reason to believe that the versions of  $V_\alpha$  in the different universes are the same. In Section 5 we will address this issue in detail.

### 3.3 Axioms

The axioms of the multiverse are the usual axioms of ZFC written in the vocabulary  $\{\in\}$  using first order logic, with variables ranging over sets in the multiverse, as if there were only one universe. In a sense, the variables are thought to range simultaneously over all universes This means that we do not

have any symbols for relations such as “being in the same universe”, no names for individual universes, etc. Moreover, we write the axioms in ordinary first order logic. There are no new logical operations arising from the multiverse perspective at this point. In Section 5 we consider the use of extensions of first order logic.

The meaning of the *Axiom of Extensionality*  $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$ . is that two sets are equal if they have the same elements. Since our mental picture of the universe is that it consists of parallel realities, and we think of bound variables as ranging over each universe at the same time, as if the multiverse consisted of a Cartesian product of universes, each set that we consider in the axioms has its “versions” (or “projections”, although there is no such projection map in the language of set theory) in the different universes. Thus the Axiom of Extensionality says that two sets from the multiverse are equal if they have the same elements in each of those parallel universes. Note that the empty sets of the different universes do not have elements but they are not *urelements*, for the Axiom of Extensionality implies that no universe has urelements (apart from the empty set). The meaning of the *Axiom of Pairing*  $\forall x \forall y \exists z \forall u (u \in z \leftrightarrow (u = x \vee u = y))$ . is that from any two sets  $a$  and  $b$  we can form the unordered pair in each universe separately and the result is denoted  $\{a, b\}$ . As to the *Axiom of Union*  $\forall x \exists y \forall z (z \in y \leftrightarrow \exists u (u \in x \wedge z \in u))$ ., the set  $\bigcup a$  has its version in each universe, always the union of the sets in that universe that are elements of the corresponding version of  $a$ . Concerning the *Axiom of Power set*  $\forall x \exists y \forall z (z \in y \leftrightarrow \forall u (u \in z \rightarrow u \in x))$ . as discussed above, even if the set  $a$  is the same in each universe, the power-set is allowed to be different, although the power-sets may also all be equal, as far as the ZFC axioms can tell.

In the *Axiom Schema of Subsets*

$$\forall x \forall u_1 \dots \forall u_n \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge \varphi(z, \vec{u})))$$

we have a formula  $\varphi(z, \vec{y})$  that we use to cut out a subset from a given set  $x$ . The quantifiers of the first order formula  $\varphi(z, \vec{u})$  range over the multiverse elements of  $x$ . Notice that the same definition is used in each universe to cut out a subset from  $x$ . It is the same with the *Axiom Schema of Replacement*

$$\forall x \forall w_1 \dots \forall w_n (\forall u \forall z \forall z' ((u \in x \wedge \varphi(u, z, \vec{w}) \wedge \varphi(u, z', \vec{z})) \rightarrow z = z') \rightarrow \exists y \forall z (z \in y \leftrightarrow \exists u (u \in x \wedge \varphi(u, z, \vec{w}))))).$$

The *Axiom of Infinity*  $\exists x (\exists y (y \in x \wedge \forall z \neg (z \in y)) \wedge \forall y (y \in x \rightarrow \exists z (y \in z \wedge z \in x)))$  says that every universe of the multiverse has an infinite set. This has, of course, nothing to do with the question whether there are infinitely many universes. We have no axiom (yet) which would imply that. The *Axiom of Foundation*  $\forall x \exists y (x \cap y = \emptyset)$  arises in multiverse set theory for the same reason as in one-universe set theory: sets are formed in stages and the stages are well-founded, so every set  $x$  has an element  $y$  which was formed earliest. The “choice” involved in the *Axiom of Choice* is in multiverse set theory spread

over the multiverse. The ideology of multiverse set theory is that “choosing” happens simultaneously in all universes. It is exactly because the “choosing” is problematic even in one universe when there are infinitely many sets to choose from, that we allow many universes, each with its own mode of choosing. The extra complication arising from choosing simultaneously in many universes is simply not part of the setup of multiverse set theory.

## 4 Multiverse logic

Let us now discuss the metamathematics of multiverse set theory, that is, we take the (codes of the) ZFC axioms as a set inside set theory, define what it means for another set to be a multiverse model of the set of ZFC axioms, again inside set theory, and then investigate in set theory this relationship. It makes no difference whether we study the metamathematics of ZFC in one-universe set theory or multiverse set theory, because we identify truth in both cases with truth in all (or in *the one*, if there is only one) universes.

Instead of the concept of a single model with one universe satisfying ZFC we have the concept of a *multiverse model* satisfying ZFC. Let us first define the general concept of a multiverse model in first order logic:

**Definition 1** *Suppose  $L$  is a vocabulary. A multiverse  $L$ -model is a set  $\mathbb{M}$  of  $L$ -structures.*

So our concept of a multiverse model has the same degree of generality as the concept of a model in first order logic. There need not be any connections between the individual models constituting a multiverse model.

**Example 2** *Examples of  $L$ -multiverse structures are*

1. The **empty** multiverse. This is a singular case but permitted by the definition.
2. The **one universe** multiverse  $\{M\}$  for any  $L$ -structure  $M$ . This is the classical one universe structure.
3. The **full** multiverse: Suppose  $T$  is a countable first order  $L$ -theory. The set of all models of  $T$ , with domain  $\mathbb{N}$ , is an  $L$ -multiverse. This is *mutatis mutandis* the set of complete consistent extensions of  $T$ .
4. A **bifurcated** multiverse: Suppose  $T$  is a countable first order  $L$ -theory and  $\varphi$  is an  $L$ -sentence such that  $T \not\vdash \varphi$  and  $T \not\vdash \neg\varphi$ . Let  $M_0 \models T \cup \{\varphi\}$  and  $M_1 \models T \cup \{\neg\varphi\}$ . Then  $\mathbb{M} = \{M_0, M_1\}$  is a multiverse  $L$ -model of  $T$ . When we define truth in a multiverse (see Definition 4), we shall see that  $T$  holds in  $\mathbb{M}$  but neither  $\varphi$  nor  $\neg\varphi$  does.
5. Woodin’s **generic** multiverse of set theory ([33]): Suppose  $M$  is a countable transitive model of ZFC. Let  $\mathbb{M}$  be the smallest set of countable transitive sets such that  $M \in \mathbb{M}$  and such that for all pairs,  $(M_1, M_2)$ , of

countable transitive models of ZFC such that  $M_2$  is a generic extension of  $M_1$ , if either  $M_1 \in \mathbb{M}$  or  $M_2 \in \mathbb{M}$  then both  $M_1$  and  $M_2$  are in  $\mathbb{M}$ . It is a remarkable property of this multiverse model that if we start to build it replacing  $M$  by any  $N \in \mathbb{M}$ , the same  $\mathbb{M}$  results.

6. Steel's **generic** multiverse of set theory (See [26]). Let  $M$  be a transitive model of ZFC, and let  $G$  be  $M$ -generic for  $\text{Col}(\omega, < \text{OR}^M)$ . The worlds of the multiverse  $\mathbb{M}^G$  are all those  $W$  such that  $W[H] = M[G \upharpoonright \alpha]$ , for some  $H$  set generic over  $W$ , and some  $\alpha \in \text{OR}^M$ . Again, if we start to build this replacing  $M$  by any  $N \in \mathbb{M}^G$ , the same  $\mathbb{M}^G$  results.
7. The set of countable computably saturated models of ZFC, with domain  $\mathbb{N}$ , is a multiverse of set theory introduced in [8], where it is shown to satisfy the hypotheses for multiverse models of set theory of [5].

## 4.1 Metamathematics

We proceed now towards the truth definition in multiverse logic. We work inside multiverse set theory, treating formulas and multiverse models as sets on a par with other sets. So a multiverse model  $\mathbb{M}$  exists intuitively at the same time in possibly more than one universe. However, we need not worry about this because the metamathematical study of the multiverse uses only first order logic, that is, we do not use here the new logical operations of Section 5, and everything we say is consistent with there being just one universe. So we can avoid the infinite regress of multiverse, multimultiverse,...etc.

Since a multiverse model is just a set of models, the set-theoretic operations make sense. The *union* of two multiverse  $L$ -models  $\mathbb{M}$  and  $\mathbb{M}'$  is their set-theoretic union, denoted  $\mathbb{M} \cup \mathbb{M}'$ .

**Definition 3** Suppose  $\mathbb{M}$  is an  $L$ -multiverse model and  $M \in \mathbb{M}$ . An assignment into  $M$  is a mapping  $s$  such that  $\text{dom}(s)$  is a set of variables and  $s(x) \in M$  for each  $x \in \text{dom}(s)$ . An assignment in  $\mathbb{M}$  is a mapping  $\mathfrak{s}$  such that  $\text{dom}(\mathfrak{s}) = \mathbb{M}$  and if  $M \in \mathbb{M}$  then  $\mathfrak{s}(M)$ , denoted  $\mathfrak{s}_M$ , is an assignment into  $M$ .

We write  $M \models_{\mathfrak{s}} \varphi$  if, according to the usual textbook definition, the assignment  $s$  satisfies the first order formula  $\varphi$  in the one universe model  $M$ .

As is the custom with multiverses (e.g. [33]), the truth in a multiverse simply means truth in each structure of the multiverse:

**Definition 4** Suppose  $\mathbb{M}$  is an  $L$ -multiverse structure,  $\mathfrak{s}$  an assignment in  $\mathbb{M}$ , and  $\varphi$  a first order  $L$ -formula. We define

$$\mathbb{M} \models_{\mathfrak{s}} \varphi \iff_{\text{def}} \forall M \in \mathbb{M} (M \models_{\mathfrak{s}(M)} \varphi). \quad (2)$$

For sentences  $\varphi$  we ignore  $\mathfrak{s}$ .

An immediate consequence of the definition is that validity in all multiverse models is equivalent to validity in all one universe models.

If we consider truth in a multiverse as an *independent* concept, we can note the following consequence of the definition.  $\mathfrak{s} \upharpoonright \mathbb{M}_0$  means the restriction of the mapping  $\mathfrak{s}$  to the set  $\mathbb{M}_0$ . For the interpretation of the quantifiers we adopt the following notation: If  $F$  is a mapping such that  $\text{dom}(F) = \mathbb{M}$  and  $F(M) \in M$  for each  $M \in \mathbb{M}$ , then  $\mathfrak{s}(F/x)$  denotes the modification of  $\mathfrak{s}$  such that  $\mathfrak{s}(F/x)(M)(x) = F(M)$  and  $\mathfrak{s}(F/x)(M)(y) = \mathfrak{s}(M)(y)$  for  $y \neq x$ . The value of a term  $t$  in a (one-universe) model  $M$  under the assignment  $s$  is denoted by  $t^M \langle s \rangle$ .

**Proposition 5** *Truth in a multiverse has the following properties:*

1.  $\mathbb{M} \models_{\mathfrak{s}} t = t'$  iff  $t^M \langle \mathfrak{s}_M \rangle = t'^M \langle \mathfrak{s}_M \rangle$  for all  $M \in \mathbb{M}$ .
2.  $\mathbb{M} \models_{\mathfrak{s}} \neg t = t'$  iff  $t^M \langle \mathfrak{s}_M \rangle \neq t'^M \langle \mathfrak{s}_M \rangle$  for all  $M \in \mathbb{M}$ .
3.  $\mathbb{M} \models_{\mathfrak{s}} R(t_1, \dots, t_n)$  iff  $(t_1^M \langle \mathfrak{s}_M \rangle, \dots, t_n^M \langle \mathfrak{s}_M \rangle) \in R^M$  for all  $M \in \mathbb{M}$ .
4.  $\mathbb{M} \models_{\mathfrak{s}} \neg R(t_1, \dots, t_n)$  iff  $(t_1^M \langle \mathfrak{s}_M \rangle, \dots, t_n^M \langle \mathfrak{s}_M \rangle) \notin R^M$  for all  $M \in \mathbb{M}$ .
5.  $\mathbb{M} \models_{\mathfrak{s}} \varphi \wedge \psi$  if and only if  $(\mathbb{M} \models_{\mathfrak{s}} \varphi$  and  $\mathbb{M} \models_{\mathfrak{s}} \psi)$ .
6.  $\mathbb{M} \models_{\mathfrak{s}} \varphi \vee \psi$  if and only if there are  $\mathbb{M}_0 \subseteq \mathbb{M}$  and  $\mathbb{M}_1 \subseteq \mathbb{M}$  such that  $\mathbb{M} = \mathbb{M}_0 \cup \mathbb{M}_1$ ,  $\mathbb{M}_0 \models_{\mathfrak{s} \upharpoonright \mathbb{M}_0} \varphi$  and  $\mathbb{M}_1 \models_{\mathfrak{s} \upharpoonright \mathbb{M}_1} \psi$ .
7.  $\mathbb{M} \models_{\mathfrak{s}} \exists x \varphi$  if and only if there is  $F$  such that  $\text{dom}(F) = \mathbb{M}$ ,  $F(M) \in M$  for all  $M \in \mathbb{M}$ , and  $\mathbb{M} \models_{\mathfrak{s}(F/x)} \varphi$ .
8.  $\mathbb{M} \models_{\mathfrak{s}} \forall x \varphi$  if and only if for all  $F$  such that  $\text{dom}(F) = \mathbb{M}$  and  $F(M) \in M$  for all  $M \in \mathbb{M}$ , we have  $\mathbb{M} \models_{\mathfrak{s}(F/x)} \varphi$ .

We could have used the above conditions, instead of Definition 4, to define truth. The “value” of a term  $t$  in a multiverse  $\mathbb{M}$  under the assignment  $\mathfrak{s}$  can be thought of as the mapping  $M \mapsto t^M \langle \mathfrak{s}_M \rangle$ , where  $M \in \mathbb{M}$ .

The intuition behind the property (6) above is the following: For the disjunction  $\varphi \vee \psi$  to hold in the multiverse, every model  $M \in \mathbb{M}$  has to decide of  $\{\varphi, \psi\}$ , which is true in  $M$ . We collect into  $\mathbb{M}_0$  those  $M$  which pick  $\varphi$ , and into  $\mathbb{M}_1$  those that pick  $\psi$ . Since conceivably no  $M$  picks, say,  $\psi$ , we may end up with  $\mathbb{M}_1 = \emptyset$ . This is the reason for allowing the empty multiverse.

For the negation  $\mathbb{M} \models_{\mathfrak{s}} \neg \varphi$  the usual  $\mathbb{M} \not\models_{\mathfrak{s}} \varphi$  will not work, since it is possible in view of Definition 4 that

$$\mathbb{M} \not\models_{\mathfrak{s}} \varphi \text{ and } \mathbb{M} \not\models_{\mathfrak{s}} \neg \varphi.$$

For example, for the generic multiverse of Example 2 (5) we have

$$\mathbb{M} \not\models CH \text{ and } \mathbb{M} \not\models \neg CH.$$

Negated formulas are therefore handled by pushing the negation into the formula by means of the de Morgan laws<sup>19</sup>.

<sup>19</sup> $\neg(\varphi \wedge \psi) \equiv \neg \varphi \vee \neg \psi$ ,  $\neg \exists x \varphi \equiv \forall x \neg \varphi$ , etc.

As validity in all multiverse models is equivalent to validity in all one universe models, a first order sentence is valid in all multiverse models if and only if it has a proof in the ordinary sense of first order logic. Until we extend first order logic with new logical operations (Section 5) it is only metamathematics and intuition that distinguishes multiverse logic from one universe logic. Proving things is the same in each case. The different intuition is not manifested in proofs in any way.

After this introduction to multiverse logic let us return to the metamathematics of multiverse set theory.

We have the first order axioms ZFC and the concept of a multiverse model of them. The generic multiverses are certainly very good examples of multiverse models of ZFC. As was mentioned, a particularly attractive feature of them is their invariance under permutation of the initial model. So if one works inside one of the universes  $M$  of a generic multiverse  $\mathbb{M}$  one can really forget about the particular  $M$ , be oblivious as to which of the many universes in  $\mathbb{M}$  this  $M$  happens to be. Moreover, in the above two generic multiverses one can also express the truth of any given  $\varphi$  in all the universes of the multiverse with a translation  $\varphi^*$ . Thus

$$M \models \varphi^* \iff \mathbb{M} \models \varphi.$$

Steel [26] presents a formal system  $MV$  for multiverse set theory. He has two sorts for variables, set sort and world sort, and axioms which dictate that any multiverse model of  $MV$  is of the form of a generic multiverse. The difference to our multiverse set theory is first of all that we have no variables for worlds, as the worlds are in our system completely hidden, and secondly that Steel has axioms which force the multiverse to be a generic multiverse, while at least for the time being we allow any kind of multiverse, even empty or full. We will say more of the connections to generic multiverse in Section 5.3.

## 4.2 Truth in the multiverse

Having introduced the axioms of multiverse set theory and having taken the first steps in the study of the metamathematics of multiverse ZFC, we return to the basic questions of truth and justification in the multiverse framework.

Propositions of multiverse set theory are of the form

$$\Phi(a_1, \dots, a_n), \tag{3}$$

where  $\Phi(x_1, \dots, x_n)$  is a first order formula<sup>20</sup> and  $a_1, \dots, a_n$  are definable<sup>21</sup> terms. The meaning of (3) is intuitively that  $\Phi(a_1, \dots, a_n)$  is true in the multiverse  $\mathbb{V}$ , which for first order formulas means intuitively simply truth in each universe separately. The bound variables of  $\Phi(x_1, \dots, x_n)$  range intuitively over all the universes of  $\mathbb{V}$  simultaneously. We use the ordinary rules of classical logic

<sup>20</sup>In Section 5 we allow propositions that arise naturally in the multiverse framework but go beyond first order logic.

<sup>21</sup>The concept of a definable term is not itself definable but for each quantifier-rank separately this concept is definable.

to derive truths from the axioms. Even though we do not accept the principle that non-truth of  $\neg\varphi$  implies  $\varphi$ , all the rules of logic are valid. In particular,  $\neg\neg\varphi$  logically implies  $\varphi$ .

The proposition (3) has definable terms  $a_1, \dots, a_n$  and in the spirit of the multiverse framework we think of them as existing simultaneously but independently in each universe. First order logic does not provide the means to force such terms to be equal across the universes. Later in Section 5 we introduce tools to overcome this.

Let us then consider the question under what circumstances are we justified in asserting (3). The gold standard of justification in set theory—also multiverse set theory—is a proof of (3) from the ZFC axioms. A widely accepted addition to ZFC are large cardinal axioms in one form or another. This justification, using large cardinals or not, gives more than is needed, for if we subject ZFC to a metamathematical study, a proof in ZFC justifies truth in *all* multiverses which satisfy the axioms. However, from the point of view of multiverse set theory as a foundation of mathematics there is just one “true” multiverse.

By the laws of classical logic, still valid in the multiverse framework, we accept  $\varphi \vee \neg\varphi$  as true for every  $\varphi$ . How is this justified? In terms of justification by proof the matter is not resolved because  $\varphi \vee \neg\varphi$  is either an axiom or follows from essentially equivalent axioms with a similar need for justification. The justification of the Law of Excluded Middle  $\varphi \vee \neg\varphi$  for first order  $\varphi$  reduces to its justification, which we assume, in the one universe framework, for the meaning of  $\varphi \vee \neg\varphi$  is that every single universe satisfies  $\varphi$  or  $\neg\varphi$ .

By Gödel’s Incompleteness Theorem  $Con(ZFC)$  cannot be decided on the basis of ZFC alone, unless ZFC is inconsistent. From the perspective of the *full* multiverse,  $Con(ZFC)$  is absolutely undecidable. But this is no basis to consider  $Con(ZFC)$  absolutely undecidable in  $\mathbb{V}$ . In fact, the large cardinal axioms decide it. Still, the method by which incompleteness arises in Gödel’s Incompleteness Theorem manifests a kind of absolute undecidability, not of a particular proposition, but of the informal expression “the method of justification is itself consistent”.

## 5 Multiverse and team semantics

The semantics of multiverse logic suggests new logical operations with applications to multiverse set theory.

The basic idea is the following. Even if we think that the universe of set theory may be bifurcated, we need *not* think that every possible complete extension of ZFC is realized in the multiverse. It is very plausible to think that the natural numbers and perhaps even the real numbers are the same in all universes of the multiverse. (In the one universe set theory the whole idea that the natural (or the real) numbers are the same is of course meaningless.)

We may also have doubts about the truth-value of CH but we may be convinced that whether CH holds is independent of whether there are inaccessible cardinals. Again, in the one universe set theory it is meaningless to ask whether

CH is independent of the existence of inaccessible cardinals. In one universe set theory CH is true or false and there are inaccessible cardinals or there are none, but from these facts one cannot derive conclusions about mutual independence. The fact that there are models of ZFC which demonstrate the independence of ZFC from large cardinals is of no help because those models are just *some* models, not necessarily models in *our* multiverse.

There are several ways one can “homogenize” the multiverse, distinguish it from the trivializing full multiverse of all possible models of ZFC, and bring it closer to the world of one universe. One approach is to have variables for worlds, as Steel (this volume) does, and specify axioms which dictate that all the worlds are related to each other by forcing. We choose another route, that of extending the logical power of first order logic by new logical operations, based on *team semantics*.

Team semantics<sup>22</sup> is a variation of ordinary Tarski semantics of first order logic. In team semantics the basic concept is not that of an assignment  $s$  satisfying a formula  $\varphi$  in a model  $M$  but a set  $X$  of assignments, called a *team*, satisfying a formula. For first order formulas a team satisfies the formula simply if all the assignments satisfy the formula. What is gained by the introduction of teams? In a team one can manifest dependence and independence phenomena. In the extension of first order logic called *dependence logic* [27] the following new atomic formulas are added to first order logic:

$$=(\vec{y}, \vec{x}),$$

with the so-called Armstrong Axioms [1], governing the intuition “the values of  $\vec{y}$  functionally determine the values of  $\vec{x}$  in the team”. Semantically, the truth of  $=(\vec{y}, \vec{x})$  in a team  $X$  is defined as

$$\forall s, s' \in X (s(\vec{y}) = s'(\vec{y}) \rightarrow s(\vec{x}) = s'(\vec{x})).$$

For example, the sentence

$$\forall z \forall x \exists y (=(y, x) \wedge \neg y = z)$$

says that there is a one-one function from the universe to a proper subset i.e. the universe is infinite. The idea goes back to A. Ehrenfeucht, according to [13]. The idea is the following: In an infinite universe we can pick a one-one function  $f$  into the complement of the element  $z$ . Then we let  $y = f(x)$ . Because  $f$  is one-one, the value of  $y$  completely determines the value of  $x$ . Conversely, if for given  $x$  we can find  $y \neq z$  such that  $x$  is a function  $g(y)$  of  $y$ , then the mapping  $x \mapsto y$  must be one-one, and hence the domain must be infinite. Thus validity in dependence logic is non-axiomatizable. However, the first order logical consequences of dependence logic sentences can be axiomatized and such axioms are given in [14].

There are many other new atomic formulas that suggest themselves in the team semantics context, for example the *independence atom* [12]  $x \perp y$  with the

<sup>22</sup>See [29] for the origin of team semantics and [27] for a detailed study of it.



meaning that  $x$  and  $y$  are “independent” (the values of  $x$  reveal nothing about the values of  $y$ ), and the *inclusion atom* [7]  $x \subseteq y$  with the meaning that values of  $x$  occur also as values of  $y$ ,

The connection between multiverse logic and dependence logic is that the same feature of team semantics which allows us to express the dependence and independence of variables, allows us in the multiverse framework to talk about dependence and independence of formulas, and about absolute undecidability. We accomplish this by thinking of a team of assignments as a multiverse of copies of a single model with single assignments. We now make that shift:

**Definition 6** Multiverse dependence logic, *MD* for short, is the extension of first order logic by the dependence atoms  $=(\vec{y}, \vec{x})$ . (Later we add other new logical operations.)

Inference in MD takes place as in first order logic, using special axioms and rules<sup>23</sup> for  $=(\vec{y}, \vec{x})$ .

The semantics of MD in the multiverse setup is as follows: To account for the fact that there may be many identical copies of the same model in the multiverse, we assume that multiverses are indexed  $\mathbb{M} = \{M_i : i \in I\}$ . The semantics is clearly independent of what indexing is used. Assignments  $\mathfrak{s}$  are defined on  $I$ :  $\mathfrak{s}_i$  is the assignment of the model  $M_i$ .

Adapting the conditions of Proposition 3 to the team semantics yields<sup>24</sup>:

**Definition 7** Suppose  $\mathbb{M} = \{M_i : i \in I\}$  is an  $L$ -multiverse structure,  $\mathfrak{s}$  a team in  $\mathbb{M}$ , and  $\varphi$  a dependence logic  $L$ -formula. We define  $\mathbb{M} \models_{\mathfrak{s}} \varphi$  as follows:

- (i)  $\mathbb{M} \models_{\mathfrak{s}} t = t'$  iff  $t^M \langle \mathfrak{s}_i \rangle = t'^M \langle \mathfrak{s}_i \rangle$  for all  $i \in I$ .
- (ii)  $\mathbb{M} \models_{\mathfrak{s}} \neg t = t'$  iff  $t^M \langle \mathfrak{s}_i \rangle \neq t'^M \langle \mathfrak{s}_i \rangle$  for all  $i \in I$ .
- (iii)  $\mathbb{M} \models_{\mathfrak{s}} R(t_1, \dots, t_n)$  iff  $(t_1^{M_i} \langle \mathfrak{s}_i \rangle, \dots, t_n^{M_i} \langle \mathfrak{s}_i \rangle) \in R^{M_i}$  for all  $i \in I$ .
- (iv)  $\mathbb{M} \models_{\mathfrak{s}} \neg R(t_1, \dots, t_n)$  iff  $(t_1^{M_i} \langle \mathfrak{s}_i \rangle, \dots, t_n^{M_i} \langle \mathfrak{s}_i \rangle) \notin R^{M_i}$  for all  $i \in I$ .
- (v)  $\mathbb{M} \models_{\mathfrak{s}} =(\vec{y}, \vec{x})$  iff  $\forall i, i' \in I (\mathfrak{s}_i(\vec{y}) = \mathfrak{s}_{i'}(\vec{y}) \rightarrow \mathfrak{s}_i(\vec{x}) = \mathfrak{s}_{i'}(\vec{x}))$ .
- (vi)  $\mathbb{M} \models_{\mathfrak{s}} \varphi \wedge \psi$  if and only if  $(\mathbb{M} \models_{\mathfrak{s}} \varphi$  and  $\mathbb{M} \models_{\mathfrak{s}} \psi)$ .
- (vii)  $\mathbb{M} \models_{\mathfrak{s}} \varphi \vee \psi$  if and only if there are  $\mathbb{M}_0 \subseteq \mathbb{M}$  and  $\mathbb{M}_1 \subseteq \mathbb{M}$  such that  $\mathbb{M} = \mathbb{M}_0 \cup \mathbb{M}_1$ ,  $\mathbb{M}_0 \models_{\mathfrak{s}|_{\mathbb{M}_0}} \varphi$  and  $\mathbb{M}_1 \models_{\mathfrak{s}|_{\mathbb{M}_1}} \psi$ .
- (viii)  $\mathbb{M} \models_{\mathfrak{s}} \exists x \varphi$  if and only if there is  $F$  such that  $\text{dom}(F) = I$ ,  $F(i) \in M_i$  for all  $i \in I$ , and  $\mathbb{M} \models_{\mathfrak{s}', \mathfrak{s}'_i} \varphi$ , where for each  $i \in I$ ,  $\mathfrak{s}'_i$  is the assignment  $\{\mathfrak{s}_i(F(i)/x) : i \in I\}$ .

<sup>23</sup>Given in [14].

<sup>24</sup>It makes again no difference whether we investigate the semantics of dependence logic in one-universe set theory or multiverse set theory. The reason is, that although dependence logic goes beyond first order logic and dependence logic truth cannot be reduced to truth in all universes, we do *not* use dependence logic in metatheory. Our metatheory is just first order logic.

(ix)  $\mathbb{M} \models_{\mathfrak{s}} \forall x \varphi$  if and only if  $\mathbb{M}' \models_{\mathfrak{s}'}$   $\varphi$ , where  $\mathbb{M}' = \{M'_i : i \in I'\}$ ,  $I' = \{(M, a) : M \in \mathbb{M}, a \in M\}$ ,  $M'_i = M$ ,  $\mathfrak{s}'_i = \mathfrak{s}_i(a/x)$ , for  $i = (M, a)$ .

For example,  $\mathbb{M} \models_{\mathfrak{s}} \exists x \varphi$  if and only if  $\forall M, M' \in \mathbb{M} (\mathfrak{s}_M(x) = \mathfrak{s}_{M'}(x))$ , that is, if and only if  $x$  has a constant value over all models. Thus  $\mathbb{M} \models \exists x \varphi$  means that all models have a common element.

Compared to the conditions in Proposition 5, there is a difference. Condition (8) of Proposition 5 has become condition (ix) of Definition 7. The intuition behind the new condition (ix) is that the team  $\mathfrak{s}'$  presents all the possible value of  $x$  at once, in one big multiverse  $\mathbb{M}'$ , where every possible value for  $x$  occurs in some universe, while the old (8) goes through different choices  $F(M)$  for  $x$ , one function at a time. So the new (ix) has all the functions  $F$  at once in a big Cartesian product, while the old (8) checked each function  $F$  separately. From the point of view of first order truth there is no difference whether one checks these one at a time or all at once. But from the point of view of dependence logic it is necessary to use the new all-at-once definition (ix). Another change to Proposition 5 is the inclusion of the case (v) for the dependence atom.

## 5.1 Homogenization

We now introduce new logical operations which are multiverse-specific. The goal is to homogenize the multiverse, because the intuition about the multiverse is not that everything that is logically possible should also happen in some universe (which would lead to the *full* multiverse), but that the multiverse is one universe the boundaries of which are “verschwommen” (“blurred”), as von Neumann wrote (see above).

The first new logical operation is the following:

**Definition 8 (Boolean Disjunction)**  $\mathbb{M} \models_{\mathfrak{s}} \varphi \vee_B \psi$  if and only if  $\mathbb{M} \models_{\mathfrak{s}} \varphi$  or  $\mathbb{M} \models_{\mathfrak{s}} \psi$

This may look a bit surprising—do we not already have disjunction in first order logic? However, our prior definition of the semantics of disjunction was different:  $\forall M \in \mathbb{M} (M \models_{\mathfrak{s}} \varphi \vee \psi)$ , which in the case of first order logic means  $\forall M \in \mathbb{M} (M \models_{\mathfrak{s}} \varphi \text{ or } M \models_{\mathfrak{s}} \psi)$ , certainly different from “ $\forall M \in \mathbb{M} (M \models_{\mathfrak{s}} \varphi)$  or  $\forall M \in \mathbb{M} (M \models_{\mathfrak{s}} \psi)$ ”.

Let us write

$$=(\varphi) \text{ for } \varphi \vee_B \neg\varphi.$$

This says that  $\varphi$  has a truth-value i.e. it is either true in the entire multiverse or false in the entire multiverse. If  $\varphi$  says “ $0^\#$  exists”, then the generic multiverse satisfies  $=(\varphi)$ . Note that  $=(\varphi) \vee =(\varphi)$  and  $=(\varphi)$  are in general non-equivalent. The first says that the multiverse can be divided into two parts in both of which  $\varphi$  has a (possibly different) truth-value, while  $=(\varphi)$  says  $\varphi$  has the same truth-value in the entire multiverse. For example, if  $\varphi$  says “ $0^\#$  exists”, and the multiverse is the union of two generic universes, one with  $\varphi$  and another

with  $\neg\varphi$ , then the multiverse satisfies  $=(\varphi)\vee =(\varphi)$  but not  $\varphi$ , and also not  $=(\varphi)\vee_B =(\varphi)$ .

In terms of logical rules the difference between disjunction and Boolean disjunction is:

$$\frac{\begin{array}{c} [\varphi] \\ \vdots \\ \varphi \vee \psi \\ \theta \end{array} \quad \begin{array}{c} [\psi] \\ \vdots \\ \theta \end{array}}{\theta \vee \theta} \quad \frac{\begin{array}{c} [\varphi] \\ \vdots \\ \varphi \vee_B \psi \\ \theta \end{array} \quad \begin{array}{c} [\psi] \\ \vdots \\ \theta \end{array}}{\theta}$$

We have allowed, mainly for technical reasons, the possibility that the multiverse is empty. Now we introduce a logical constant **NE** to remedy this:

**Definition 9 (Non-emptiness)**  $\mathbb{M} \models_s \text{NE} \iff \mathbb{M} \neq \emptyset$ .

This seems to be a most unnecessary addition, because we could have assumed all along that multiverses are non-empty. However, since we did not make this assumption, we have to introduce **NE**. There is no first order formula which would be able to say that the multiverse is non-empty. For example, the empty multiverse satisfies  $\varphi \wedge \neg\varphi$  for all  $\varphi$ .

With **NE** we can introduce a new logical operation. We write

$$\neq(\varphi) \text{ for } (\varphi \wedge \text{NE}) \vee (\neg\varphi \wedge \text{NE}).$$

The meaning of  $\neq(\varphi)$  is that  $\varphi$  is *absolutely undecidable*, true in some models and false in some. So if we add  $\neq(\text{CH})$  to the axioms of set theory we are committed to the idea that the Continuum Hypothesis will never be solved.

What is the evidence we can give to  $\neq(\text{CH})$ ? If we adopted it as an axiom, we should think that it is self-evident. The only thing that is evident, however, is that despite over 130 years of attempts, there is no unanimous opinion among experts about it. Note that  $\neq(\text{CH})$  cannot have first order consequences which would not already follow from ZFC.

On the other hand, adding  $=(\text{CH})$  to the axioms of set theory would indicate conviction that CH has a truth-value, although we do not know—and may never know—what the truth-value is.

We can also say that the universes have the same arithmetic theory of natural numbers by stipulating  $=(\varphi^{\mathbb{N}})$  for every arithmetical sentence  $\varphi$ .

Using the Boolean disjunction and **NE** we can also define for first order  $\varphi$  and  $\psi$

$$\varphi \perp \psi$$

with the intuitive meaning that  $\varphi$  and  $\psi$  are *independent* in the multiverse. More exactly,

$$\mathbb{M} \models_s \varphi \perp \psi \iff \forall M, M' \in \mathbb{M} \exists M'' \in \mathbb{M} [(M'' \models_{s, M''} \varphi \iff M \models_{s, M} \varphi)$$

$$\text{and } (M'' \models_{\mathfrak{s}_{M''}} \psi \iff M' \models_{\mathfrak{s}_{M'}} \psi).$$

This is reminiscent of the formula  $\vec{x} \perp \vec{y}$  in [12] (see also [7]). The idea is that  $M''$  picks the truth value of  $\varphi$  from  $M$  and the truth value of  $\psi$  from  $M'$ . Thus, in the light of the multiverse  $\mathbb{M}$ , knowing the truth value of  $\varphi$  gives no clue as to the truth value of  $\psi$ . It is still possible that  $\mathbb{M}$  is a one universe multiverse but if there are several models then there have to be enough models to satisfy the independence.

**Definition 10** *Suppose  $\mathbb{M} = \{M_i : i \in I\}$  is an  $L$ -multiverse structure,  $\mathfrak{s}$  an assignment in  $\mathbb{M}$ , and  $\varphi(\vec{x})$  a first order  $L$ -formula. We define  $\mathbb{M} \models_{\mathfrak{s}} =(\vec{x} : \varphi(\vec{x}))$  as follows:*

$$\begin{aligned} \mathbb{M} \models_{\mathfrak{s}} =(\vec{x} : \varphi(\vec{x})) &\iff \\ \forall M, N \in \mathbb{M} : \{\vec{a} \in M^n : M \models \varphi(\vec{a})\} &= \{\vec{a} \in N^n : N \models \varphi(\vec{a})\}. \end{aligned}$$

For example,  $=(\varphi)$  is equivalent to  $=(: \varphi)$  for first order sentences  $\varphi$ . Notice, that  $=(x : x = x)$  says that all the models of the multiverse have the same domain, and  $=(\vec{x} : R(\vec{x}))$  says that they have the same interpretation of the relation  $R$ . Moreover,  $=(x : \forall y \neg y \in x)$  says all models in the multiverse have the same  $\emptyset$ . We can continue like this writing axioms which say that all models in the multiverse have the same natural numbers, real numbers, etc.

We include all the above operations in MD, and now turn to the problem how to justify the truth of a sentence of MD in the multiverse. The meaning of a sentence of MD of the form  $=(\vec{x} : \varphi(\vec{x}))$  can be given in terms of the different models constituting the multiverse *only* on the level of intuition, or as a mathematical property when the relationship  $\mathbb{M} \models_{\mathfrak{s}} =(\vec{x} : \varphi(\vec{x}))$  is studied as a mathematical relation. For verification of a proposition of the form  $=(\vec{x} : \varphi(\vec{x}))$  we need axioms and rules of proof.

## 5.2 Axiomatization

We have defined the new logical operations by giving their semantics. This should be complemented by the elimination and introduction rules for such operations. For details of a complete and explicit axiomatization of first order consequence in dependence logic, we refer to [14].

In Section 4.2 we discussed the problem how to justify claims of truth in the multiverse of set theory, and we gave the ZFC axioms, perhaps extended by Axioms of Infinity, as a criterion. The same applies to the problem of justification in multiverse dependence logic. The rules of classical logic can be freely applied to first order formulas but with dependence formulas more care is needed, as, for example  $=(\varphi) \vee =(\varphi)$  and  $=(\varphi)$  need not be equivalent. Reduction to first order logic gives the following result:

**Theorem 11** *The relation  $T \models \varphi$  for recursive dependence logic theories  $T$  and first order sentences  $\varphi$  is effectively axiomatizable in the vocabulary  $\{\in\}$ .*

**Proof.** We essentially reconstruct the multiverse in (one universe) first order logic by means of new predicates. The details of this are straightforward, but since we have also new logical operations, we try to be as detailed as possible. Notice that the new predicates that we introduce are only for the sake of the proof, they are not part of the axiomatization.

Suppose  $U$  is a unary predicate symbol,  $W$  a binary predicate symbol and  $S$  an  $n + 1$ -ary predicate symbol. Intuitively,  $U$  is the set of universes,  $W$  codes the domain of each universe, and  $S$  is an assignment. Let  $\Theta_n(U, W, S)$  be the first order sentence

$$\begin{aligned} & \forall x(U(x) \rightarrow \exists yW(x, y)) \wedge \forall u\forall x(W(u, x) \rightarrow U(x)) \wedge \\ & \forall u\forall x_1 \dots \forall x_n(S(u, x_1, \dots, x_n) \rightarrow (U(u) \wedge \bigwedge_{i=1}^n W(u, x_i))) \wedge \\ & \forall u\forall x_1 \dots \forall x_{2n}((S(u, x_1, \dots, x_n) \wedge S(u, x_{n+1}, \dots, x_{2n})) \rightarrow \bigwedge_{i=1}^n x_i = x_{n+i}) \wedge \\ & \forall u(U(u) \rightarrow \exists x_1 \dots \exists x_n S(u, x_1, \dots, x_n)). \end{aligned}$$

We associate with every dependence logic formula  $\varphi(x_{i_1}, \dots, x_{i_n})$  first order sentences  $\tau_{1,\varphi}(U, W, S)$  and  $\tau_{0,\varphi}(U, W, S)$ , with the intended meaning that  $\varphi$  is true (respectively, false) in the multiverse coded by  $U, W$  and  $S$ , as follows:

**Case 1:** Suppose  $\varphi(x_{i_1}, \dots, x_{i_n})$  is an atomic formula. We let  $\tau_{1,\varphi}(U, W, S)$  be

$$\forall u\forall x_{i_1} \dots \forall x_{i_n}(S(u, x_{i_1}, \dots, x_{i_n}) \rightarrow \varphi^*(u, x_{i_1}, \dots, x_{i_n}))$$

and we let  $\tau_{0,\varphi}(U, W, S)$  be

$$\forall u\forall x_{i_1} \dots \forall x_{i_n}(S(u, x_{i_1}, \dots, x_{i_n}) \rightarrow \neg\varphi^*(u, x_{i_1}, \dots, x_{i_n}))$$

where  $\varphi^*(u, x_{i_1}, \dots, x_{i_n})$  is obtained from  $\varphi(x_{i_1}, \dots, x_{i_n})$  by replacing every predicate symbol  $R(t_1, \dots, t_k)$  by  $R^*(u, t_1, \dots, t_k)$ , every function symbol  $F(t_1, \dots, t_k)$  by  $F^*(u, t_1, \dots, t_k)$ , and every constant symbol  $c$  by  $c^*(u)$ .

**Case 2:** Suppose  $\varphi(x_{i_1}, \dots, x_{i_n})$  is  $(t_1(x_{i_1}, \dots, x_{i_n}), \dots, t_m(x_{i_1}, \dots, x_{i_n}))$ , where  $i_1 < \dots < i_n$ . We define  $\tau_{1,\varphi}(S)$  as follows:

**Subcase 2.1:**  $m = 0$ . We let  $\tau_{1,\varphi}(U, W, S) = \forall u(u = u)$  and  $\tau_{0,\varphi}(U, W, S) = \forall u\neg U(u)$ .

**Subcase 2.2:**  $m = 1$ . Now  $\varphi(x_{i_1}, \dots, x_{i_n})$  is  $(t_1(x_{i_1}, \dots, x_{i_n}))$ . We let  $\tau_{1,\varphi}(U, W, S)$  be the formula

$$\begin{aligned} & \forall u\forall u'\forall x_{i_1} \dots \forall x_{i_n} \forall x_{i_n+1} \dots \forall x_{i_n+n}((S(u, x_{i_1} \dots x_{i_n}) \wedge S(u', x_{i_n+1} \dots x_{i_n+n})) \\ & \rightarrow t_1(x_{i_1}, \dots, x_{i_n}) = t_1(x_{i_n+1}, \dots, x_{i_n+n})) \end{aligned}$$

and we further let  $\tau_{0,\varphi}(U, W, S)$  be the formula<sup>25</sup>  $\forall u\neg U(u)$ .

**Subcase 2.3:** If  $m > 1$  we let  $\tau_{1,\varphi}(U, W, S)$  be the formula

$$\begin{aligned} & \forall u\forall u'\forall x_{i_1} \dots \forall x_{i_n} \forall x_{i_n+1} \dots \forall x_{i_n+n}((S(u, x_{i_1} \dots x_{i_n}) \wedge S(u', x_{i_n+1} \dots x_{i_n+n}) \wedge \\ & t_1(x_{i_1}, \dots, x_{i_n}) = t_1(x_{i_n+1}, \dots, x_{i_n+n}) \wedge \\ & \dots \\ & t_{m-1}(x_{i_1}, \dots, x_{i_n}) = t_{m-1}(x_{i_n+1}, \dots, x_{i_n+n})) \\ & \rightarrow t_m(x_{i_1}, \dots, x_{i_n}) = t_m(x_{i_n+1}, \dots, x_{i_n+n})) \end{aligned}$$

<sup>25</sup>The reason for making the negation of the dependence atom false except in the empty multiverse is that the dependence atom does not have any natural negation.

and we further let  $\tau_{0,\varphi}(U, W, S)$  be the formula  $\forall u \neg U(u)$ .

**Case 3:** Suppose  $\varphi(x_{i_1}, \dots, x_{i_n})$  is

$$=(x_{i_1} \dots x_{i_n} : \psi(x_{i_1}, \dots, x_{i_n})),$$

where  $\psi(x_{i_1}, \dots, x_{i_n})$  is first order. We let  $\tau_{1,\varphi}(U, W, S)$  be the formula

$$\forall u, u' \forall x_{i_1} \dots \forall x_{i_n} \{ (U(u) \wedge U(u')) \rightarrow \\ [ (W(u, x_{i_1}) \wedge \dots \wedge W(u, x_{i_n}) \wedge (\psi^*(u, x_{i_1}, \dots, x_{i_n}))) \leftrightarrow \\ (W(u', x_{i_1}) \wedge \dots \wedge W(u', x_{i_n}) \wedge (\psi^*(u, x_{i_1}, \dots, x_{i_n}))) ] \}$$

where  $\varphi^*(u, x_{i_1}, \dots, x_{i_n})$  is obtained from  $\varphi(x_{i_1}, \dots, x_{i_n})$  by replacing every predicate symbol  $R(t_1, \dots, t_k)$  by  $R^*(u, t_1, \dots, t_k)$ , every function symbol  $F(t_1, \dots, t_k)$  by  $F^*(u, t_1, \dots, t_k)$ , every constant symbol  $c$  by  $c^*(u)$ , and restricting every quantifier to  $W(u, \cdot)$ . We further let  $\tau_{0,\varphi}(U, W, S)$  be the formula  $\forall u \neg U(u)$ .

**Case 4:** Suppose  $\varphi$  is NE. We let  $\tau_{1,\varphi}(U, W, S)$  be the formula  $\exists u U(u)$  and we further let  $\tau_{0,\varphi}(U, W, S)$  be the formula  $\forall u \neg U(u)$ .

**Case 5:** Suppose  $\varphi(x_{i_1}, \dots, x_{i_n})$  is the disjunction

$$\psi(x_{j_1}, \dots, x_{j_p}) \vee \theta(x_{k_1}, \dots, x_{k_q}),$$

where  $\{i_1, \dots, i_n\} = \{j_1, \dots, j_p\} \cup \{k_1, \dots, k_q\}$ . We let the sentence  $\tau_{1,\varphi}(U, W, S)$  be

$$\exists U_1 \exists U_2 \exists W_1 \exists W_2 \exists S_1 \exists S_2 [ \Theta_p(U_1, W_1, S_1) \wedge \Theta_q(U_2, W_2, S_2) \wedge \\ \tau_{1,\psi}(U_1, W_1, S_1) \wedge \tau_{1,\theta}(U_2, W_2, S_2) \wedge \\ \forall u (U(u) \leftrightarrow (U_1(u) \vee U_2(u))) \wedge \\ \forall u \forall x \bigwedge_{i=1}^2 (W_i(u, x) \leftrightarrow (W(u, x) \wedge U_i(u))) \wedge \\ \forall u \forall x_{i_1} \dots \forall x_{i_n} (S(u, x_{i_1}, \dots, x_{i_n}) \rightarrow \\ (S_1(u, x_{j_1}, \dots, x_{j_p}) \vee S_2(u, x_{k_1}, \dots, x_{k_q})) ) ]$$

and we let the sentence  $\tau_{0,\varphi}(U, W, S)$  be

$$\exists S_1 \exists S_2 [ \Theta_p(U, W, S_1) \wedge \Theta_q(U, W, S_2) \wedge \\ \tau_{0,\psi}(U, W, S_1) \wedge \tau_{0,\theta}(U, W, S_2) \\ \forall u \forall x_{i_1} \dots \forall x_{i_n} (S(u, x_{i_1}, \dots, x_{i_n}) \rightarrow \\ (S_1(u, x_{j_1}, \dots, x_{j_p}) \wedge S_2(u, x_{k_1}, \dots, x_{k_q})) ) ].$$

**Case 6:** Conjunction is handled as disjunction.

**Case 7:** Suppose  $\varphi(x_{i_1}, \dots, x_{i_n})$  is the Boolean disjunction

$$\psi(x_{j_1}, \dots, x_{j_p}) \vee_B \theta(x_{k_1}, \dots, x_{k_q}),$$

where  $\{i_1, \dots, i_n\} = \{j_1, \dots, j_p\} \cup \{k_1, \dots, k_q\}$ . We let the sentence  $\tau_{1,\varphi}(U, W, S)$  be

$$\exists S_1 \exists S_2 [ \Theta_p(U, W, S_1) \wedge \Theta_q(U, W, S_2) \wedge \\ (\tau_{0,\psi}(U, W, S_1) \vee \tau_{0,\theta}(U, W, S_2)) \\ \forall u \forall x_{i_1} \dots \forall x_{i_n} (S(u, x_{i_1}, \dots, x_{i_n}) \rightarrow \\ (S_1(u, x_{j_1}, \dots, x_{j_p}) \wedge S_2(u, x_{k_1}, \dots, x_{k_q})) ) ].$$

We further let  $\tau_{0,\varphi}(U, W, S)$  be the formula  $\forall u \neg U(u)$ .

**Case 8:**  $\varphi$  is  $\neg\psi$ .  $\tau_{d,\varphi}(U, W, S)$  is the formula  $\tau_{1-d,\psi}(U, W, S)$ .

**Case 9:** Suppose  $\varphi(x_{i_1}, \dots, x_{i_n})$  is the formula  $\exists x_{i_{n+1}} \psi(x_{i_1}, \dots, x_{i_{n+1}})$ . Then  $\tau_{1,\varphi}(U, W, S_1)$  is the formula

$$\exists S_1(\tau_{1,\psi}(U, W, S_1) \wedge \Theta_{n+1}(U, W, S_1) \wedge \forall u \forall x_{i_1} \dots \forall x_{i_n} (S(u, x_{i_1}, \dots, x_{i_n}) \rightarrow \exists x_{i_{n+1}} (S_1(u, x_{i_1}, \dots, x_{i_{n+1}}))))$$

and  $\tau_{0,\varphi}(U, W, S)$  is the formula

$$\begin{aligned} \exists U_1 \exists W_1 \exists S_1 \exists F [ & \tau_{0,\psi}(U_1, W_1, S_1) \wedge \Theta_{n+1}(U_1, W_1, S_1) \wedge \\ & \forall u \forall x ((U(u) \wedge W(u, x)) \leftrightarrow U_1(F(u, x))) \wedge \\ & \forall u \forall u' \forall x \forall x' (F(u, x) = F(u', x') \rightarrow (u = u' \wedge x = x')) \wedge \\ & \forall u \forall x ((U(u) \wedge W(u, x)) \leftrightarrow U_1(F(u, x))) \wedge \\ & \forall x (U_1(x) \rightarrow \exists u \exists y (x = F(u, y))) \wedge \\ & \forall z \forall u \forall x (S_1(F(u, z), x_1, \dots, x_{n+1}) \leftrightarrow (S(u, x_1, \dots, x_n) \wedge x_{n+1} = z))] \end{aligned}$$

**Case 10:** The universal quantifier is handled as the existential one.

It is now straightforward to prove the equivalence of the following two statements for first order  $\psi$ :

- $T \not\models \psi$
- The first order theory  $\{\tau_{1,\varphi}(U, W, S) : \varphi \in T\} \cup \{\Theta_0(U, W, S)\} \cup \{\exists u U(u)\} \cup \{\forall u (U(u) \rightarrow \neg\psi^*(u))\}$ , where  $\psi^*(u)$  is obtained from  $\psi$  by replacing every predicate symbol  $R(t_1, \dots, t_k)$  by  $R^*(u, t_1, \dots, t_k)$ , every function symbol  $F(t_1, \dots, t_k)$  by  $F^*(u, t_1, \dots, t_k)$ , every constant symbol  $c$  by  $c^*(u)$ , and restricting every quantifier to  $W(u, \cdot)$ , has a model.

So the claim follows from the Completeness Theorem of first order logic.  $\square$

When the above theorem is applied to multiverse set theory we get an axiomatization of first order consequences of our desired theory, for example any of the below:

1.  $ZFC$  : Pure  $ZFC$ .
2.  $ZFC + =(\varphi) + \neq(\psi)$  :  $ZFC$  plus “ $\varphi$  has a truth value but  $\psi$  is absolutely undecidable”.
4.  $ZFC + \varphi \perp \psi$  :  $ZFC$  plus “ $\varphi$  is independent of  $\psi$ ”.
- 5 :  $ZFC + \{=(\varphi^{\mathbb{N}}) : \varphi \text{ number theoretic}\}$  :  $ZFC$  plus “no independence in number theory”

In many individual cases one can show that the first order consequences are the same as first order consequences of  $ZFC$ . In the last case we do not escape

the force of Gödel's Incompleteness Theorem, although it may seem so. If  $\theta$  is the relevant Gödel-sentence, then  $ZFC + \{=(\varphi^{\aleph}) : \varphi \text{ number theoretic}\}$  has two multiverse models, one with  $\theta^{\aleph}$  and another with  $\neg\theta^{\aleph}$ . There is no contradiction with the fact that both satisfy  $=(\theta^{\aleph})$ .

We can further use dependence sentences such as  $=(x : On(x))$  to say in set theory that all the universes have the same ordinals. Furthermore, there is a dependence sentence  $\Theta_{wf}$  which essentially says that the ordinals are non-well-founded. Thus for first order  $\varphi$ :

$$ZFC + =(x : On(x)) \models \Theta_{wf} \vee \varphi$$

if and only if  $\varphi$  is true in all well-founded models of  $ZFC$ . This shows<sup>26</sup> that we cannot hope to axiomatize entire MD.

Finally we may adopt the ultimate uniformization axiom

$$=(x, y : x \in y)$$

which in multiverse set theory says that, after all, there is just one universe.

### 5.3 The generic multiverse

We shall now show that we can capture the generic multiverse of Woodin and Steel with multiverse dependence logic. Notice that truth in their generic multiverse can be even captured by first order logic in the sense that truth of a given sentence  $\varphi$  in the generic multiverse can be expressed as the truth of another sentence  $\varphi^*$  in  $V$ . So multiverse dependence logic is not needed in this case. However, the method by which we characterize the generic multiverse in MD is so general that it applies to any similar situation.

We first recall an interesting result of Laver [16, Theorem 3]. There is a formula  $\varphi(x, y)$  of set theory with the following property. Suppose  $P$  is a forcing notion,  $\delta = |P|$ , and  $G$  is  $P$ -generic over  $V$ . Then in  $V[G]$  the formula  $\varphi(x, y)$  defines the ground model  $V$  with the set  $V_{\delta+1}$  as a parameter, that is, in  $V[G]$  the following holds:

$$V = \{a : \varphi(a, V_{\delta+1})\}.$$

The formula  $\varphi(x, y)$  is by no means trivial, but it can be explicitly written down. Note that although in  $V$  the universe is trivially definable by the formula  $x = x$ , after the forcing by  $P$  the old universe may a priori be completely hidden.

We now define a logical operation GMV (really a new atomic formula) so that in the context of multiverse set theory:

$\mathbb{M} \models \text{GMV}$  if and only if the following are true

- For any universe  $M$  in  $\mathbb{M}$  and any po-set  $P$  in  $M$  there is a generic extension of  $M$  by  $P$  in  $\mathbb{M}$ .

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<sup>26</sup>And there are stronger results, reducing the truth of a  $\Pi_2$ -sentence (in  $V$ ) consequence in MD.



- For any universe  $M$  in  $\mathbb{M}$  and any po-set  $P$  in  $M$ , if  $M$  is a generic extension of  $N$  by  $P$ , then  $N \in \mathbb{M}$
- For any universes  $M, M' \in \mathbb{M}$  there are generic extensions  $M[G]$  and  $M'[G']$  of  $M$  and  $M'$  in  $\mathbb{M}$  such that  $M[G] = M'[G']$ .

The logical operation GMV can be added to MD without losing Theorem 11, but in the case of ZFC we do not get any new first order consequences. The conditions in the definition of GMV are the conditions that characterize Steel's generic multiverse ([26]). So adding GMV to the ZFC axioms means in multiverse set theory the same as restricting the multiverse to the generic multiverse generated by  $V$ .

## 6 Conclusion

The working mathematician need not worry whether he or she is working in a one universe setup or a multiverse setup, because the two, as I have explained them, are in harmony with each other.

But if the mathematician wants to incorporate in his or her investigation the firm conviction that a certain proposition has a determined truth-value, although this conviction does not lead to a conclusion as to whether the true-value is true or false, he or she can use the operation  $=(\varphi)$  to add a new axioms to this effect. Respectively, if a mathematician has a firm conviction that a certain proposition lacks a truth-value i.e. is absolutely undecidable, he or she can use the operation  $\neq(\varphi)$  to add an axiom to this effect.

The sentences  $=(\varphi)$  and  $\neq(\varphi)$  are examples of sentences in a new *multiverse dependence logic* which provides a whole arsenal of methods to inject order into the multiverse.

## References

- [1] W. W. Armstrong. Dependency structures of data base relationships. *Information Processing*, 74, 1974.
- [2] Jon Barwise (ed.). *Handbook of mathematical logic*. North-Holland.
- [3] Paul Benacerraf and Hilary Putnam (eds.). *Philosophy of mathematics. Selected readings*. 1st edition Prentice-Hall, 1964. 2nd edition Cambridge University Press, 1964.
- [4] George Boolos. The iterative conception of set. *The Journal of Philosophy*, 68(8):pp. 215–231, 1971.
- [5] Joel David Hamkins. The set-theoretic multiverse: a natural context for set theory. *Ann. Japan Assoc. Philos. Sci.*, 19:37–55, 2011.

- [6] John E. Doner, Andrzej Mostowski, and Alfred Tarski. The elementary theory of well-ordering—a metamathematical study. In *Logic Colloquium '77 (Proc. Conf., Wrocław, 1977)*, volume 96 of *Stud. Logic Foundations Math.*, pages 1–54. North-Holland, Amsterdam, 1978.
- [7] Pietro Galliani. Inclusion and exclusion dependencies in team semantics—on some logics of imperfect information. *Ann. Pure Appl. Logic*, 163(1):68–84, 2012.
- [8] Victoria Gitman and Joel David Hamkins. A natural model of the multi-verse axioms. *Notre Dame J. Form. Log.*, 51(4):475–484, 2010.
- [9] Kurt Gödel. What is Cantor’s continuum problem? *Amer. Math. Monthly*, 54:515–525, 1947. Reprinted with a supplement in [3].
- [10] Kurt Gödel. *Collected works. Vol. II.* The Clarendon Press Oxford University Press, New York, 1990. Publications 1938–1974, Edited and with a preface by Solomon Feferman.
- [11] Kurt Gödel. *Collected works. Vol. III.* The Clarendon Press Oxford University Press, New York, 1995. Unpublished essays and lectures, With a preface by Solomon Feferman, Edited by Feferman, John W. Dawson, Jr., Warren Goldfarb, Charles Parsons and Robert M. Solovay.
- [12] Erich Grädel and Jouko Väänänen. Dependence and independence. *Studia Logica*, 101(2):233–236, 2013.
- [13] L. Henkin. Some remarks on infinitely long formulas. In *Infinitistic Methods (Proc. Sympos. Foundations of Math., Warsaw, 1959)*, pages 167–183. Pergamon, Oxford, 1961.
- [14] Juha Kontinen and Jouko Väänänen. Axiomatizing first order consequences in dependence logic. <Link to paper>, Also: <http://arxiv.org/abs/1208.0176>.
- [15] G. Kreisel. Informal rigour and completeness proofs. In *Proceedings of the International Colloquium in the Philosophy of Science, London, 1965, Vol. 1. Edited by Imre Lakatos*, pages 138–157. North-Holland Publishing Co., Amsterdam, 1967.
- [16] Richard Laver. Certain very large cardinals are not created in small forcing extensions. *Ann. Pure Appl. Logic*, 149(1-3):1–6, 2007.
- [17] Per Lindström. Quasi-realism in mathematics. *The Monist*, 83(1):pp. 122–149, 2000.
- [18] L. Löwenheim. Über Möglichkeiten im Relativkalkül. *Math. Ann.*, 76:447–470, 1915.

- [19] D. Mirimanoff. Les antinomies de Russell et de Burali-Forti et le problème fondamental de la théorie des ensembles. *L'Enseignement Mathématique*, 19:37–52, 1917.
- [20] Charles Parsons. What is the iterative conception of set? In *Logic, foundations of mathematics and computability theory (Proc. Fifth Internat. Congr. Logic, Methodology and Philos. of Sci., Univ. Western Ontario, London, Ont., 1975), Part I*, pages 335–367. Univ. Western Ontario Ser. Philos. Sci., Vol. 9. Reidel, Dordrecht, 1977.
- [21] Saharon Shelah. The future of set theory. In *Set theory of the reals (Ramat Gan, 1991)*, volume 6 of *Israel Math. Conf. Proc.*, pages 1–12. Bar-Ilan Univ., Ramat Gan, 1993.
- [22] Saharon Shelah. *Cardinal arithmetic*, volume 29 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.
- [23] Saharon Shelah. Logical dreams. *Bull. Amer. Math. Soc. (N.S.)*, 40(2):203–228 (electronic), 2003.
- [24] Joseph R. Shoenfield. *Mathematical logic*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1967.
- [25] Th. Skolem. Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre. (English translation in [31]). Akateeminen Kirjakauppa, 1923.
- [26] John Steel. Gödel’s program. *This volume*.
- [27] Jouko Väänänen. *Dependence logic*, volume 70 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2007. A new approach to independence friendly logic.
- [28] Jouko Väänänen. Second order logic or set theory? *Bull. Symbolic Logic*, 18(1):91–121, 2012.
- [29] Jouko Väänänen and Wilfrid Hodges. Dependence of variables construed as an atomic formula. *Ann. Pure Appl. Logic*, 161(6):817–828, 2010.
- [30] Mark van Atten and Juliette Kennedy. “Gödel’s modernism: on set-theoretic incompleteness,” revisited. In *Logicism, intuitionism, and formalism*, volume 341 of *Synth. Libr.*, pages 303–355. Springer, Dordrecht, 2009.
- [31] Jean van Heijenoort. *From Frege to Gödel. A source book in mathematical logic, 1879–1931*. Harvard University Press, Cambridge, Mass., 1967.
- [32] J. von Neumann. Eine Axiomatisierung der Mengenlehre. (English translation in [31]). *Journal für die reine und angewandte Mathematik*, 154:219–240, 1925.

- [33] W. Hugh Woodin. The continuum hypothesis, the generic-multiverse of sets, and the  $\Omega$  conjecture. In *Set theory, arithmetic, and foundations of mathematics: theorems, philosophies*, volume 36 of *Lect. Notes Log.*, pages 13–42. Assoc. Symbol. Logic, La Jolla, CA, 2011.
- [34] E. Zermelo. Über Grenzzahlen und Mengenbereiche. Neue Untersuchungen über die Grundlagen der Mengenlehre. *Fundamenta Mathematicae*, 16:29–47, 1930.

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