Inner models from extended logics

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Constructible hierarchy generalized

\[ L'_0 = \emptyset \]
\[ L'_{\alpha+1} = \text{Def}_{L^*}(L'_\alpha) \]
\[ L'_\nu = \bigcup_{\alpha < \nu} L'_\alpha \text{ for limit } \nu \]

We use \( C(L^*) \) to denote the class \( \bigcup_{\alpha} L'_\alpha \).
Thus a typical set in $L'_{\alpha+1}$ has the form

$$X = \{ a \in L'_\alpha : (L'_\alpha, \in) \models \varphi(a, \vec{b}) \}$$
Introduction

The cof-model

The aa-model

HOD

Higher order logics

Infinitary languages

Generalized quantifiers
Examples

- $C(\mathcal{L}_{\omega\omega}) = L$
- $C(\mathcal{L}_{\omega_1\omega}) = L(\mathbb{R})$
- $C(\mathcal{L}_{\omega_1\omega_1}) = \text{Chang model}$
- $C(\mathcal{L}^2) = \text{HOD}$
Possible attributes of inner models

- Forcing absolute.
- Support large cardinals.
- Satisfy Axiom of Choice.
- Arise "naturally".
- Decide questions such as CH.
Inner models we have

- $L$: Forcing-absolute but no large cardinals (above WC)
- HOD: Has large cardinals but forcing-fragile
- $L(R)$: Forcing-absolute, has large cardinals, but no AC
- Extender models: Tailor made to support given large cardinals
Absolute logics—nothing new

Theorem
Suppose $\mathcal{L}^*$ is ZFC+V=L-absolute with parameters from L, and the syntax of $\mathcal{L}^*$ is ZFC+V=L-absolute with parameters from L. Then $C(\mathcal{L}^*) = L$.

Corollary
$C(\mathcal{L}(Q^L_\alpha)) = L$
Magidor-Malitz quantifier

Definition
Magidor-Malitz quantifier of dimension $n$:

$$
\mathcal{M} \models Q_{\alpha}^{\text{MM},n} x_1, \ldots, x_n \varphi(x_1, \ldots, x_n) \iff \\
\exists X \subseteq M(|X| \geq \aleph_\alpha \land \forall a_1, \ldots, a_n \in X : \mathcal{M} \models \varphi(a_1, \ldots, a_n)).
$$
Magidor-Malitz quantifier, assuming $0^\#$

Consistently, $C(Q_{1}^{MM,2}) \neq L$, but:

**Theorem**

*If $0^\#$ exists, then $C(Q_{\alpha}^{MM,<\omega}) = L.$*

**Lemma**

*Suppose $0^\#$ exists and $A \in L$, $A \subseteq [\alpha]^2$. If there is an uncountable $B$ such that $[B]^2 \subseteq A$, then there is such a set $B$ in $L.$*
Shelah’s cofinality quantifier

Definition
The cofinality quantifier $Q^\text{cf}_{\omega}$ is defined as follows:

$$\mathcal{M} \models Q^\text{cf}_{\omega} xy \varphi(x, y, \bar{a}) \iff \{(c, d) : \mathcal{M} \models \varphi(c, d, \bar{a})\}$$

is a linear order of cofinality $\omega$.

- Axiomatizable
- Fully compact
- Downward Löwenheim-Skolem down to $\aleph_1$
The "cof-model" $C^*$

**Definition**

$$C^* = \text{def } C(Q^c_\omega)$$

**Example:**

$$\{\alpha < \beta : \text{cf}^V(\alpha) > \omega \} \in C^*$$
Theorem

*If $0^\#$ exists, then $0^\# \in C^*$.*

Proof.
Let

$$X = \{\xi < \aleph_\omega : \xi \text{ is a regular cardinal in } L \text{ and } \text{cf}(\xi) > \omega\}$$

Now $X \in C^*$ and

$$0^\# = \{\neg \varphi(x_1, \ldots, x_n) \vdash L_{\aleph_\omega} \models \varphi(\gamma_1, \ldots, \gamma_n) \text{ for some } \gamma_1 < \ldots < \gamma_n \text{ in } X\}.$$
• More generally, the above argument shows that $x^\# \in C^*(x)$ for any $x \in C^*$ such that $x^\#$ exists.
• Hence $C^* \neq L(x)$ whenever $x$ is a set of ordinals such that $x^\#$ exists in $V$. 
Theorem
The Dodd-Jensen Core model is contained in $C^*$.

Theorem
Suppose $L^\mu$ exists. Then some $L^\nu$ is contained in $C^*$. 
**Theorem**

*If there is a measurable cardinal $\kappa$, then $V \neq C^*$.*

**Proof.**

Suppose $V = C^*$ and $\kappa$ is a measurable cardinal. Let $i : V \to M$ with critical point $\kappa$ and $M^\kappa \subseteq M$. Now $(C^*)^M = (C^*)^V = V$, whence $M = V$. This contradicts Kunen’s result that there cannot be a non-trivial $i : V \to V$. 

$\square$
Theorem

If there is an infinite set $E$ of measurable cardinals (in $V$), then $E \not\in C^*$. Moreover, then $C^* \neq \text{HOD}$.

Proof.

As Kunen’s result that if there are uncountably many measurable cardinals, then AC is false in the Chang model.
Stationary Tower Forcing

Suppose $\lambda$ is Woodin.

- There is a forcing $\mathbb{Q}$ such that in $V[G]$ there is $j : V \to M$ with $V[G] \models M^\omega \subseteq M$ and $j(\omega_1) = \lambda$.

- For all regular $\omega_1 < \kappa < \lambda$ there is a cofinality $\omega$ preserving forcing $\mathbb{P}$ such that in $V[G]$ there is $j : V \to M$ with $V[G] \models M^\omega \subseteq M$ and $j(\kappa) = \lambda$. 
Theorem

If there is a Woodin cardinal, then $\omega_1$ is (strongly) Mahlo in $C^*$. 

Proof.

Let $Q$, $G$ and $j : V \rightarrow M$ with $M^\omega \subset M$ and $j(\omega_1) = \lambda$ be as above.

Now,

$$(C^*)^M = C^*_\lambda \subseteq V.$$
Theorem
Suppose there is a Woodin cardinal \( \lambda \). Then every regular cardinal \( \kappa \) such that \( \omega_1 < \kappa < \lambda \) is weakly compact in \( C^* \).

Proof.
Suppose \( \lambda \) is a Woodin cardinal, \( \kappa > \omega_1 \) is regular and \( < \lambda \). To prove that \( \kappa \) is strongly inaccessible in \( C^* \) we can use the “second” stationary tower forcing \( \mathbb{P} \) above. With this forcing, cofinality \( \omega \) is not changed, whence \((C^*)^M = C^*\). \( \Box \)
Theorem

If there is a proper class of Woodin cardinals, then the regular cardinals $\geq \aleph_2$ are indiscernible in $C^*$. 

Proof.

We use the “second” stationary tower forcing $\mathbb{P}$ to show first that the Woodin cardinals are indiscernible, and after that the regular cardinals $\geq \aleph_2$ are indiscernible. Remember that the here $\mathbb{P}$ and $j$ preserve $C^*$. 

$\square$
Theorem

If \( V = L^\mu \), then \( C^* \) is exactly the inner model \( M_{\omega^2}[E] \), where \( M_{\omega^2} \) is the \( \omega^2 \)th iterate of \( V \) and \( E = \{ \kappa_{\omega \cdot n} : n < \omega \} \).

Proof.

1. \( C^* \subseteq M_{\omega^2}[E] \): In \( M_{\omega^2}[E] \) we can detect which ordinals have cofinality \( \omega \) in \( V \).

2. \( M_{\omega^2}[E] \subseteq C^* \): The set \( E \) is the set of ordinals \( \kappa_{\omega^2} \) which have cofinality \( \omega \) in \( V \) but are regular in the core model. The measure \( i_{0\omega^2}(\mu) \) on \( \kappa_{\omega^2} \) can be defined from \( E \) by \( \mu'(X) = 1 \) if and only if \( \exists \alpha \in E \forall \beta \in E (\alpha < \beta \rightarrow \beta \in X) \).
Theorem

Suppose there is a proper class of Woodin cardinals. Suppose $\mathcal{P}$ is a forcing notion and $G \subseteq \mathcal{P}$ is generic. Then

$$\text{Th}((C^*)^V) = \text{Th}((C^*)^{V[G]}).$$
Proof.

Let $H_1$ be generic for $\mathbb{Q}$. Now

$$j_1 : (C^*)^V \to (C^*)^{M_1} = (C^*)^{V[H_1]} = (C^*_{<\lambda})^V.$$ 

Let $H_2$ be generic for $\mathbb{Q}$ over $V[G]$. Then

$$j_2 : (C^*)^{V[G]} \to (C^*)^{M_2} = (C^*)^{V[H_2]} = (C^*_{<\lambda})^{V[G]} = (C^*_{<\lambda})^V.$$
Theorem
\[ |\mathcal{P}(\omega) \cap C^*| \leq \aleph_2. \]

Proof.
Suppose \( a \subseteq \omega \) and \( a \in C^* \). We build \((M_\alpha)_{\alpha<\omega_1}\) such that

1. \( a \in M_0, M_0 |= a \in C^*, |M_\alpha| \leq \omega, M_\alpha < H(\mu). \)
2. \( M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha \), if \( \gamma = \bigcup \gamma \).
3. If \( \beta \in M_\alpha \) and \( \text{cf}^V(\beta) = \omega \), then \( M_{\alpha+1} \) contains an \( \omega \)-sequence from \( H(\mu) \), cofinal in \( \beta \).
4. If \( \beta \in M_\alpha \) and \( \text{cf}^V(\beta) > \omega \) then for unboundedly many \( \gamma < \omega_1 \) there is \( \rho \in M_{\gamma+1} \) with \( \sup(\bigcup_{\xi<\gamma}(M_\xi \cap \beta)) < \rho < \beta \).

Let \( M = \bigcup_{\alpha<\omega_1} M_\alpha \), \( N \) the transitive collapse of \( M \), and \( \zeta < \omega_2 \) the ordinal \( N \cap \text{On} \). An ordinal in \( N \) has cofinality \( \omega \) in \( V \) iff it has cofinality \( \omega \) in \( N \). Thus \( (L_\xi')^N = L_\xi' \) for all \( \xi < \zeta \). Since \( N |= a \in C^* \), we have \( a \in L_\zeta' \). The claim follows. \( \square \)
Theorem

If there are infinitely many Woodin cardinals and a measurable cardinal above them, then there is a cone of reals $x$ such that $C^*(x)$ satisfies CH.
If two reals $x$ and $y$ are Turing-equivalent, then $C^*(x) = C^*(y)$. Hence the set

$$\{ y \subseteq \omega : C^*(y) \models CH \}$$

is closed under Turing-equivalence. Need to show that

(I) The set (1) is projective.

(II) For every real $x$ there is a real $y$ such that $x \leq_T y$ and $y$ is in the set (1).
Lemma

Suppose there is a Woodin cardinal and a measurable cardinal above it. The following conditions are equivalent:

(i) $C^*(y) \models CH$.

(ii) There is a countable iterable structure $M$ with a Woodin cardinal such that $y \in M$, $M \models \exists \alpha ("L'_\alpha(y) \models CH")$ and for all countable iterable structures $N$ with a Woodin cardinal such that $y \in N$: $\mathcal{P}(\omega)^{(C^*)_N} \subseteq \mathcal{P}(\omega)^{(C^*)_M}$. 
Consistency results about $C^*$, I

Theorem

Suppose $V = L$ and $\kappa$ is a cardinal of cofinality $> \omega$. There is a forcing notion $\mathbb{P}$ which forces $C^* \models 2^\omega = \kappa$ and preserves cardinals between $L$ and $C^*$. 
Consistency results about $C^*$, II

Theorem

*It is consistent, relative to the consistency of an inaccessible cardinal, that $V = C^*$ and $2^\aleph_0 = \aleph_2$.*
Stationary logic

Definition
\( \mathcal{M} \models \text{aaS}\varphi(s) \iff \{ A \in [\mathcal{M}]^{\leq \omega} : \mathcal{M} \models \varphi(A) \} \) contains a club of countable subsets of \( M \). (i.e. almost all countable subsets \( A \) of \( M \) satisfy \( \varphi(A) \).) We denote \( \neg\text{aaS}\neg\varphi \) by \( \text{stat}s\varphi \).

\[
C(aa) = C(\mathcal{L}(aa))
\]

\( C^* \subseteq C(aa) \)
Definition

1. A first order structure $\mathcal{M}$ is *club-determined* if

$$\mathcal{M} \models \forall \vec{s} \forall \vec{x}[\text{aa} \vec{t} \varphi(\vec{x}, \vec{s}, \vec{t}) \lor \text{aa} \vec{t} \neg \varphi(\vec{x}, \vec{s}, \vec{t})],$$

where $\varphi(\vec{x}, \vec{s}, \vec{t})$ is any formula of $L(\text{aa})$.

2. We say that the inner model $C(\text{aa})$ is *club-determined* if every level $L'_{\alpha}$ is.
Theorem

If there are a proper class of measurable Woodin cardinals or $\text{MM}^{++}$ holds, then $C(aa)$ is club-determined.

Proof.

Suppose $L'_\alpha$ is the least counter-example. W.l.o.g $\alpha < \omega_2^V$. Let $\delta$ be measurable Woodin, or $\omega_2$ in the case of $\text{MM}^{++}$. The hierarchies

$$C(aa)^M, C(aa)^{V[G]}, C(aa_{<\delta})^V$$

are all the same and the (potential) failure of club-determinateness occurs in all at the same level.
Lemma

1. If $\delta$ is Woodin, $S \subseteq \delta$ is in $M$ and $M$ thinks that $S$ is stationary, then $V[G]$ thinks that $S$ is stationary.

2. If $MM^{++}$ holds and $S$ is a set of countable subsets of $\omega_2^V$ in $M$ and $M$ thinks that $S$ is stationary, then $V$ thinks that $S$ is a stationary set of subsets of size $\leq \aleph_1^V$ of $\omega_2^V$. 
Theorem
Suppose there are a proper class of measurable Woodin cardinals or $\text{MM}^{++}$. Then every regular $\kappa \geq \kappa_1$ is measurable in $C(aa)$. 
Theorem

Suppose there are a proper class of measurable Woodin cardinals. Then the theory of $C(aa)$ is (set) forcing absolute.

Proof.

Suppose $\mathbb{P}$ is a forcing notion and $\delta$ is a Woodin cardinal $> |\mathbb{P}|$. Let $j : V \rightarrow M$ be the associated elementary embedding. Now

$$C(aa) \equiv (C(aa))^M = (C(aa_{<\delta}))^V.$$  

On the other hand, let $H \subseteq \mathbb{P}$ be generic over $V$. Then $\delta$ is still Woodin, so we have the associated elementary embedding $j' : V[H] \rightarrow M'$. Again

$$(C(aa))^V[H] \equiv (C(aa))^M' = (C(aa_{<\delta}))^{V[H]}.$$  

Finally, we may observe that $(C(aa_{<\delta}))^{V[H]} = (C(aa_{<\delta}))^V$. Hence

$$(C(aa))^V[H] \equiv (C(aa))^V.$$
Definition

$C(aa')$ is the extension of $C(aa)$ obtained by allowing “implicit” definitions.

- $C^* \subseteq C(aa) \subseteq C(aa')$.
- The previous results about $C(aa)$ hold also for $C(aa')$.

Theorem

If there is a proper class of Woodin cardinals, or $MM^{++}$, then $C(aa')$ satisfies CH (even $\diamond$).
Shelah’s stationary logic

Definition

\[ M \models Q^{St}xyz \varphi(x, \bar{a})\psi(y, z, \bar{a}) \] if and only if \((M_0, R_0)\), where

\[ M_0 = \{ b \in M : M \models \varphi(b, \bar{a}) \} \]

and

\[ R_0 = \{ (b, c) \in M : M \models \psi(b, c, \bar{a}) \}, \]

is an \(\aleph_1\)-like linear order and the set \(\mathcal{I}\) of initial segments of 

\((M_0, R_0)\) with an \(R_0\)-supremum in \(M_0\) is stationary in the set \(\mathcal{D}\) of all (countable) initial segments of \(M_0\) in the following sense:

If \(\mathcal{J} \subseteq \mathcal{D}\) is unbounded in \(\mathcal{D}\) and \(\sigma\)-closed in \(\mathcal{D}\), then \(\mathcal{J} \cap \mathcal{I} \neq \emptyset\).
• The logic $\mathcal{L}(Q^{St})$, a sublogic of $\mathcal{L}(aa)$, is recursively axiomatizable and $\aleph_0$-compact. We call this logic **Shelah’s stationary logic**, and denote $C(\mathcal{L}(Q^{St}))$ by $C(aa^-)$.

• We can say in the logic $\mathcal{L}(Q^{St})$ that a formula $\varphi(x)$ defines a stationary (in $V$) subset of $\omega_1$ in a transitive model $M$ containing $\omega_1$ as an element as follows:

$$M \models \forall x (\varphi(x) \rightarrow x \in \omega_1) \land Q^{St}xyz \varphi(x)(\varphi(y) \land \varphi(z) \land y \in z).$$

Hence

$$C(aa^-) \cap NS_{\omega_1} \in C(aa^-).$$
Theorem

If there is a Woodin cardinal or MM holds, then the filter
\( D = C(aa^-) \cap NS_{\omega_1} \) is an ultrafilter in \( C(aa^-) \) and

\[ C(aa^-) = L[D]. \]
Theorem

If there is a proper class of Woodin cardinals, then for all set forcings $P$ and generic sets $G \subseteq P$

$$Th(C(aa^-)^V) = Th(C(aa^-)^{V[G]})$$
We write

\[ \text{HOD}_1 = \text{df } C(\Sigma^1_1). \]

Note:

- \( \{ \alpha < \beta : \text{cf}^V(\alpha) = \omega \} \in \text{HOD}_1 \)
- \( \{ (\alpha, \beta) \in \gamma^2 : |\alpha|^V \leq |\beta|^V \} \in \text{HOD}_1 \)
- \( \{ \alpha < \beta : \alpha \text{ cardinal in } V \} \in \text{HOD}_1 \)
- \( \{ (\alpha_0, \alpha_1) \in \beta^2 : |\alpha_0|^V \leq (2^{\alpha_1})^V \} \in \text{HOD}_1 \)
- \( \{ \alpha < \beta : (2^{\alpha})^V = (|\alpha|^+)^V \} \in \text{HOD}_1 \)
Lemma

1. $C^* \subseteq HOD_1$.
2. $C(Q^{MM,<_\omega}_1) \subseteq HOD_1$
3. If $0^\#$ exists, then $0^\# \in HOD_1$
Theorem

It is consistent, relative to the consistency of infinitely many weakly compact cardinals that for some $\lambda$:

$$\{\kappa < \lambda : \kappa \text{ weakly compact (in } V)\} \notin HOD_1,$$

and, moreover, $HOD_1 = L \neq HOD$. 
Open questions

- $C^*$ has small large cardinals, is forcing absolute (assuming PCW).
- **OPEN**: Can $C^*$ have a measurable cardinal?
- $C^*$ has some elements of GCH
- **OPEN**: Does $C^*$ satisfy CH if large cardinals are present?
- $C^{(aa)}$ has measurable cardinals.
- **OPEN**: Bigger cardinals in $C^{(aa)}$?
- $C^{(aa)}$ satisfies CH.
- **OPEN**: Does $C^{(aa)}$ satisfy GCH?
Thank you!

Happy Birthday Menachem!