Inner models from extended logics

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February 2016

(Corrections made 2023)

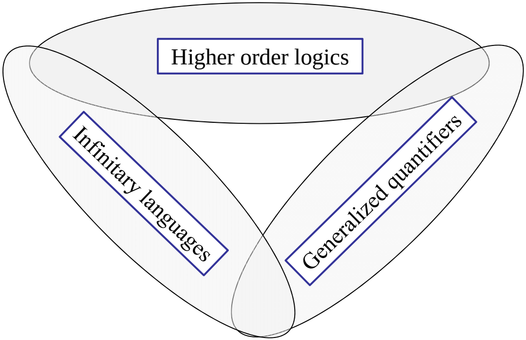
Constructible hierarchy generalized

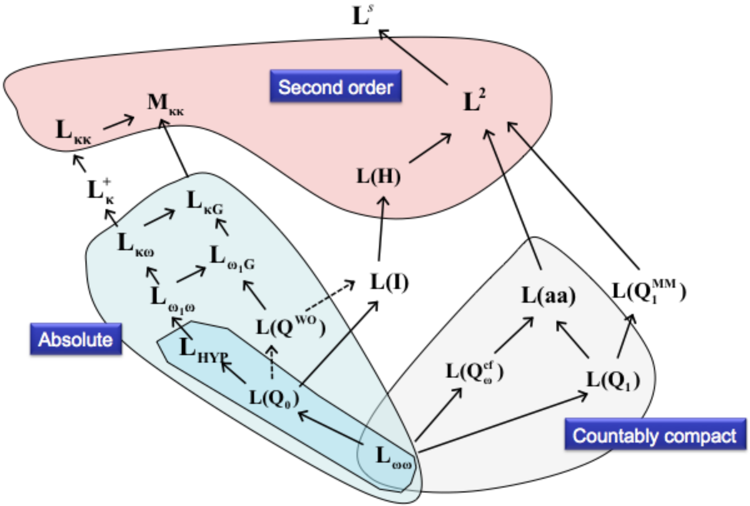
$$
\begin{array}{lcl}
L'_0 & = & \emptyset \\
L'_{\alpha+1} & = & \mathrm{Def}_{\mathcal{L}^*}(L'_\alpha) \\
L'_\nu & = & \bigcup_{\alpha<\nu} L'_\alpha \text{ for limit } \nu
\end{array}
$$

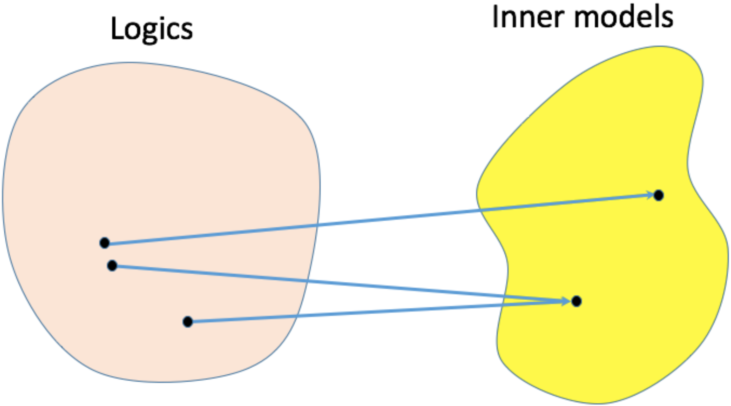
We use $C(\mathcal{L}^*)$ to denote the class $\bigcup_\alpha L'_\alpha$.

Thus a typical set in $L'_{\alpha+1}$ has the form

$$X = \{a \in L'_\alpha : (L'_\alpha, \in) \models \varphi(a, \vec{b})\}$$


Higher order logics
Infinitary languages
Generalized quantifiers


L^s
Second order
$M_{\kappa\kappa}$
$L_{\kappa\kappa}$
L^2
$L(H)$
L^+_κ
$L_{\kappa G}$
$L_{\kappa\omega}$
$L_{\omega_1 G}$
$L(I)$
$L(aa)$
$L(Q_1^{MM})$
$L_{\omega_1\omega}$
$L(Q^{WO})$
Absolute
L_{HYP}
$L(Q^{cf}_\omega)$
$L(Q_1)$
$L(Q_0)$
Countably compact
$L_{\omega\omega}$


Logics
Inner models

Examples

- $C(\mathcal{L}_{\omega\omega}) = L$
- $C(\mathcal{L}_{\omega_1\omega}) = L(\mathbb{R})$
- $C(\mathcal{L}_{\omega_1\omega_1}) =$ Chang model
- $C(\mathcal{L}^2) = \mathrm{HOD}$

Possible attributes of inner models

- Forcing absolute.
- Support large cardinals.
- Satisfy Axiom of Choice.
- Arise “naturally".
- Decide questions such as CH.

Inner models we have

- L: Forcing-absolute but no large cardinals (above WC)
- HOD: Has large cardinals but forcing-fragile
- $L(\mathbb{R})$: Forcing-absolute, has large cardinals, but no AC
- Extender models: Tailor made to support given large cardinals

Absolute logics—nothing new

Theorem

Suppose $\mathcal{L}^*$ is ZFC[redacted]-absolute with parameters from L, and the syntax of $\mathcal{L}^*$ is ZFC[redacted]-absolute with parameters from L. Then $C(\mathcal{L}^*) = L$.

Corollary

$C(\mathcal{L}(Q_\alpha)) = L$

Magidor-Malitz quantifier

Definition

Magidor-Malitz quantifier of dimension n:

$$\mathcal{M} \models Q^{\mathrm{MM},n}_{\alpha} x_1, \ldots, x_n \varphi(x_1, \ldots, x_n) \iff$$

$$\exists X \subseteq M(|X| \geq \aleph_\alpha \wedge \forall a_1, \ldots, a_n \in X : \mathcal{M} \models \varphi(a_1, \ldots, a_n)).$$

Magidor-Malitz quantifier, assuming $0^\sharp$

Consistently, $C(Q_1^{\mathrm{MM},2}) \neq L$, but:

Theorem

If $0^\sharp$ exists, then $C(Q_\alpha^{MM,<\omega}) = L$.

Lemma

Suppose $0^\sharp$ exists and $A \in L$, $A \subseteq [\alpha]^2$. If there is an uncountable B such that $[B]^2 \subseteq A$, then there is such a set B in L.

Shelah's cofinality quantifier

Definition

The cofinality quantifier Q^{cf}_{ω} is defined as follows:

$$\mathcal{M} \models Q^{\mathrm{cf}}_{\omega} xy\varphi(x, y, \vec{a}) \iff \{(c, d) : \mathcal{M} \models \varphi(c, d, \vec{a})\}$$

is a linear order of cofinality ω.

- Axiomatizable
- Fully compact
- Downward Löwenheim-Skolem down to $\aleph_1$

The “cof-model" C^*

Definition

$$C^* =_{def} C(Q_\omega^{\mathrm{cf}})$$

Example:

$$\{\alpha < \beta : \mathrm{cf}^V(\alpha) > \omega\} \in C^*$$

Theorem

If $0^\sharp$ exists, then $0^\sharp \in C^*$.

Proof.

Let

$$X = \{\xi < \aleph_\omega : \xi \text{ is a regular cardinal in } L \text{ and } \mathrm{cf}(\xi) > \omega\}$$

Now $X \in C^*$ and

$$0^\sharp = \{\ulcorner\varphi(x_1, ..., x_n)\urcorner : L_{\aleph_\omega} \models \varphi(\gamma_1, ..., \gamma_n) \text{ for some } \gamma_1 < ... < \gamma_n \text{ in } X\}.$$

$\square$

- More generally, the above argument shows that $x^\sharp \in C^*(x)$ for any $x \in C^*$ such that $x^\sharp$ exists.
- Hence $C^* \neq L(x)$ whenever x is a set of ordinals such that $x^\sharp$ exists in V.

Theorem

The Dodd-Jensen Core model is contained in C^*.

Theorem

Suppose L^μ exists. Then some L^ν is contained in C^*.

Theorem

If there is a measurable cardinal κ, then $V \neq C^*$.

Proof.

Suppose $V = C^*$ and κ is a measurable cardinal. Let $i : V \to M$ with critical point κ and $M^\kappa \subseteq M$. Now $(C^*)^M = (C^*)^V = V$, whence $M = V$. This contradicts Kunen's result that there cannot be a non-trivial $i : V \to V$. $\square$

Theorem

If there is an infinite set E of measurable cardinals (in V), then $E \notin C^*$. Moreover, then $C^* \neq \mathrm{HOD}$.

Proof.

As Kunen's result that if there are uncountably many measurable cardinals, then AC is false in the Chang model. □

Stationary Tower Forcing

Suppose λ is Woodin.

- There is a forcing $\mathbb{Q}$ such that in $V[G]$ there is $j : V \to M$ with $V[G] \models M^\omega \subseteq M$ and $j(\omega_1) = \lambda$.
- For all regular $\omega_1 < \kappa < \lambda$ there is a cofinality ω preserving forcing $\mathbb{P}$ such that in $V[G]$ there is $j : V \to M$ with $V[G] \models M^\omega \subseteq M$ and $j(\kappa) = \lambda$.

Theorem

If there is a Woodin cardinal, then ω_1 is (strongly) Mahlo in C^*.

Proof.

Let $\mathbb{Q}$, G and $j : V \to M$ with $M^\omega \subset M$ and $j(\omega_1) = \lambda$ be as above.
Now,

$$(C^*)^M = C^*_{<\lambda} \subseteq V.$$

$\square$

Theorem

Suppose there is a Woodin cardinal λ. Then every regular cardinal κ such that $\omega_1 < \kappa < \lambda$ is weakly compact in C^*.

Proof.

Suppose λ is a Woodin cardinal, $\kappa > \omega_1$ is regular and $< \lambda$. To prove that κ is strongly inaccessible in C^* we can use the "second" stationary tower forcing $\mathbb{P}$ above. With this forcing, cofinality ω is not changed, whence $(C^*)^M = C^*$. $\square$

Theorem

If there is a proper class of Woodin cardinals, then the regular cardinals $\geq \aleph_2$ are indiscernible in C^*.

Proof.

We use the “second" stationary tower forcing $\mathbb{P}$ to show first that the Woodin cardinals are indiscernible, and after that the regular cardinals $\geq \aleph_2$ are indiscernible. Remember that the here $\mathbb{P}$ and j preserve C^*. $\square$

Theorem

If $V = L^\mu$, then C^* is exactly the inner model $M_{\omega^2}[E]$, where M_{ω^2} is the ω^2th iterate of V and $E = \{\kappa_{\omega\cdot n} : n < \omega\}$.

Proof.

1. $C^* \subseteq M_{\omega^2}[E]$: In $M_{\omega^2}[E]$ we can detect which ordinals have cofinality ω in V.
2. $M_{\omega^2}[E] \subseteq C^*$: The set E is the set of ordinals $< \kappa_{\omega^2}$ which have cofinality ω in V but are regular in the core model. The measure $i_{0\omega^2}(\mu)$ on κ_{ω^2} can be defined from E by $\mu'(X) = 1$ if and only if $\exists\alpha \in E \forall\beta \in E(\alpha < \beta \to \beta \in X)\}$.

□

Theorem

Suppose there is a proper class of Woodin cardinals. Suppose $\mathcal{P}$ is a forcing notion and $G \subseteq \mathcal{P}$ is generic. Then

$$Th((C^*)^V) = Th((C^*)^{V[G]}).$$

Proof.

Let H_1 be generic for $\mathbb{Q}$. Now

$$j_1 : (C^*)^V \to (C^*)^{M_1} = (C^*)^{V[H_1]} = (C^*_{<\lambda})^V.$$

Let H_2 be generic for $\mathbb{Q}$ over $V[G]$. Then

$$j_2 : (C^*)^{V[G]} \to (C^*)^{M_2} = (C^*)^{V[H_2]} = (C^*_{<\lambda})^{V[G]} = (C^*_{<\lambda})^V.$$

$\square$

Theorem

$|\mathcal{P}(\omega) \cap C^*| \leq \aleph_2$.

Proof.

Suppose $a \subseteq \omega$ and $a \in C^*$. We build $(M_\alpha)_{\alpha<\omega_1}$ such that

1. $a \in M_0$, $M_0 \models a \in C^*$, $|M_\alpha| \leq \omega$, $M_\alpha \prec H(\mu)$.
2. $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$, if $\gamma = \cup\gamma$.
3. If $\beta \in M_\alpha$ and $\mathrm{cf}^V(\beta) = \omega$, then $M_{\alpha+1}$ contains an ω-sequence from $H(\mu)$, cofinal in β.
4. If $\beta \in M_\alpha$ and $\mathrm{cf}^V(\beta) > \omega$ then for unboundedly many $\gamma < \omega_1$ there is $\rho \in M_{\gamma+1}$ with $\sup(\bigcup_{\xi<\gamma}(M_\xi \cap \beta)) < \rho < \beta$.

Let M be $\bigcup_{\alpha<\omega_1} M_\alpha$, N the transitive collapse of M, and $\zeta < \omega_2$ the ordinal $N \cap On$. An ordinal in N has cofinality ω in V iff it has cofinality ω in N. Thus $(L'_\xi)^N = L'_\xi$ for all $\xi < \zeta$. Since $N \models a \in C^*$, we have $a \in L'_\zeta$. The claim follows. $\square$

Theorem

If there are infinitely many Woodin cardinals and a measurable cardinal above them, then there is a cone of reals x such that $C^*(x)$ satisfies CH.

If two reals x and y are Turing-equivalent, then $C^*(x) = C^*(y)$. Hence the set

$$\{y \subseteq \omega : C^*(y) \models CH\} \tag{1}$$

is closed under Turing-equivalence. Need to show that

(I) The set (1) is projective.

(II) For every real x there is a real y such that $x \leq_T y$ and y is in the set (1).

Lemma

Suppose there is a Woodin cardinal and a measurable cardinal above it. The following conditions are equivalent:

(i) $C^*(y) \models CH$.

(ii) *There is a countable iterable structure M with a Woodin cardinal such that $y \in M$, $M \models \exists\alpha(\text{“}L'_\alpha(y) \models CH\text{”})$ and for all countable iterable structures N with a Woodin cardinal such that $y \in N$: $\mathcal{P}(\omega)^{(C^*)^N} \subseteq \mathcal{P}(\omega)^{(C^*)^M}$.*

Consistency results about C^*, I

Theorem

Suppose $V = L$ and κ is a cardinal of cofinality $> \omega$. There is a forcing notion $\mathbb{P}$ which forces $C^* \models 2^\omega = \kappa$ and preserves cardinals between L and C^*.

Consistency results about C^*, II

Theorem

It is consistent, relative to the consistency of an inaccessible cardinal, that $V = C^*$ and $2^{\aleph_0} = \aleph_2$.

Stationary logic

Definition

$\mathcal{M} \models \mathtt{aa}\, s\varphi(s) \iff \{A \in [M]^{\leq\omega} : \mathcal{M} \models \varphi(A)\}$ contains a club of countable subsets of M. (i.e. almost all countable subsets A of M satisfy $\varphi(A)$.) We denote $\neg\mathtt{aa}\, s\neg\varphi$ by $\mathtt{stat}\, s\varphi$.

$$C(aa) = C(\mathcal{L}(\mathtt{aa}))$$

$$C^* \subseteq C(aa)$$

Definition

1. A first order structure $\mathcal{M}$ is *club-determined* if

$$\mathcal{M} \models \forall \vec{s} \forall \vec{x} [\mathtt{aa} \vec{t} \varphi(\vec{x}, \vec{s}, \vec{t}) \vee \mathtt{aa} \vec{t} \neg \varphi(\vec{x}, \vec{s}, \vec{t})],$$

   where $\varphi(\vec{x}, \vec{s}, \vec{t})$ is any formula of $\mathcal{L}(\mathtt{aa})$.
2. We say that the inner model $C(aa)$ is *club-determined* if every level L'_α is.

Theorem

If there are a proper class of [redacted] *Woodin cardinals or* MM^{++} *holds, then $C(aa)$ is club-determined.*

Proof.

Suppose L'_α is the least counter-example. W.l.o.g $\alpha < \omega_2^V$. Let δ be [redacted] Woodin, or ω_2 in the case of MM^{++}. The hierarchies

$$C(aa)^M, C(aa)^{V[G]}, C(aa_{<\delta})^V$$

are all the same and the (potential) failure of club-determinateness occurs in all at the same level. $\square$

Lemma

1. *If δ is [redacted] Woodin, $S \subseteq \delta$ is in M and M thinks that S is stationary, then $V[G]$ thinks that S is stationary.*
2. *If MM^{++} holds and S is a set of countable subsets of ω_2^V in M and M thinks that S is stationary, then V thinks that S is a stationary set of subsets of size $\leq \aleph_1^V$ of ω_2^V.*

Theorem

Suppose there are a proper class of [redacted] Woodin cardinals or MM^{++}. Then every regular $\kappa \geq \aleph_1$ is measurable in $C(aa)$.

Theorem

Suppose there are a proper class of [redacted] Woodin cardinals. Then the theory of $C(aa)$ is (set) forcing absolute.

Proof.

Suppose $\mathbb{P}$ is a forcing notion and δ is a Woodin cardinal $> |\mathbb{P}|$. Let $j : V \to M$ be the associated elementary embedding. Now

$$C(aa) \equiv (C(aa))^M = (C(aa_{<\delta}))^V.$$

On the other hand, let $H \subseteq \mathbb{P}$ be generic over V. Then δ is still Woodin, so we have the associated elementary embedding $j' : V[H] \to M'$. Again

$$(C(aa))^{V[H]} \equiv (C(aa))^{M'} = (C(aa_{<\delta}))^{V[H]}.$$

Finally, we may observe that $(C(aa_{<\delta}))^{V[H]} = (C(aa_{<\delta}))^V$. Hence

$$(C(aa))^{V[H]} \equiv (C(aa))^V.$$

Definition

$C(aa')$ is the extension of $C(aa)$ obtained by allowing “implicit" definitions.

- $C^* \subseteq C(aa) \subseteq C(aa')$.
- The previous results about $C(aa)$ hold also for $C(aa')$.

Theorem

If there is a proper class of [redacted] Woodin cardinals, or MM^{++}, then $C(aa')$ satisfies CH (even $\Diamond$).

Shelah's stationary logic

Definition

$\mathcal{M} \models Q^{St} xyz\varphi(x,\vec{a})\psi(y,z,\vec{a})$ if and only if (M_0, R_0), where

$$M_0 = \{b \in M : \mathcal{M} \models \varphi(b,\vec{a})\}$$

and

$$R_0 = \{(b,c) \in M : \mathcal{M} \models \psi(b,c,\vec{a})\},$$

is an $\aleph_1$-like linear order and the set $\mathcal{I}$ of initial segments of (M_0, R_0) with an R_0-supremum in M_0 is stationary in the set $\mathcal{D}$ of all (countable) initial segments of M_0 in the following sense: If $\mathcal{J} \subseteq \mathcal{D}$ is unbounded in $\mathcal{D}$ and σ-closed in $\mathcal{D}$, then $\mathcal{J} \cap \mathcal{I} \neq \emptyset$.

- The logic $\mathcal{L}(Q^{St})$, a sublogic of $\mathcal{L}(\mathtt{aa})$, is recursively axiomatizable and $\aleph_0$-compact. We call this logic *Shelah's stationary logic*, and denote $C(\mathcal{L}(Q^{St}))$ by $C(\mathtt{aa}^-)$.
- We can say in the logic $\mathcal{L}(Q^{St})$ that a formula $\varphi(x)$ defines a stationary (in V) subset of ω_1 in a transitive model M containing ω_1 as an element as follows:

$$M \models \forall x(\varphi(x) \to x \in \omega_1) \wedge Q^{St}xyz\varphi(x)(\varphi(y) \wedge \varphi(z) \wedge y \in z).$$

Hence

$$C(\mathtt{aa}^-) \cap \mathrm{NS}_{\omega_1} \in C(\mathtt{aa}^-).$$

Theorem

If there is a Woodin cardinal or MM holds, then the filter $D = C(\mathtt{aa}^-) \cap \mathrm{NS}_{\omega_1}$ is an ultrafilter in $C(\mathtt{aa}^-)$ and

$$C(\mathtt{aa}^-) = L[D].$$

Theorem

If there is a proper class of Woodin cardinals, then for all set forcings P and generic sets $G \subseteq P$

$$Th(C(\mathtt{aa}^-)^V) = Th(C(\mathtt{aa}^-)^{V[G]}).$$

We write

$$\mathrm{HOD}_1 =_{\mathrm{df}} C(\Sigma^1_1).$$

Note:

- $\{\alpha < \beta : \mathrm{cf}^V(\alpha) = \omega\} \in \mathrm{HOD}_1$
- $\{(\alpha, \beta) \in \gamma^2 : |\alpha|^V \leq |\beta|^V\} \in \mathrm{HOD}_1$
- $\{\alpha < \beta : \alpha \text{ cardinal in } V\} \in \mathrm{HOD}_1$
- $\{(\alpha_0, \alpha_1) \in \beta^2 : |\alpha_0|^V \leq (2^{|\alpha_1|})^V\} \in \mathrm{HOD}_1$
- $\{\alpha < \beta : (2^{|\alpha|})^V = (|\alpha|^+)^V\} \in \mathrm{HOD}_1$

Lemma

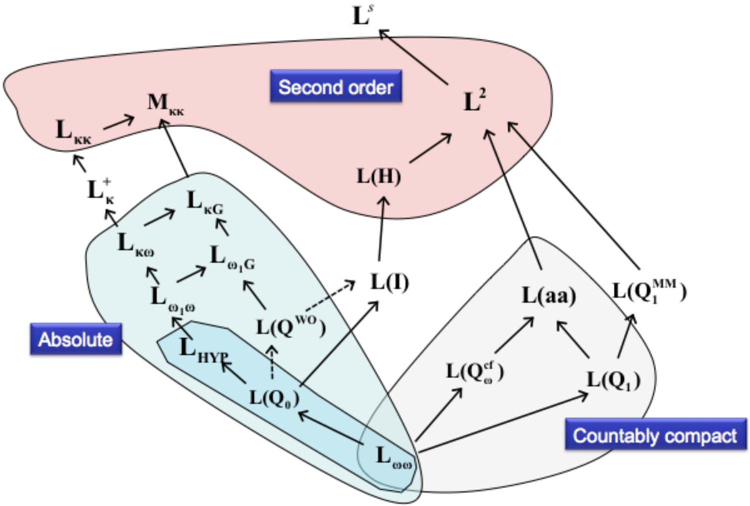
1. $C^* \subseteq \mathrm{HOD}_1$.
2. $C(Q_1^{MM,<\omega}) \subseteq \mathrm{HOD}_1$
3. *If* $0^\sharp$ *exists, then* $0^\sharp \in \mathrm{HOD}_1$

Theorem

It is consistent, relative to the consistency of infinitely many weakly compact cardinals that for some λ:

$$\{\kappa < \lambda : \kappa \text{ weakly compact (in } V)\} \notin \mathrm{HOD}_1,$$

and, moreover, $\mathrm{HOD}_1 = L \neq \mathrm{HOD}$.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L(Q_1)
Countably compact
L_ωω

Open questions

- C^* has small large cardinals, is forcing absolute (assuming PCW).
- OPEN: Can C^* have a measurable cardinal?
- C^* has some elements of GCH
- OPEN: Does C^* satisfy CH if large cardinals are present?
- $C(\mathtt{aa})$ has measurable cardinals.
- OPEN: Bigger cardinals in $C(\mathtt{aa})$?
- $C(\mathtt{aa})$ satisfies CH.
- ~~OPEN~~: Does $C(\mathtt{aa})$ satisfy GCH?

Thank you!

Happy Birthday Menachem!