Modal Dependence Logic

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Abstract

We introduce a modal language which involves the concept of *dependence*. We give two game-theoretic definitions for the semantics of the language, and one inductive, and prove the equivalence of all three.

1 Introduction

Is it possible that in the future currency exchange rates depend only on government decisions? It is perhaps possible, but it is certainly not necessary. In (Väänänen, 2007) we outlined the basics of the logic of dependence. In this paper we take it upon ourselves to start a study of the logic of "possible dependence".

By *dependence* we mean dependence as it occurs in the following contexts: Dependence of

- a move of a player in a game on previous moves
- an attribute of a database on other attributes
- an event in history on other events
- a variable of an expression on other variables
- a choice of an agent on choices by other agents.

We claim that there is a coherent theory of such dependence with applications to games, logic, computer science, linguistics, economics, etc.

There is an earlier study of the closely related concept of independence in the form of the independence friendly logic, by Jaakko Hintikka (1996). In that approach independence is tied up with quantifiers. We find dependence a more basic and a more tractable concept than independence. Also, we find that dependence (or independence) is not really a concept limited to quantifiers but a more fundamental property of individuals. Likewise, we do

Krzysztof R. Apt, Robert van Rooij (eds.). *New Perspectives on Games and Interaction.*
Texts in Logic and Games 4, Amsterdam University Press 2008, pp. 237–254.

not study here dependence or independence of modal operators from each other.

The basic concept of our approach to the logic of dependence is the dependence atom:

$$=(p_1, \ldots, p_n, q). \tag{1.1}$$

with the intuitive meaning that q depends only on $p_1 \ldots p_n$. The quantities $p_1 \ldots p_n$ and q can be propositions or individuals, in this paper they are propositions.

Definition 1.1. The *modal language of dependence* has formulas of the form:

1. $p, q, \ldots$ proposition symbols
2. $=(p_1, \ldots, p_n, q)$ meaning "q depends only on $p_1 \ldots p_n$"
3. $A \vee B$
4. $\neg A$
5. $\Diamond A$

The logical operations $\Box A$ (i.e., $\neg\Diamond\neg A$) and $A \wedge B$ (i.e., $\neg A \vee \neg B$), $A \to B$ (i.e., $\neg A \vee B$), $A \leftrightarrow B$ (i.e., $(A \to B) \wedge (B \to A)$), are treated as abbreviations.

The intuition is that a set of nodes of a Kripke structure satisfies the formula $=(p_1, \ldots, p_n, q)$ if in these nodes the truth value of q depends only on the truth values of $p_1 \ldots p_n$. Note that this criterion really assumes, as emphasized in a similar context in (Hodges, 1997), a *set* of nodes, for one cannot meaningfully claim that the propositional symbols true or false in one single node manifest any kind of dependence. Figures 1 and 2 give examples of dependence and lack of it.

We think of the sentence

$$\Box\Diamond(=(p, q) \wedge A)$$

as being true in a Kripke structure if every node accessible from the root has access to a node with A in such a way that in these nodes q depends only on p. A practical example of such a statement could be:

> *Whatever decisions the governments make in the next 10 years, it is possible that by the year 2050 the sea levels rise and whether the rise is over 50 cm depends only on how many countries have reduced their greenhouse gas emissions.*

We define now the game-theoretical semantics of our modal dependence language:

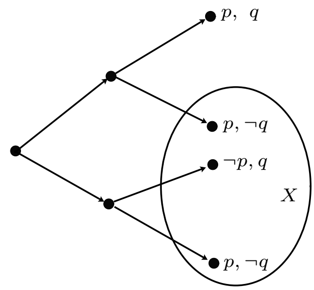

FIGURE 1. q depends only on p in X.

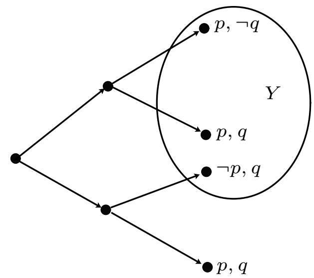

FIGURE 2. q does not depend only on p in Y.

Definition 1.2. The semantic game $G_1(A)$ is defined as follows: Positions are of the form (s, B, d), where s is a node of the Kripke structure, B is a modal formula and d is a player (I or II). In the beginning of $G_{\text{sem}}(A)$, played at s_0, the position is (s_0, A, II). The rules of the game are:

1. Position is (s, p, d): Player d wins if p is true in s, otherwise the opponent wins.

2. Position is $(s, =(p_1, \ldots, p_n, q), d)$: Player d wins.

3. Position is $(s, \neg A, d)$: The next position is (s, A, d^*), where d^* is the opponent of d.

4. Position is $(s, A \vee B, d)$: Player d chooses C from $\{A, B\}$. The next position is (s, C, d).

5. Position is $(s, \Diamond A, d)$: Player d chooses a node s', accessible from s. The next position is (s', A, d).

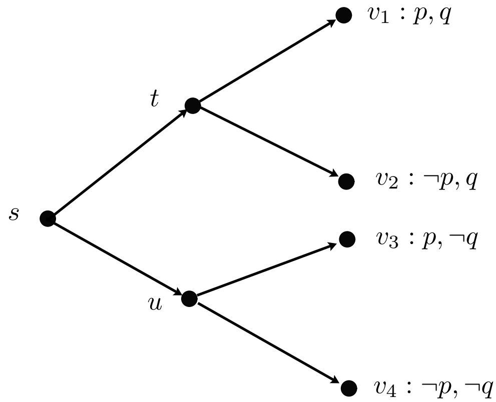

FIGURE 3. A Kripke model M.

A strategy σ of d is *uniform* if in any two plays where d uses σ and the game reaches a position $(s, =(p_1, \ldots, p_n, q), d)$ in the first play and $(s', =(p_1, \ldots, p_n, q), d)$ in the second play, with the same subformula $=(p_1, \ldots, p_n, q)$ of A and the same truth values of $p_1, \ldots, p_n$, the truth value of q is also the same. (By the "same subformula" we mean the same formula occurring in the same position in A.) In the extreme case of $=(p)$, the truth value of p has to be the same every time the game ends in position $(s, =(p), d)$ with the same $=(p)$.

Note that the game $G_{\text{sem}}(A)$ is determined and a perfect information game. Thus one of the players has always a winning strategy. However, there is no guarantee that this winning strategy is uniform (see Section 2). Thus the requirement of uniformity changes the nature of the game from determined to non-determined. In a sense the game loses the perfect information characteristic as the player who counts on a dependence atom $=(p_1, \ldots, p_n, q)$ being true has to choose the possible worlds without looking at other parameters than $p_1, \ldots, p_n$, as far as the truth of q is concerned. Rather than putting explicit information related restrictions on the moves of the players, we simply follow how they play and check whether the moves *seem* to depend on parameters not allowed by the winning positions $(s, =(p_1, \ldots, p_n, q), d)$. In a sense, a player is allowed to know everything all the time, but is not allowed to use the knowledge.

Definition 1.3. A is true at a node s if player II has a uniform winning strategy in the game $G_{\text{sem}}(A)$ at s.

The sentences

$$\Diamond\Box q$$
$$\Diamond\Box{=}(p,q)$$
$$\Box\Box(p \vee \neg p)$$

are all true at the root of the Kripke model of Figure 3. By the definition of the meaning of the negation, $\neg A$ is true in a node s if and only if player I has a uniform winning strategy in position (s, A, II). By a *logical consequence* $A \Rightarrow B$ in this context we mean that the formula B is true in every Kripke model at every node where A is true. Respectively, $A \Leftrightarrow B$ means that both $A \Rightarrow B$ and $B \Rightarrow A$ hold. Finally, A is called valid if it is true in every Kripke structure at every node.

Example 1.4.

1. $A \wedge (A \rightarrow B) \Rightarrow B$
2. $A \Rightarrow (B \rightarrow A)$
3. $(A \rightarrow (B \rightarrow C)) \wedge (A \rightarrow B) \Rightarrow A \rightarrow C$
4. $\neg B \rightarrow \neg A \Rightarrow A \rightarrow B$
5. $A \vee B \Leftrightarrow B \vee A$
6. $A \wedge B \Leftrightarrow B \wedge A$
7. $A \wedge A \Leftrightarrow A$
8. $A \wedge (B \wedge C) \Leftrightarrow (A \wedge B) \wedge C$
9. $A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C$
10. ${=}(p,q,r) \Leftrightarrow\ {=}(q,p,r)$
11. $({=}(p,q)\ \wedge\ {=}(q,r)) \Rightarrow\ {=}(p,r)$
12. ${=}(p,r) \Rightarrow\ {=}(p,q,r)$
13. If A is valid, then so is $\Box A$
14. $\Box(A \rightarrow B) \wedge \Box A \Rightarrow \Box B$
15. $\Box A \wedge \Box B \Leftrightarrow \Box(A \wedge B)$
16. $\Diamond A \vee \Diamond B \Leftrightarrow \Diamond(A \vee B)$

2 An example of non-determinacy

Consider the Kripke model M of Figure 3. The sentence

$$\Box\Diamond(p \leftrightarrow q)$$

is clearly true at the root s of the model, as both extensions of s have an extension in which p and q have the same truth value. On the other hand, the sentence

$$A: \ \Box\Diamond(=(p) \wedge (p \leftrightarrow q))$$

is not true at the root for the following reason. After the move of player I, the node is t or u. Suppose it is t. Now player II, in order not to lose right away, has to commit herself to $=(p)$ and the node with $p \wedge q$. Suppose the game is played again but Player I decides to move to node u. Now player II has to commit herself to $=(p)$ and the node with $\neg p \wedge \neg q$. At this point we see that the strategy that Player II is using is not uniform, for two plays have reached the same dependence atom $=(p)$ with a different truth value for p. This contradicts the very definition of uniformity. However, the sentence

$$\neg A: \ \neg\Box\Diamond(=(p) \wedge (p \leftrightarrow q))$$

is not true either, that is, neither does Player I have a uniform winning strategy in position (s, A, II). To see why this is so, let us assume I has a winning strategy (uniform or non-uniform) in position (s, A, II) and derive a contradiction. The position

$$(s, \Box\Diamond(=(p) \wedge (p \leftrightarrow q)), \text{II})$$

is actually the position

$$(s, \neg\Diamond\neg\Diamond(=(p) \wedge (p \leftrightarrow q)), \text{II}),$$

from which the game moves automatically to position

$$(s, \Diamond\neg\Diamond(=(p) \wedge (p \leftrightarrow q)), \text{I}).$$

So in this position, according to the rules, Player I makes a move and chooses according to his strategy, say, t. We are in position

$$(t, \neg\Diamond(=(p) \wedge (p \leftrightarrow q)), \text{I})$$

from which the game moves automatically to position

$$(t, \Diamond(=(p) \wedge (p \leftrightarrow q)), \text{II}).$$

Now it is Player II's turn to make a choice. We let her choose the node with $p \wedge q$. So we are in position

$$(v_1, =(p) \wedge (p \leftrightarrow q), \mathrm{II})$$

which leads to the position

$$(v_1, \neg{=}(p) \vee \neg(p \leftrightarrow q), \mathrm{I}).$$

Player I is to move. He does not want to play $\neg(p \leftrightarrow q)$ for that would lead to position

$$(v_1, \neg(p \leftrightarrow q), \mathrm{I}),$$

that is,

$$(v_1, p \leftrightarrow q, \mathrm{II}),$$

which is a winning position for Player II. So Player I is forced to play $\neg{=}(p)$, leading to position

$$(v_1, \neg{=}(p), \mathrm{I}),$$

that is,

$$(v_1, =(p), \mathrm{II}).$$

But this is a winning position for Player II, too. So again I has lost. If Player I moved u instead of t, the argument would be essentially the same. So we may conclude that I simply does not have a winning strategy in position (s, A, II). The game $G_{\mathrm{sem}}(A)$ is in this case *non-determined.*

We may conclude that the sentence $A \vee \neg A$ is not true at the root of M. Thus the *Law of Excluded Middle* is not valid in this logic. Also, the implication $A \to A$ is not valid. How can this be understood? The explanation lies in our game-theoretic concept of truth. For Player II to have a uniform winning strategy in position $(s, A \to A, \mathrm{II})$, she has to count on herself or Player I having a uniform winning strategy in position (s, A, II). As we have seen, the game $G_{\mathrm{sem}}(A)$ has no Gale-Stewart Theorem to guarantee it being determined. We have to give up—in the context of dependence logic—the idea that the meaning of $A \to B$ is that if A is true then B is true. Rather, we should think of $A \to B$ meaning that if Player I does not have a uniform winning strategy in $G_{\mathrm{sem}}(A)$, then Player II has a uniform winning strategy in $G_{\mathrm{sem}}(B)$.

3 A non-idempotency phenomenon

Consider the Kripke model N of Figure 4 and the sentence

$$B : \Box{=}(p).$$

It is clear that although Player II trivially wins every round of the game $G_{\mathrm{sem}}(B)$ at s, she does not have a uniform winning strategy at s, because

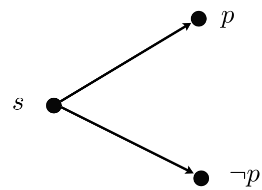

FIGURE 4. A Kripke model N.

depending on which extension of s Player I chooses, the value of p is true or false. On the other hand, Player I does have a uniform winning strategy, namely he simply plays the node with p during every round of the game.

Let us then look at

$$C : \Box(=(p) \vee =(p)).$$

Now Player II has a uniform winning strategy: *If I plays the node with p, she plays the left disjunct, and otherwise the right disjunct.* So we have shown that

$$\Box(D \vee D) \not\Rightarrow \Box D.$$

4 Inductive truth definition

There is an alternative but equivalent truth definition, similar to the inductive truth definition of Hodges (1997) for Hintikka's IF logic. The basic concept here is a set X of nodes satisfying a formula, rather than a single node. We define:

- p is true in X if p is true in every node in X.
- p is true in X if p is false in every node in X.
- $=(p_1, \ldots, p_n, q)$ is true in X if any two nodes in X that agree about $p_1, \ldots, p_n$ also agree about q.
- $\neg{=}(p_1, \ldots, p_n, q)$ is true in X if $X = \varnothing$.
- $A \vee B$ is true in X if X is the union of a set where A is true and a set where B is true (see Figure 5).
- $A \wedge B$ is true in X if both A and B are.
- $\Diamond A$ is true in X if A is true in some set Y such that every node in X has an extension in Y (see Figure 5).
- $\Box A$ is true in X if A is true in the set consisting of all extensions of all nodes in X (see Figure 5).

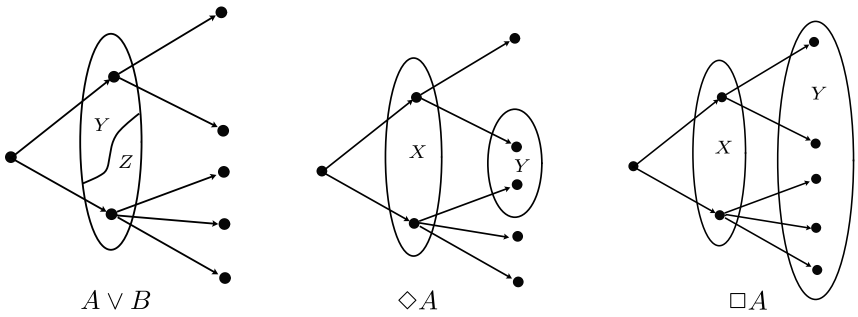

FIGURE 5. Truth definition.

More formally:

Definition 4.1. A is true in X if and only if $(X, A, \mathrm{II}) \in \mathcal{T}$, where the set $\mathcal{T}$ is defined as follows:

(T1) $(X, p, \mathrm{II}) \in \mathcal{T}$ iff p is true in every node in X.

(T2) $(X, p, \mathrm{I}) \in \mathcal{T}$ iff $\neg p$ is true in every node in X.

(T3) $(X, =(p_1, \ldots, p_n, q), \mathrm{II}) \in \mathcal{T}$ iff any two nodes in X that agree about $p_1, \ldots, p_n$ also agree about q.

(T4) $(X, =(p_1, \ldots, p_n, q), \mathrm{I}) \in \mathcal{T}$ iff $X = \emptyset$.

(T5) $(X, \neg A, d) \in \mathcal{T}$ iff $(X, A, d^*) \in \mathcal{T}$

(T6) $(X, A \vee B, \mathrm{II}) \in \mathcal{T}$ iff X is contained in the union of a set Y and a set Z such that $(Y, A, \mathrm{II}) \in \mathcal{T}$ and $(Z, B, \mathrm{II}) \in \mathcal{T}$.

(T7) $(X, A \vee B, \mathrm{I}) \in \mathcal{T}$ iff X is contained in the intersection of a set Y and a set Z such that $(Y, A, \mathrm{I}) \in \mathcal{T}$ and $(Z, B, \mathrm{I}) \in \mathcal{T}$.

(T8) $(X, \Diamond A, \mathrm{II}) \in \mathcal{T}$ iff $(Y, A, \mathrm{II}) \in \mathcal{T}$ for some set Y such that every node in X has an extension in Y.

(T9) $(X, \Diamond A, \mathrm{I}) \in \mathcal{T}$ iff $(Y, A, \mathrm{I}) \in \mathcal{T}$ for the set Y consisting of all extensions of all nodes in X.

An easy induction shows that, as shown in (Hodges, 1997):

Lemma 4.2.

1. $(X, A, d) \in \mathcal{T}$ implies $(Y, A, d) \in \mathcal{T}$ for all $Y \subseteq X$. *The Downward Closure Property.*

2. $(X, A \wedge \neg A, \mathrm{II}) \in T$ implies $X = \varnothing$. *The Consistency Property.*

From the Downward Closure Property it follows that (T6) can be replaced by

(T6)' $(X, A \vee B, \mathrm{II}) \in \mathcal{T}$ iff X is the union of a set Y and a set Z such that $(Y, A, \mathrm{II}) \in \mathcal{T}$ and $(Z, B, \mathrm{II}) \in \mathcal{T}$.

and (T7) can be replaced by

(T7)' $(X, A \vee B, \mathrm{I}) \in \mathcal{T}$ iff $(X, A, \mathrm{I}) \in \mathcal{T}$ and $(X, B, \mathrm{I}) \in \mathcal{T}$.

The way we defined the game $G_{\text{sem}}(A)$ there was always an initial node from which the game started. We can generalize the setup up a little by allowing a set X of initial nodes. A strategy of a player d in $G_{\text{sem}}(A)$ is a *winning strategy in* X if the player wins every game started from a position (s, A, II), where $s \in X$. The strategy is *uniform in* X if in any two plays p_1 and p_2, started from positions (x_1, A, II) and (x_2, A, II), with $x_1, x_2 \in X$, where d uses the strategy and the game reaches a position $(s, {=}(p_1, \ldots, p_n, q), d)$, with the same ${=}(p_1, \ldots, p_n, q)$ and the same truth values of $p_1, \ldots, p_n$, the truth value of q is also the same. Thus a player has a uniform winning strategy (in the original sense) at s iff he or she has a uniform winning strategy in $\{s\}$.

Theorem 4.3. If in the game $G_{\text{sem}}(A)$ Player II has a uniform winning strategy in the set X, then $(X, A, \mathrm{II}) \in \mathcal{T}$, i.e., A is true in the set X.

Proof. Suppose II has a uniform winning strategy σ in $G_{\text{sem}}(A_0)$ in the set X_0. We prove by induction on subformulas A of A_0 that if $\Gamma(A, d)$ denotes the set of nodes s such that position (s, A, d) is reached while $G_{\text{sem}}(A_0)$ is being played, II following σ, then $(\Gamma(A, d), A, d) \in \mathcal{T}$. This will suffice, for the initial position (s, A_0, II) can be reached for any $s \in X_0$ and so it will follow that A_0 is true in X_0. When dealing with $\Gamma(A, d)$ we have consider different occurrences of the same subformula of A_0 as separate. So, e.g., ${=}(p)$ may occur in A_0 in two different places and $\Gamma({=}(p), d)$ is computed separately for each of them.

Case i: $X = \Gamma(p, \mathrm{II})$. Since σ is a winning strategy, p is true at every $s \in X$. Thus $(X, p, \mathrm{II}) \in \mathcal{T}$ by (T1).

Case ii: $X = \Gamma(p, \mathrm{I})$. Since σ is a winning strategy, $\neg p$ is true at every $s \in X$. Thus $(X, p, \mathrm{I}) \in \mathcal{T}$ by (T2).

Case iii: $X = \Gamma({=}(p_1, \ldots, p_n, q), \mathrm{II})$. Let us consider $s, t \in X$ that agree about $p_1, \ldots, p_n$. Since σ is a uniform strategy, s and t agree about q. By (T3), $(X, {=}(p_1, \ldots, p_n, q), \mathrm{II}) \in \mathcal{T}$.

Case iv: $X = \Gamma({=}(p_1, \ldots, p_n, q), \mathrm{I})$. Since σ is a winning strategy of II, $X = \varnothing$. By (T4), $(X, {=}(p_1, \ldots, p_n, q), \mathrm{I}) \in \mathcal{T}$.

Case v: $X = \Gamma(\neg A, d)$. Note that $X = \Gamma(A, d^*)$. By induction hypothesis, $(X, A, d^*) \in \mathcal{T}$, and hence $(X, \neg A, d) \in \mathcal{T}$.

Case vi: $X = \Gamma(A \vee B, \mathrm{II})$. Note that $X \subseteq Y \cup Z$, where $Y = \Gamma(A, \mathrm{II})$ and $Z = \Gamma(B, \mathrm{II})$. By induction hypothesis, $(Y, A, \mathrm{II}) \in \mathcal{T}$ and $(Z, B, \mathrm{II}) \in \mathcal{T}$. Thus $(X, A \vee B, \mathrm{II}) \in T$ by (T6).

Case vii: $X = \Gamma(A \vee B, \mathrm{I})$. Note that $X \subseteq Y \cap Z$, where $Y = \Gamma(A, \mathrm{I})$ and $Z = \Gamma(B, \mathrm{I})$. By induction hypothesis, $(Y, A, \mathrm{I}) \in \mathcal{T}$ and $(Z, B, \mathrm{I}) \in \mathcal{T}$. Thus $(X, A \vee B, \mathrm{I}) \in T$ by (T7).

Case viii: $X = \Gamma(\Diamond A, \mathrm{II})$. For each $s \in X$ there is some s' reachable from s that II chooses according to her winning strategy σ in position $(s, \Diamond A, \mathrm{II})$. Let Y be the set of all such s'. Note that then $Y \subseteq \Gamma(A, \mathrm{II})$. By induction hypothesis, $(\Gamma(A, \mathrm{II}), A, \mathrm{II}) \in \mathcal{T}$. By (T8), $(X, \Diamond A, \mathrm{II}) \in \mathcal{T}$.

Case ix: $X = \Gamma(\Diamond A, \mathrm{I})$. For each $s \in X$ there may be some s' reachable from s that I could choose in position $(s, \Diamond A, \mathrm{I})$. Let Y be the set of all such possible s' (i.e., Y is the set of all possible extensions of all $s \in X$). By induction hypothesis $(Y, A, \mathrm{I}) \in \mathcal{T}$. By (T9), $(X, \Diamond A, \mathrm{I}) \in \mathcal{T}$. Q.E.D.

Corollary 4.4. If II has a uniform winning strategy in $G_{\text{sem}}(A)$ at s, then A is true in $\{s\}$.

5 Truth strategy

We define a new game $G_2(A)$, which we call the *set game* as follows: Positions are of the form (X, B, d), where X is a set of nodes, B is a modal dependence formula, and d is either I or II. The rules of the game are as follows:

(S1) (X, p, II): Player II wins if p is true at every node in X, otherwise I wins.

(S2) (X, p, I): Player II wins if p is false at every node in X, otherwise I wins.

(S3) $(X, {=}(p_0, \ldots, p_n, q), \mathrm{II})$: Player II wins if any two nodes in X that agree about $p_1, \ldots, p_n$ also agree about q. Otherwise I wins.

(S4) $(X, {=}(p_0, \ldots, p_n, q), \mathrm{I})$: Player II wins if $X = \varnothing$, otherwise I wins.

(S5) $(X, \neg A, d)$: The game continues from (X, A, d^*).

(S6) $(X, A \vee B, \mathrm{II})$: Player II chooses Y and Z such that $X \subseteq Y \cup Z$. Then Player I chooses whether the game continues from (Y, A, II) or (Z, B, II).

(S7) $(X, A \vee B, \mathrm{I})$: Player II chooses Y and Z such that $X \subseteq Y \cap Z$. Then Player I chooses whether the game continues from (Y, A, I) or (Z, B, I).

(S8) $(X, \Diamond A, \mathrm{II})$: Player II chooses a set Y such that every node in X has an extension in Y. The next position is (Y, A, II).

(S9) $(X, \Diamond A, \mathrm{I})$: The next position is (Y, A, I), where Y consists of every extension of every node in X.

An easy induction shows that if Player II has a winning strategy in position (X, A, d), and $Y \subseteq X$, then she has in position (Y, A, d), too. From this fact it follows that (S6) can be replaced by

(S6)' $(X, A \vee B, \mathrm{II})$: Player II chooses Y and Z such that $X = Y \cup Z$. Then Player I chooses whether the game continues from (Y, A, II) or (Z, B, II).

and (S7) can be replaced by

(S7)' $(X, A \vee B, \mathrm{I})$: Player I chooses whether the game continues from (X, A, I) or (X, B, I).

Theorem 5.1. If $(X, A, \mathrm{II}) \in \mathcal{T}$ (i.e., A is true in X), then Player II has a winning strategy in $G_{\mathrm{set}}(A)$ in position (X, A, II).

Proof. Suppose that $(X_0, A_0, \mathrm{II}) \in \mathcal{T}$. The strategy of II in $G_{\mathrm{set}}(A_0)$ is to play in such a way that if the play is in $G_{\mathrm{set}}(A_0)$ in position $P = (X, A, d)$, then $\tau(P) = (X, A, d) \in \mathcal{T}$. In the beginning the position is (X_0, A_0, II) and indeed A_0 is true at X_0. After this we have different cases before the game ends:

Case 1: $P = (X, \neg A, d)$. By assumption, $\tau(P) = (X, \neg A, d) \in \mathcal{T}$. By (T5) $(X, A, d^*) \in \mathcal{T}$. Now the game continues from position $P' = (t, A, d^*)$ and $\tau(P') = (X, A, d^*) \in \mathcal{T}$.

Case 2: $P = (X, A \vee B, \mathrm{II})$. By assumption, $\tau(P) = (X, A \vee B, \mathrm{II}) \in \mathcal{T}$. By (T6) there are Y and Z such that $X \subseteq Y \cup Z$, $(Y, A, \mathrm{II}) \in \mathcal{T}$ and $(Z, B, \mathrm{II}) \in \mathcal{T}$. So II plays Y and Z in $G_{\mathrm{set}}(A_0)$. Now I decides whether the game continues from position (Y, A, II) or from position (Z, B, II). Whichever the decision is, we have $(Y, A, \mathrm{II}) \in \mathcal{T}$ and $(Z, B, \mathrm{II}) \in \mathcal{T}$.

Case 3: $P = (t, A \vee B, \mathrm{I})$. By assumption, $\tau(P) = (X, A \vee B, \mathrm{I}) \in \mathcal{T}$. By (T7)', $(X, A, \mathrm{I}) \in \mathcal{T}$ and $(X, B, \mathrm{I}) \in \mathcal{T}$. Now the set game continues from position (X, A, I) or from position $(Y, B, \mathrm{I}$, according to the decision of I. Whichever the decision is, we have $(X, A, \mathrm{I}) \in \mathcal{T}$ and $(X, B, \mathrm{I}) \in \mathcal{T}$.

Case 4: $P = (t, \Diamond A, \mathrm{II})$. By assumption, $\tau(P) = (X, \Diamond A, \mathrm{II}) \in \mathcal{T}$. By (T8), $(Y, A, \mathrm{II}) \in \mathcal{T}$ for some set Y of nodes accessible from nodes in X. This set Y is the choice of II in $G_{\mathrm{set}}(A_0)$. Now the game continues from position $P' = (Y, A, \mathrm{II})$ and $\tau(P') = (Y, A, \mathrm{II}) \in \mathcal{T}$.

Case 5: $P = (t, \Diamond A, \mathrm{I})$. By assumption, $\tau(P) = (X, \Diamond A, \mathrm{I}) \in \mathcal{T}$. By (T9), $(Y, A, \mathrm{I}) \in \mathcal{T}$ for the set Y of all nodes accessible from nodes in X. Now the game continues from position $P' = (Y, A, \mathrm{I})$ and $\tau(P') = (Y, A, \mathrm{I}) \in \mathcal{T}$.

At the end of the game $G_{\text{set}}(A_0)$ we have to check that II indeed has won. There are again several cases:

Case 6: $P = (X, p, \mathrm{II})$. Since $\tau(P) = (X, p, \mathrm{II}) \in \mathcal{T}$, p is true at every $t \in X$ by (T1). So II has won.

Case 7: $P = (X, p, \mathrm{I})$. Since $\tau(P) = (X, p, \mathrm{I}) \in \mathcal{T}$, $\neg p$ is true at every $t \in X$ by (T2). So II has won.

Case 8: $P = (X, {=}(p_1, \ldots, p_n, q), \mathrm{II})$. Let $s, t \in X$ agree about $p_1, \ldots, p_n$. Since $(X, {=}(p_1, \ldots, p_n, q), \mathrm{II}) \in \mathcal{T}$, we can conclude from (T3) that s and t agree about q. Player II has won.

Case 9: $P = (X, {=}(p_1, \ldots, p_n, q), \mathrm{I})$. So $\tau(P) = (X, {=}(p_1, \ldots, P_n, q), \mathrm{I}) \in \mathcal{T}$. By (T4), $X = \varnothing$. Player II has won. Q.E.D.

6 Power strategy

We shall describe a strategy in $G_{\text{sem}}(A)$ which is based on playing $G_{\text{set}}(A)$ in the power set of the Kripke model, hence the name *power strategy.* The advantage of playing in the power set is that we can in a sense play many games in parallel and use this to get a uniform strategy in $G_{\text{sem}}(A)$ (see Figure 6).

Theorem 6.1. If Player II has a winning strategy in $G_{\text{set}}(A)$ in position (X, A, II), then in $G_{\text{sem}}(A)$, she has a uniform winning strategy in X.

Proof. Suppose σ is a winning strategy of II in $G_{\text{set}}(A_0)$ in position (X_0, A_0, II). The strategy of II in $G_{\text{sem}}(A_0)$ is to play so that if the play is in position $P = (t, A, d)$, then II is in the game $G_{\text{set}}(A_0)$, playing σ, in position $\tau(P) = (X, A, d)$ with $t \in X$. In the beginning the position in $G_{\text{sem}}(A_0)$ can be any (s, A_0, II), where $s \in X_0$. In $G_{\text{set}}(A_0)$ the initial position is (X_0, A_0, II). So whichever $P = (s, A_0, \mathrm{II})$ the game $G_{\text{sem}}(A_0)$ starts with, we can let $\tau(P) = (X_0, A_0, \mathrm{II})$. After this we have different cases before the game ends:

Case 1: $P = (t, \neg A, d)$. By assumption, $\tau(P) = (X, \neg A, d)$ with $t \in X$. Now the game continues from position $P' = (t, A, d^*)$ in $G_{\text{sem}}(A_0)$ and from position $\tau(P') = (X, A, d^*)$ in $G_{\text{set}}(A_0)$.

Case 2: $P = (t, A \vee B, \mathrm{II})$. By assumption, $\tau(P) = (X, A \vee B, \mathrm{II})$ such that $t \in X$. By (S6) the strategy σ gives two sets Y and Z such that $X \subseteq Y \cup Z$, the game $G_{\text{set}}(A_0)$ continues from (Y, A, II) or (Z, B, II). Since $t \in Y \cup Z$, we have either $t \in Y$ or $t \in Z$. In the first case II lets $C = A$, $U = Y$ and in the second case $C = B$, $U = Z$. Now the game $G_{\text{sem}}(A_0)$ continues from position $P' = (t, C, \mathrm{II})$ and $\tau(P') = (U, C, \mathrm{II})$.

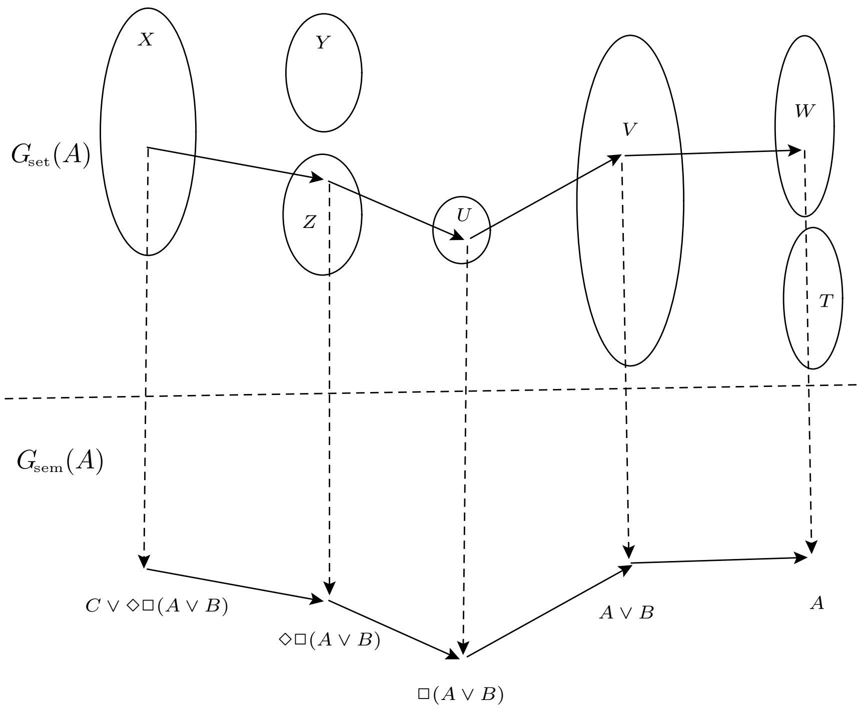

FIGURE 6. Power strategy.

Case 3: $P = (t, A \vee B, \mathrm{I})$. By assumption, $\tau(P) = (X, A \vee B, \mathrm{I})$. By (S7)', the game $G_{\text{set}}(A_0)$ can continue from either (X, A, I) or (X, B, I). Now the game $G_{\text{sem}}(A_0)$ continues from position (t, C, I), where $C = A$ or $C = B$, according to the choice of I. In either case we let $\tau(P') = (X, C, \mathrm{I})$.

Case 4: $P = (t, \Diamond A, \mathrm{II})$. By assumption, $\tau(P) = (X, \Diamond A, \mathrm{II})$. By (S8), the strategy σ gives a set Y of nodes accessible from nodes in X and the game $G_{\text{set}}(A_0)$ continues from (Y, A, II). Since $t \in X$, there is an extension u of t in Y. This is the choice of II in $G_{\text{sem}}(A_0)$. Now the game continues from position $P' = (u, A, \mathrm{II})$ and we define $\tau(P') = (Y, A, \mathrm{II})$.

Case 5: $P = (t, \Diamond A, \mathrm{I})$. By assumption, $\tau(P) = (X, \Diamond A, \mathrm{I})$. By (S9), the game $G_{\text{set}}(A_0)$ continues from position (Y, A, I) for the set Y of all nodes accessible from nodes in X. Since $t \in X$, the extension u of t chosen by I is bound to be in Y. Now the game continues from position $P' = (u, A, \mathrm{I})$ and we let $\tau(P') = (Y, A, \mathrm{I})$.

At the end of the game $G_{\text{sem}}(A_0)$ we have to check that II indeed has won. There are again several cases:

Case 6: $P = (t, p, \mathrm{II})$. Since $\tau(P) = (X, p, \mathrm{II})$ and σ is a winning strategy, p is true at t. So II has won $G_{\mathrm{sem}}(A_0)$.

Case 7: $P = (t, p, \mathrm{I})$. Since $\tau(P) = (X, p, \mathrm{I})$ and σ is a winning strategy, $\neg p$ is true at t. So II has won $G_{\mathrm{sem}}(A_0)$.

Case 8: $P = (t, {=}(p_1, \ldots, p_n, q), \mathrm{II})$. Player II has won $G_{\mathrm{sem}}(A_0)$.

Case 9: $P = (t, {=}(p_1, \ldots, p_n, q), \mathrm{I})$. Now $\tau(P) = (X, {=}(p_1, \ldots, p_n, q), \mathrm{I})$. Since σ is a winning strategy, $X = \varnothing$. On the other hand, by assumption, $t \in X$. So this case simply cannot occur.

Now that we know that this strategy is a winning strategy, we have to show that it is a uniform strategy. Suppose therefore that two plays

$$P_0, \ldots, P_m, \text{where } P_i = (t_i, A_i, d_i)$$

$$P'_0, \ldots, P'_{m'}, \text{where } P'_i = (t'_i, A'_i, d'_i)$$

end in the same formula $A_m = A'_{m'}$ which is of the form ${=}(p_1, \ldots, p_n, q)$ and that the nodes t_m and $t'_{m'}$ give $p_1, \ldots, p_n$ the same value. Let

$$\tau(P_i) = (X_i, A_i, d_i), i = 1, \ldots, m$$

$$\tau(P'_i) = (X'_i, A'_i, d'_i), i = 0, \ldots, m'$$

be the corresponding positions in $G_{\mathrm{set}}(A_0)$. We show now by induction on i that $m = m'$, $X_i = X'_i$, $A_i = A'_i$ and $d_i = d'_i$. The case $i = 0$ is clear: $A_0 = A'_0$, $X_0 = X'_0$ and $d_0 = d'_0 = \mathrm{II}$. The inductive proof is trivial, apart from the case $P_i = (t_i, A \vee B, d_i)$, $d_i = \mathrm{II}$. By assumption, $\tau(P_i) = (X_i, A \vee B, \mathrm{II})$. The strategy σ has given the two sets Y and Z such that $X \subseteq Y \cup Z$, and the game $G_{\mathrm{set}}(A_0)$ continues from (Y, A, II) or (Z, B, II). Since $t \in Y \cup Z$, we have either $t \in Y$ or $t \in Z$. In the first case II lets $C = A$, $U = Y$ and in the second case $C = B$, $U = Z$. Now the game $G_{\mathrm{sem}}(A_0)$ continues from position $P_{i+1} = (t, C, \mathrm{II})$ and $\tau(P_{i+1}) = (U, C, \mathrm{II})$. Respectively, $P'_i = (t'_i, A \vee B, \mathrm{II})$ and $\tau(P'_i) = (X_i, A \vee B, \mathrm{II})$. The strategy σ (which does not depend on the elements t_i and t'_i) has given the same two sets Y and Z, as above, and the game $G_{\mathrm{set}}(A_0)$ continues after $\tau(P_i) = \tau(P'_i)$ from (Y, A, II) or (Z, B, II), according to whether $t \in Y$ or $t \in Z$. So $X'_{i+1} = X_{i+1}$, $A'_{i+1} = A_{i+1}$ and $d'_{i+1} = d_{i+1}$.

Thus t_m and t'_m are in X_m and give the same value to $p_1, \ldots, p_n$. Because σ is a winning strategy of II, the nodes t_m and t'_m must give the same value also to q. We have demonstrated the uniformity of the strategy. Q.E.D.

7 The main result

Putting Theorems 4.3, 5.1 and 6.1 together, we obtain:

Theorem 7.1. Suppose A is a sentence of the modal dependence language, and X is a set of nodes of a Kripke structure. The following are equivalent:

1. $(X, A, \mathrm{II}) \in \mathcal{T}$ (i.e., A is true in the set X).

2. Player II has a uniform winning strategy in $G_{\mathrm{sem}}(A)$ in the set X.

3. Player II has a winning strategy in $G_{\mathrm{set}}(A)$ in X.

Corollary 7.2. Suppose A is a sentence of the modal dependence language, and s is a node of a Kripke structure. The following are equivalent:

1. $(\{s\}, A, \mathrm{II}) \in \mathcal{T}$ (i.e., A is true in the set $\{s\}$).

2. Player II has a uniform winning strategy in $G_{\mathrm{sem}}(A)$ at s.

3. Player II has a winning strategy in $G_{\mathrm{set}}(A)$ in $\{s\}$.

The proved equivalence leads to easy proofs of the logical consequences and equivalences of Example 1.4. Let us consider, as an example

$$\Box(A \to B) \wedge \Box A \Rightarrow \Box B.$$

Let X be a set of nodes of a Kripke model. Suppose $\Box(A \to B)$ and $\Box A$ are true in X. Let X' be the set of nodes accessible from nodes in X. Thus $A \to B$ and A are true in X'. Then by (T6)', $X' = Y \cup Z$ such that $\neg A$ is true in Y and B is true in Z. By Lemma 4.2 and (T7), $A \wedge \neg A$ is true in Y. By Lemma 4.2, $Y = \varnothing$. So $X' = Z$ and B is true in X'. We have demonstrated that $\Box B$ is true in X.

The point of Theorem 7.1 is that the first game G_1 with positions of the form (s, A, d) is non-determined and of imperfect information. The set game G_2 is determined and of perfect information. In an obvious sense the two games are *equivalent*. **So we have been able to replace a non-determined game of imperfect information with a determined game of perfect information.** The cost of this operation is that the determined game of perfect information is played on sets rather than elements. So in a sense there is an exponential cost.

8 Further developments

We can define $=(p_1, \ldots, p_n, q)$ in terms of $=(q)$ if we allow exponential growth of the formula size: $=(p_1, \ldots, p_n, q)$ is true in a set X if and only if the following formula is:

$$\left.\begin{array}{l} (\ \ p_1 \wedge \cdots \wedge \ \ p_n \wedge =(q)) \vee \\ (\neg p_1 \wedge \cdots \wedge \ \ p_n \wedge =(q)) \vee \\ \qquad\quad \vdots \qquad\qquad\qquad \vee \\ (\neg p_1 \wedge \cdots \wedge \neg p_n \wedge =(q)) \end{array}\right\} 2^n \text{ disjuncts.}$$

We can define $=(p)$ if we add to our modal dependence language a Boolean disjunction $A \vee_{\mathbf{B}} B$ with the obvious meaning that $A \vee_{\mathbf{B}} B$ is true in a set iff A is true in the set or B is, (and $\neg(A \vee_{\mathbf{B}} B)$ is true only[1] if $X = \varnothing$). In terms of the game $G_{\text{sem}}(A_0)$ this means that in position $(s, A \vee_{\mathbf{B}} B, \text{II})$ Player II chooses A or B, and in position $(s, A \vee_{\mathbf{B}} B, \text{I})$ Player I wins. A uniform winning strategy of II is required to satisfy the extra condition that player II has to make the same move *every* time the position $(s, A \vee_{\mathbf{B}} B, \text{II})$ is encountered, however many times the game is played. With these conventions $=(p)$ is logically equivalent to $p \vee_{\mathbf{B}} \neg p$.

Merlijn Sevenster (2008) has proved a normal form for modal dependence language and used it to show that the modal dependence language has in fact a translation into basic modal language, but again at exponential cost. He also shows that the satisfaction problem of modal dependence language is NEXP complete.

The finite information logic (Parikh and Väänänen, 2005) is based on dependence formulas of the type $=(A_1, \ldots, A_n, x)$, with the meaning that the value of the variable x is chosen on the basis of the truth values of the formulas $A_1, \ldots, A_n$ only. The formulas $A_1, \ldots, A_n$ are assumed to be quantifier free first order formulas (in fact they can be Δ_2 formulas). Quantifiers are allowed only in a "guarded" situation such as $\exists x(=(A_1, \ldots, A_n, x) \wedge B)$ and $\forall x(=(A_1, \ldots, A_n, x) \rightarrow B)$. This is equivalent to the existential-universal fragment of first order logic, but at exponential cost in the length of the formula. The point of this logic is that it captures the concept of *social software* in the sense that people in social situations often make decisions on the basis of finite information about the parameters, indeed on the basis of the truth-values of some predicates, like "has a valid visa", "speaks Dutch," etc.

In full *dependence logic* (Väänänen, 2007) first order logic is extended by dependence formulas $=(y_1, \ldots, y_n, x)$ with the meaning that the value of x depends only on the values of $y_1, \ldots, y_n$. This logic is equivalent to the existential second order logic, and is thus quite powerful.

If dependence formulas are added to second order logic, again no proper extension results. We may thus conclude that adding dependence to a logic increases the expressive power in the "middle range" of first order logic, but not in the case of the relatively weak modal logic and the relatively strong second order logics.

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[1] This follows by symmetry from (T6) and (T7) if we add the condition "one of Y and Z is empty".

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