THE JOURNAL OF SYMBOLIC LOGIC
Volume 55, Number 3, Sept. 1990

ON SCOTT AND KARP TREES OF UNCOUNTABLE MODELS

TAPANI HYTTINEN AND JOUKO VÄÄNÄNEN

Abstract. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two countable relational models of the same first order language. If the models are nonisomorphic, there is a unique countable ordinal α with the property that

$$\mathfrak{A} \equiv^{\alpha}_{\infty\omega} \mathfrak{B} \text{ but not } \mathfrak{A} \equiv^{\alpha+1}_{\infty\omega} \mathfrak{B},$$

i.e. $\mathfrak{A}$ and $\mathfrak{B}$ are $L_{\infty\omega}$-equivalent up to quantifier-rank α but not up to $\alpha + 1$. In this paper we consider models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality ω_1 and construct trees which have a similar relation to $\mathfrak{A}$ and $\mathfrak{B}$ as α above. For this purpose we introduce a new ordering $T \ll T'$ of trees, which may have some independent interest of its own. It turns out that the above ordinal α has two qualities which coincide in countable models but will differ in uncountable models. Respectively, two kinds of trees emerge from α. We call them Scott trees and Karp trees, respectively. The definition and existence of these trees is based on an examination of the Ehrenfeucht game of length ω_1 between $\mathfrak{A}$ and $\mathfrak{B}$. We construct two models of power ω_1 with 2^{ω_1} mutually noncomparable Scott trees.

§1. Introduction. It is well known that ω_1-like dense linear orderings without endpoints are $L_{\infty\omega_1}$-equivalent but not necessarily isomorphic. We provide a framework for measuring to what extent such nonisomorphic structures of cardinality ω_1 are "similar".

In the range of countable structures we can measure "similarity" with countable ordinals by looking for the largest ordinal α for which the models are in the relation $\equiv^{\alpha}_{\infty\omega}$. We show in this paper that when we move on to uncountable models, the notion $\equiv^{\alpha}_{\infty\omega}$ splits in a natural way into two different notions: Scott trees and Karp trees. Moreover we show that while we can order trees like Scott and Karp trees in a way which remotely resembles the ordering of ordinal numbers, the situation is still fundamentally different: for example, the "nonsimilarity" of some uncountable models can be measured with two noncomparable Scott trees.

The structure of this paper is as follows. §1 contains some necessary prerequisites about Ehrenfeucht games. §2 presents basic properties of a game-theoretically oriented ordering of trees. This paves the way to the definition and existence of Karp and Scott trees presented in §3. §4 is devoted to a detailed study of some game-theoretic properties of stationary sets. §5 contains calculations of Scott trees of certain dense ω_1-like linear orderings.

Received March 13, 1989.

0022-4812/90/5503-0001/$02.20

Throughout the paper $\mathfrak{A}$ and $\mathfrak{B}$ denote structures of cardinality ω_1 for a fixed finite finitary relational vocabulary L. We assume $A \cap B = \varnothing$ for simplicity. We use $G(\mathfrak{A}, \mathfrak{B})$ to denote the *Ehrenfeucht game* of length ω_1 associated with $\mathfrak{A}$ and $\mathfrak{B}$. This game is defined as follows. There are two players $\forall$ and $\exists$, and both make ω_1 moves. Player $\forall$ always moves first. The players pick elements of the models $\mathfrak{A}$ and $\mathfrak{B}$. We use x_α to denote the element chosen by $\forall$ and y_α to denote the element chosen by $\exists$ at move α. The only rules of the game are:

$$x_\alpha \in A \Rightarrow y_\alpha \in B \quad \text{and} \quad x_\alpha \in B \Rightarrow y_\alpha \in A.$$

Suppose a sequence $(x_\alpha, y_\alpha)_{\alpha < \omega_1}$ is played in $G(\mathfrak{A}, \mathfrak{B})$. Player $\exists$ *wins* if for all atomic L-formulas $\varphi(x_0, \dots, x_n)$ and all $\alpha_0, \dots, \alpha_n < \omega_1$ we have

$$(*) \qquad \mathfrak{A} \models \varphi(a_{\alpha_0}, \dots, a_{\alpha_n}) \Leftrightarrow \mathfrak{B} \models \varphi(b_{\alpha_0}, \dots, b_{\alpha_n}),$$

where

$$a_\alpha = \begin{cases} x_\alpha & \text{if } x_\alpha \in A \\ y_\alpha & \text{if } x_\alpha \in B \end{cases} \quad \text{and} \quad b_\alpha = \begin{cases} x_\alpha & \text{if } x_\alpha \in B, \\ y_\alpha & \text{if } x_\alpha \in A. \end{cases}$$

Otherwise player $\forall$ *wins*.

A *strategy of player* $\exists$ in $G(\mathfrak{A}, \mathfrak{B})$ is a sequence $S = (f_\alpha(x_0, \dots, x_\alpha))_{\alpha < \omega_1}$ of functions on $A \cup B$ such that the moves $y_\alpha = f_\alpha(x_0, \dots, x_\alpha)$ are legal moves of $\exists$ in the game. This strategy is a *winning strategy* if for all sequences $(x_\alpha)_{\alpha < \omega_1}$ from $A \cup B$, for all atomic L-formulas $\varphi(x_0, \dots, x_n)$ and all $\alpha_0, \dots, \alpha_n < \omega_1$, $(*)$ holds.

A *strategy of player* $\forall$ in $G(\mathfrak{A}, \mathfrak{B})$ is a sequence $S = (f_\alpha(y_0, \dots, y_\beta, \dots)_{\beta < \alpha})_{\alpha < \omega_1}$ of functions on $A \cup B$. This is a *winning strategy* if for all sequences $(y_\alpha)_{\alpha < \omega_1}$ of legal answers of $\exists$ to the moves $x_\alpha = f_\alpha(y_0, \dots, y_\beta, \dots)_{\beta < \alpha}$ of $\forall$ there are $\alpha_0, \dots, \alpha_n < \omega_1$ and an atomic φ such that $(*)$ above fails.

LEMMA 1.1 ([Kar]). *The following conditions are equivalent*:

(1) $\mathfrak{A} \cong \mathfrak{B}$.

(2) *Player* $\exists$ *has a winning strategy in* $G(\mathfrak{A}, \mathfrak{B})$,

(3) *Player* $\forall$ *does not have a winning strategy in* $G(\mathfrak{A}, \mathfrak{B})$.

PROOF. If (1) holds, the winning strategy of $\exists$ in $G(\mathfrak{A}, \mathfrak{B})$ is based on obeying the isomorphism. That is, if $\pi\colon \mathfrak{A} \cong \mathfrak{B}$, then $\exists$ lets

$$f_\alpha(x_0, \dots, x_\alpha) = \begin{cases} \pi(x_\alpha) & \text{if } x_\alpha \in A, \\ \pi^{-1}(x_\alpha) & \text{if } x_\alpha \in B. \end{cases}$$

If (1) does not hold, then the winning strategy of $\forall$ in $G(\mathfrak{A}, \mathfrak{B})$ is based on enumerating both universes during the game. That is, if $A \cup B = \{z_\alpha \mid \alpha < \omega_1\}$, then $\forall$ lets $f_\alpha(y_0, \dots, y_\beta, \dots)_{\beta < \alpha} = z_\alpha$. Player $\exists$ cannot win such a game, for otherwise we would have (1) true. QED.

REMARK. Hintikka and Rantala [Hin] define a so-called *constituent* $C_{\mathfrak{A}}$ for a model $\mathfrak{A}$ of power ω_1. (In fact their approach is more general.) The constituent $C_{\mathfrak{A}}$ is an infinitary sentence whose syntax tree looks like the syntax tree of an $L_{\infty\omega}$-sentence, except that the tree has height ω_1. The semantics of $C_{\mathfrak{A}}$ is defined via the usual semantical game. In fact, $C_{\mathfrak{A}}$ is the natural game-sentence which renders condition (2) of Lemma 1.1 equivalent to $\mathfrak{B} \models C_{\mathfrak{A}}$. There is a dual form $D_{\mathfrak{A}}$ of $C_{\mathfrak{A}}$ with (3) of Lemma 1.1 equivalent to not $\mathfrak{B} \models D_{\mathfrak{A}}$.

By a *tree* we mean any partial ordering in which every element has a unique well-ordered set of predecessors. Let T be a tree. We use $G(\mathfrak{A}, \mathfrak{B}, T)$ to denote the modified Ehrenfeucht game in which player $\forall$ has to go up the tree T move by move. Thus during the game $G(\mathfrak{A}, \mathfrak{B}, T)$ player $\forall$ moves pairs $z_\alpha = (x_\alpha, t_\alpha)$, where $x_\alpha \in A \cup B$ and $t_\alpha \in T$, and player $\exists$ responds with elements y_α. The rules of $G(\mathfrak{A}, \mathfrak{B}, T)$ are the rules of $G(\mathfrak{A}, \mathfrak{B})$. Player $\exists$ *wins* in $G(\mathfrak{A}, \mathfrak{B}, T)$ if he has not lost the associated $G(\mathfrak{A}, \mathfrak{B})$ before the sequence $(t_\alpha)_{\alpha<\omega_1}$ fails to be an ascending chain in T.

Thus a *strategy* of $\exists$ in $G(\mathfrak{A}, \mathfrak{B}, T)$ is a sequence $S = (f_\alpha(z_0, \ldots, z_\alpha))_{\alpha<\omega_1}$ of functions with the property that if $z_\alpha = (x_\alpha, t_\alpha)$, where $x_\alpha \in A \cup B$ and $t_\alpha \in T$, then $(f_\alpha(z_0, \ldots, z_\alpha))_{\alpha<\omega_1}$ is a legal sequence of moves of $\exists$ in $G(\mathfrak{A}, \mathfrak{B})$. This strategy is a *winning strategy* if for all sequences $(x_\alpha)_{\alpha<\beta}$, $\beta < \omega_1$, from $A \cup B$, for all atomic $\varphi(x_0, \ldots, x_n)$, all ascending sequences $(t_\alpha)_{\alpha<\beta}$ from T and all $\alpha_0, \ldots, \alpha_n < \beta$ we have

$$\mathfrak{A} \models \varphi(a_{\alpha_0}, \ldots, a_{\alpha_n}) \Leftrightarrow \mathfrak{B} \models \varphi(b_{\alpha_0}, \ldots, b_{\alpha_n}),$$

where

$$a_\alpha = \begin{cases} x_\alpha & \text{if } x_\alpha \in A, \\ f_\alpha((x_0, t_0), \ldots, (x_\alpha, t_\alpha)) & \text{if } x_\alpha \in B, \end{cases}$$

and

$$b_\alpha = \begin{cases} x_\alpha & \text{if } x_\alpha \in B, \\ f_\alpha((x_0, t_0), \ldots, (x_\alpha, t_\alpha)) & \text{if } x_\alpha \in A. \end{cases}$$

A *strategy* of $\forall$ in $G(\mathfrak{A}, \mathfrak{B}, T)$ is a sequence

$$S = ((t_\alpha(y_0, \ldots, y_\beta, \ldots)_{\beta<\alpha}, f_\alpha(y_0, \ldots, y_\beta, \ldots)_{\beta<\alpha}))_{\alpha<\omega_1}$$

of pairs such that $t_\alpha(y_0, \ldots, y_\beta, \ldots)_{\beta<\alpha} \in T$ whenever $(y_0, \ldots, y_\beta, \ldots)$ is a legal sequence of moves of $\exists$. This is a *winning strategy* if for any sequence $(y_\alpha)_{\alpha<\omega_1}$ of legal answers of $\exists$ to the moves $x_\alpha = f_\alpha(y_0, \ldots, y_\beta, \ldots)_{\beta<\alpha}$ of $\forall$ there are $\alpha_0, \ldots, \alpha_n < \omega_1$ and an atomic φ such that $(*)$ above fails and the sequence $((t_\alpha(y_0, \ldots, y_\beta, \ldots)_{\beta<\alpha})_{\alpha\le\gamma}$, $\gamma = \max\{\alpha_0, \ldots, \alpha_n\}$, is an ascending chain in T.

Note that $G(\mathfrak{A}, \mathfrak{B})$, as well as any $G(\mathfrak{A}, \mathfrak{B}, T)$ where T has an uncountable branch, are by Lemma 1.1 always determined.

If $\alpha \in \mathrm{On}$, let B_α be the tree of descending sequences of elements of α. Then by classical results (see, for example, [Bar]), $G(\mathfrak{A}, \mathfrak{B}, B_\alpha)$ is determined and $\exists$ has a winning strategy iff $\mathfrak{A} \equiv^\alpha_{\infty\omega} \mathfrak{B}$. Let B_∞ be the tree with just one branch, which has order-type ω. Then $G(\mathfrak{A}, \mathfrak{B}, B_\infty)$ is determined and $\exists$ has a winning strategy iff $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$. However, if T contains longer (countable) branches, the game need not be determined, as the remark after Proposition 5.1 shows.

We are indebted to Taneli Huuskonen for pointing out a mistake in an earlier version of this paper and for suggesting many improvements.

§2. A game-theoretic ordering of trees. We define a relation $T \ll T^*$ between trees. This relation enables us to think of trees as generalized ordinals. The whole idea of this paper is to try to use trees with no uncountable branches as invariants of uncountable models in the same way as trees with no infinite branches (i.e. ordinals) are used as invariants of countable models.

Let T and T^* be two trees. We say that T is *order-embeddable* in T^*, $T \leq T^*$, if there is an order-preserving $f\colon T \to T^*$. For example, $B_\alpha \leq B_\beta$ if and only if $\alpha \leq \beta$. We write $T < T^*$, if $T \leq T^*$ but $T^* \not\leq T$, and $T \equiv T^*$ if $T \leq T^*$ and $T^* \leq T$.

Let σT denote the tree of all initial segments of branches of T. Kurepa has proved that $T < \sigma T$ (see 2.2, below). Note that $\sigma B_\alpha \equiv B_{\alpha+1}$.

Let $G_1(T, T^*)$ denote the game in which $\forall$ picks elements of an ascending chain $u_0 < \cdots < u_\alpha < \cdots$ in T^* and $\exists$ has to respond with elements $t_0 < \cdots < t_\alpha < \cdots$ of T. If a player cannot move, the other player is declared the *winner*. Clearly, $T^* \leq T$ if and only if $\exists$ has a winning strategy $t_\alpha = f_\alpha(u_0, \ldots, u_\alpha)$ in this game. To define our game-theoretic ordering of trees, we write $T \ll T^*$ if $\forall$ has a winning strategy $u_\alpha = g_\alpha(t_0, \ldots, t_\beta, \ldots)_{\beta<\alpha}$ in $G_1(T, T^*)$. Note that the length of the game is only limited by the heights of T and T^*. In Proposition 4.6 we shall have an example of a nondetermined game $G_1(T, T^*)$.

LEMMA 2.1. *If $T \ll T^*$, then $T < T^*$.*

PROOF. Let $t \in T$ be given. We can play the elements $t' \leq t$ against the winning strategy of $\forall$. This yields a last move $f(t)$ of $\forall$ in T^*. Clearly f is order-preserving $T \to T^*$. Hence $T \leq T^*$. QED.

The next lemma shows that σ operates like a successor function relative to the ordering $\ll$:

LEMMA 2.2. (1) *$T \ll \sigma T$.*

(2) *There is no T^* with $T \ll T^* \ll \sigma T$.*

PROOF. Clearly, $T \ll \sigma T$, for we can let $u_\alpha(t_0, \ldots, t_\beta, \ldots)_{\beta<\alpha}$ be the branch $\{t \in T \mid t \leq t_\beta \text{ for some } \beta < \alpha\}$. Suppose then $\forall$ has a winning strategy $a_\alpha = u_\alpha(t_0, \ldots, t_\beta, \ldots)_{\beta<\alpha}$ in $G_1(T, T^*)$ and a winning strategy $c_\alpha = v_\alpha(a_0, \ldots, a_\beta, \ldots)_{\beta<\alpha}$ in $G_1(T^*, \sigma T)$. We can define a branch $(b_\alpha)_{\alpha \in \mathrm{On}}$ in T as follows. For a start let $a_0 = u_0(\ )$, $c_0 = v_0(\ )$ and $c_1 = v_1(a_0)$. We let b_0 be the element of c_1 which has lowest rank. Suppose then that $b_0 < \cdots < b_\beta < \cdots$ $(\beta < \alpha)$ have been defined. Let

$$a_\alpha = u_\alpha(b_0, \ldots, b_\beta, \ldots)_{\beta<\alpha}, \qquad c_{\alpha+1} = v_{\alpha+1}(a_0, \ldots, a_\alpha).$$

If $\alpha = \bigcup \alpha$, let also $c_\alpha = v_\alpha(a_0, \ldots, a_\beta, \ldots)_{\beta<\alpha}$. Now $c_{\alpha+1} > c_\alpha$, whence we can pick $b_\alpha \in c_{\alpha+1}$ extending each b_β, $\beta < \alpha$, and of minimal rank. There is no limit to this branch, which is a contradiction. QED.

LEMMA 2.3. (1) *$T \ll T^*$ iff $\sigma T \leq T^*$.*

(2) *$T \leq T^*$ iff $T \ll \sigma T^*$.*

PROOF. If $T \ll T^*$, then $\exists$ can play the elements of any branch in T, i.e. any element of σT, against the winning strategy of $\forall$. The next move of $\forall$ determines the required mapping $\sigma T \to T^*$. Conversely, if such a mapping exists, $\forall$ can apply it to the sequence of previous moves of $\exists$ to get his next move. This ends the proof of (1). For (2), suppose there is an order-preserving mapping $T \to T^*$. Now $\forall$ can play $G_1(T, \sigma T^*)$ by letting the sequence of images of the previous moves of $\exists$ be his next move. Finally, let us assume $T \ll \sigma T^*$. We get an order-preserving mapping f from T into T^* as follows. Let $t \in T$ be such that $f(t')$ is defined for $t' < t$. We play all $t' \leq t$ against $\forall$ in $G_1(T, \sigma T^*)$. After these moves $\forall$ makes another move $S \in \sigma T^*$. We let $f(t)$ be the minimal element in S which extends all $f(t')$, $t' < t$. QED.

LEMMA 2.4. *There is no sequence* $(T_n)_{n<\omega}$ *such that* $T_{n+1} \ll T_n$ *for all* $n < \omega$.

PROOF. By playing all the infinitely many games simultaneously one easily derives a contradiction. Indeed, let $u_\alpha = t^n_\alpha(t_0, \ldots, t_\beta, \ldots)_{\beta<\alpha}$ be a winning strategy of $\forall$ in $G_1(T_{n+1}, T_n)$. We can define branches $b^n_0 < \cdots < b^n_\alpha < \cdots (\alpha \in \mathrm{On})$ in T_n as follows:

$$b^n_\alpha = t^n_\alpha(b^{n+1}_0, \ldots, b^{n+1}_\beta, \ldots)_{\beta<\alpha}.$$

QED.

Todorčević [Tod] defines a product $T \otimes T^*$ by restricting the direct product of T and T^* to the set of all pairs (t, t^*) from $T \times T^*$ where t has the same rank in T as t^* has in T^*. The more general product $\bigotimes_{i\in I} T_i$ is defined analogously.

LEMMA 2.5. *If* T_i, $i \in I$, *is a family of trees, then* $\bigotimes_{i\in I} T_i$ *is the infimum of* T_i, $i \in I$, *relative to* $\leq$.

PROOF. If $(t_i)_{i\in I}$ is an element of $\bigotimes_{i\in I} T_i$, let $f_j((t_i)_{i\in I}) = t_j$. Then f_j is an order-preserving mapping $\bigotimes_{i\in I} T_i \to T_j$. On the other hand, if $f_i\colon T \to T_i$ is order-preserving for all $i \in I$, then the mapping

$$f(t) = (f_i(t)|_{\alpha_t})_{i\in I}, \qquad \alpha_t = \min\{\mathrm{rank}(f_i(t)) \mid i \in I\},$$

is order-preserving. QED.

Note that there is also a natural supremum $\sup\{T_i \mid i \in I\}$ of a family T_i, $i \in I$, of trees: the disjoint union of the trees.

§3. Scott and Karp trees. If $\mathfrak{A}$ and $\mathfrak{B}$ are not $L_{\infty\omega}$-equivalent, we can easily construct a countable ordinal α such that $\exists$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, B_\alpha)$ but $\forall$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, B_{\alpha+1})$. Bearing in mind that $\sigma B_\alpha \equiv B_{\alpha+1}$ and that $G(\mathfrak{A}, \mathfrak{B}, T)$ may be nondetermined, we define:

DEFINITION. (1) T is a *Scott tree* of the pair $(\mathfrak{A}, \mathfrak{B})$ if $\forall$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, \sigma T)$ but not in $G(\mathfrak{A}, \mathfrak{B}, T)$.

(2) T is a *Karp tree* of the pair $(\mathfrak{A}, \mathfrak{B})$ if $\exists$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, T)$ but not in $G(\mathfrak{A}, \mathfrak{B}, \sigma T)$.

(3) T is a *determined Scott tree* of the pair $(\mathfrak{A}, \mathfrak{B})$ if it is a Scott and a Karp tree of $(\mathfrak{A}, \mathfrak{B})$.

Note that B_α is a Scott tree of $(\mathfrak{A}, \mathfrak{B})$ iff it is a Karp tree of $(\mathfrak{A}, \mathfrak{B})$ iff [$\mathfrak{A} \equiv^\alpha_{\infty\omega} \mathfrak{B}$ and not $\mathfrak{A} \equiv^{\alpha+1}_{\infty\omega} \mathfrak{B}$]. Note also that T is a determined Scott tree of $(\mathfrak{A}, \mathfrak{B})$ iff [$\forall$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, \sigma T)$ and $\exists$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, T)$].

The results of §5 show that a Karp tree may fail to be a Scott tree, and conversely.

Note that Scott trees (of some pair $(\mathfrak{A}, \mathfrak{B})$) are *minimal* in the following sense. Suppose T is a Scott tree and $T^* \ll T$. Then $\sigma T^* \leq T$, whence T^* cannot be a Scott tree. Scott trees are also *maximal* in the following sense: If T is a Scott tree and $T \ll T^*$, then $\sigma T \leq T^*$, whence T^* cannot be a Scott tree. In the same sense Karp trees are *minimal* and *maximal*. A Scott tree T_0 is a *smallest* Scott tree if $T_0 \leq T$ holds for all Scott trees T. Neither Scott nor Karp trees can contain an uncountable branch; in particular, both have to have height $\leq \omega_1$. If T is a Scott tree, there is a subtree T' of T which is also a Scott tree and which has cardinality $\leq 2^\omega$, because there are only 2^ω possible sequences of moves of $\exists$.

PROPOSITION 3.1. *Suppose* T^* *is a Scott tree of* $(\mathfrak{A}, \mathfrak{B})$ *and* T *is a Karp tree of* $(\mathfrak{A}, \mathfrak{B})$. *Then* $T \leq T^*$.

PROOF. Let $S = (f_\alpha(y_0, \ldots, y_\beta, \ldots)_{\beta<\alpha}, h_\alpha(y_0, \ldots, y_\beta, \ldots)_{\beta<\alpha})_{\alpha<\omega_1}$ be a winning strategy of $\forall$ in $G(\mathfrak{A}, \mathfrak{B}, \sigma T^*)$ and $S^* = (r_\alpha(z_0, \ldots, z_\alpha))_{\alpha<\omega_1}$ a winning strategy of $\exists$ in $G(\mathfrak{A}, \mathfrak{B}, T)$. We describe a winning strategy $(g_\alpha(t_0, \ldots, t_\beta, \ldots)_{\beta<\alpha})_{\alpha<\omega_1}$ of $\forall$ in $G_1(T, \sigma T^*)$. Suppose $\forall$ has moved $u_0 < \cdots < u_\beta < \cdots$ $(\beta < \alpha)$ in σT^* and $\exists$ has moved $t_0 < \cdots < t_\beta < \cdots$ $(\beta < \alpha)$ in T. Suppose $(x_\beta, y_\beta)_{\beta<\alpha}$ has been produced simultaneously in such a way that $(x_\beta, u_\beta)_{\beta<\alpha}$ is a sequence of moves of $\forall$ and $(y_\beta)_{\beta<\alpha}$ a sequence of moves of $\exists$ in $G(\mathfrak{A}, \mathfrak{B}, \sigma T^*)$ when $\forall$ plays S. Assume also that $(x_\beta, t_\beta)_{\beta<\alpha}$ is a sequence of moves of $\forall$ and $(y_\beta)_{\beta<\alpha}$ a sequence of moves of $\exists$ in $G(\mathfrak{A}, \mathfrak{B}, T)$ when $\exists$ plays S^*. Now let $\forall$ play one more move, (x_α, u_α), of $G(\mathfrak{A}, \mathfrak{B}, \sigma T^*)$ according to S. The element u_α is the next move of $\forall$ in $G_1(T, \sigma T^*)$. Thus $g_\alpha(t_0, \ldots, t_\beta, \ldots)_{\beta<\alpha} = h_\alpha(y_0, \ldots, y_\beta, \ldots)_{\beta<\alpha}$, where $y_\beta = r_\beta((x_0, t_0), \ldots, (x_\beta, t_\beta))$, $\beta < \alpha$, and $x_\beta = f_\beta(y_0, \ldots, y_\gamma, \ldots)_{\gamma<\beta}$, $\beta \le \alpha$. If $\forall$ cannot move, it is because $\forall$ cannot move in $G(\mathfrak{A}, \mathfrak{B}, \sigma T^*)$ or because $\exists$ cannot move in $G(\mathfrak{A}, \mathfrak{B}, T)$. Both cases are impossible because S^* and S are winning strategies. QED.

Note that in the above proposition it suffices to assume that $\forall$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, \sigma T^*)$ and $\exists$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, T)$.

COROLLARY 3.2. *All determined Scott trees of $(\mathfrak{A}, \mathfrak{B})$ are $\equiv$-equivalent. A determined Scott tree is a smallest Scott tree.*

PROOF. If T and T^* are determined Scott trees, then both are also Karp trees, and the claim follows from Proposition 3.1. QED.

We are ready to prove the main result of this paper about Scott and Karp trees:

THEOREM 3.3. *Every pair $(\mathfrak{A}, \mathfrak{B})$ of nonisomorphic models of power ω_1 has a Scott and a Karp tree. Moreover, if $\exists$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, T_1)$ and $\forall$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, T_2)$, then there are a Karp tree T and a Scott tree T' such that $T_1 \le T \le T' \ll T_2$.*

PROOF. Note first that player $\forall$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, T)$ for some T, since $\mathfrak{A} \not\cong \mathfrak{B}$. Similarly, player $\exists$ has a winning strategy in some $G(\mathfrak{A}, \mathfrak{B}, T)$. (We allow here the special case that $T = \emptyset$.) So the first claim follows from the second claim.

Suppose $\forall$ has a winning strategy S in $G(\mathfrak{A}, \mathfrak{B}, T_2)$. Let T_3 consist of sequences $(x_\beta, y_\beta)_{\beta\le\alpha}$ of moves in $G(\mathfrak{A}, \mathfrak{B}, T_2)$ when $\forall$ plays S and $\exists$ has made the move y_α but has not yet lost the game. Clearly S extends to a winning strategy of $\forall$ in $G(\mathfrak{A}, \mathfrak{B}, \sigma T_3)$—$\forall$ can simply start with $t_0 = \emptyset$ and then follow the moves of $\exists$. Clearly also, $\sigma T_3 \le T_2$. Let T' be a $\ll$-minimal (recall that $\ll$ is well-founded) tree such that $T' = T_3$ or $T' \ll T_3$, and $\forall$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, \sigma T')$. We show that $\forall$ has no winning strategy in $G(\mathfrak{A}, \mathfrak{B}, T')$. Suppose the contrary; that is, $\forall$ indeed has a winning strategy S in $G(\mathfrak{A}, \mathfrak{B}, T')$. Let T_4 be the tree of sequences $(x_\beta, y_\beta)_{\beta\le\alpha}$ of moves in $G(\mathfrak{A}, \mathfrak{B}, T')$ when $\forall$ plays S and $\exists$ has made the move y_α, but has not yet lost the game. As above, $\forall$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, \sigma T_4)$, and $\sigma T_4 \le T'$. Thus $T_4 \ll T'$, contrary to the minimality of T'.

Player $\exists$ does not have a winning strategy in $G(\mathfrak{A}, \mathfrak{B})$. Let T consist of winning strategies of $\exists$ in games obtained from $G(\mathfrak{A}, \mathfrak{B})$ by limiting the number of moves to some $\alpha + 1$ $(<\omega_1)$. A strategy S of $\alpha + 1$ moves *precedes* another strategy S^* of $\beta + 1$ moves in T if $\alpha < \beta$ and S and S^* coincide on the first $\alpha + 1$ moves. Clearly, $\exists$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, T)$, for $\exists$ can use the strategies which $\forall$ picks from T to carry on the game. On the other hand, $\exists$ does not have a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, \sigma T)$. Indeed, suppose S were such a strategy. S generates an uncountable branch in T as follows. If $(S_\beta)_{\beta<\alpha}$ are defined, let S_α be obtained from S by letting $\forall$ play $t_\gamma = (S_\beta)_{\beta<\gamma}$ for $\gamma \leq \alpha$ and limiting the number of moves to $\alpha + 1$. Then $(S_\beta)_{\beta<\omega_1}$ is an uncountable branch in T, contrary to $\mathfrak{A} \not\equiv \mathfrak{B}$.

We can easily map T_1 into T using the winning strategy of $\exists$ in $G(\mathfrak{A}, \mathfrak{B}, T_1)$. By Proposition 3.1, $T \leq T'$. QED.

The tree of sequences of moves used in the first part of the proof was first used in [Kar] to prove an approximation theorem for a class of game-formulas containing the duals of the constituents $C_{\mathfrak{A}}$. The tree of strategies used in the second part was first used in [Hyt1] to prove an approximation theorem for a class of game-formulas containing all constituents $C_{\mathfrak{A}}$.

The families of Scott trees and Karp trees are trivially closed under suprema. We know much less about infima. However, we have the following observations, which we give without proof.

PROPOSITION 3.4. *If $\forall$ has a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, T_i)$ for $i \in I$, then $\exists$ does not have a winning strategy in $G(\mathfrak{A}, \mathfrak{B}, \bigotimes_{i\in I} T_i)$.*

PROPOSITION 3.5. *Suppose the pair $(\mathfrak{A}, \mathfrak{B})$ has a smallest Scott tree. Then the family of Scott trees of $(\mathfrak{A}, \mathfrak{B})$ is closed under infima..*

PROBLEM. Are Scott or Karp trees closed under $\bigotimes$?

REMARK. The family of all Scott trees of a pair $(\mathfrak{A}, \mathfrak{B})$ is closed under the infimum ($\bigotimes$) of a finite number of trees if the definition of the Ehrenfeucht game is modified so as to let $\forall$ move a finite number of moves at a time.

§4. Games and stationary sets. We refer to [Jech] for the definition of stationary and cub (closed unbounded) sets. We call a set A of countable ordinals *bistationary* if it is stationary and its complement is stationary. The existence of bistationary sets is a consequence of the axiom of choice (see [Jech]). The purpose of this section is to prove some game-theoretic properties of stationary sets. The model-theoretic results of the next section rely heavily on these results.

Suppose $A \subseteq \omega_1$. The game $G(A)$ is a game of two players, I and II, who pick elements of ω_1. Each move has to be a bigger element than the previous moves, and at limit moves I has to move the limit of the previous moves. Player I starts the game. Player II *wins* if all limit moves are in A; otherwise I wins. The game $G(A, T)$ is defined similarly except that I has the extra task of going up the tree T move by move. The following lemma shows that $G(A)$ is determined. However, we shall then see that $G(A, T)$ is not necessarily determined. The following lemma is essentially in [Kue]:

LEMMA 4.1. (1) I *has a winning strategy in $G(A)$ iff $\omega_1 - A$ is stationary.*

(2) II *has a winning strategy in $G(A)$ iff $\omega_1 - A$ is nonstationary.*

PROOF. (1) Suppose I has a winning strategy S. Let C be any cub set on ω_1. If II plays against S by always choosing elements of C, he will lose at some limit ordinal which is in C but not in A. Thus $C \cap (\omega_1 - A) \neq \emptyset$. This shows that $\omega_1 - A$ is stationary. For the converse, suppose $\omega_1 - A$ is stationary. The winning strategy of I is $f_{\alpha+1}(y_0, \ldots, y_\alpha) = y_\alpha + 1$.

(2) Suppose II has a winning strategy $S = (f_\alpha(x_0, \ldots, x_\alpha))_{\alpha<\omega_1}$. Let C be the cub set of ordinals $<\omega_1$ closed under the functions $f_n(x_0, \ldots, x_n)$, $n < \omega$. Clearly, $C \subseteq A$,

whence $\omega_1 - A$ is nonstationary. Suppose then C is a cub set and $C \subseteq A$. Then II has the winning strategy

$$f_\alpha(x_0, \ldots, x_\alpha) = \min\{y \in C \mid \sup\{x_0, \ldots, x_\alpha\} < y\}.$$

QED.

Let $G^*(A)$ be the game $G(A, \sigma B_\omega)$. In other words, $G^*(A)$ is a game with two players, I and II, and ω moves. Player I plays α_0, then II plays $\alpha_1 > \alpha_0$, then I plays $\alpha_2 > \alpha_1$, then II plays $\alpha_3 > \alpha_2$, etc. Player II wins if $\sup_{n<\omega}\alpha_n$ is in A; otherwise player I wins. It is proved in [Kue] that I has a winning strategy in $G^*(A)$ iff A is nonstationary, and that II has a winning strategy in $G^*(A)$ iff $\omega_1 - A$ is nonstationary. This implies that $G^*(A)$ is nondetermined for all bistationary A. More generally:

LEMMA 4.2 ([Hyt1]). *Suppose $A \subseteq \omega_1$ is bistationary. Let T be a tree of height α, where $\omega < \alpha < \omega_1$. Then $G(A, T)$ is nondetermined.*

PROOF (A. Mekler). Suppose I has a winning strategy S in $G(A, T)$. There is a forcing notion $\mathbf{P}$ which adds a cub set inside A but adds no new reals (see [Bau] for details). In this forcing extension, II has a winning strategy in $G(A, T)$. However, since no new reals are added, S is still a winning strategy of I. This contradiction ends the proof. QED.

REMARK. A consequence of Lemma 4.2 is that, given a strategy S of I and a position in the game $G(A, T)$, where A is bistationary and T arbitrary, II has, for any countable α, a nonlosing counterstrategy against S for the next α moves. We shall use this fact frequently in the sequel.

Let $A \subseteq \omega_1$. A subset C of A is *closed* if it is closed in the order topology. Let $T(A)$ be the tree of all closed sets of elements of A, ordered by $C \leq C^*$ iff C is an initial segment of C^*. If A is costationary, this tree has no uncountable branches, and if A is stationary, it has height ω_1. Properties of $T(A)$ are vital for our results. We are indebted to S. Todorčević for bringing these trees and their properties to our attention (see, for example, [Tod]).

LEMMA 4.3. *Suppose $A \subseteq \omega_1$ is stationary. Then* I *has a winning strategy in $G(A, \sigma T(A))$.*

PROOF. The winning strategy of I in $G(A, \sigma T(A))$ is the following: I starts with $\alpha_0 = \min(A)$ and $t_0 = \emptyset$. Henceforth I plays his moves from A and picks in $\sigma T(A)$ the sequence of his previous moves in A. If all limit moves are in A, we shall have a cub set in A. Hence I is bound to beat II with this strategy. QED.

LEMMA 4.4. *Suppose $A \subseteq \omega_1$ is stationary. If $B \subseteq \omega_1$ is costationary, then* I *does not have a winning strategy in $G(A, T(B))$. If, moreover, $A - B$ is stationary, then* I *does not even have a winning strategy in $G(A, \sigma T(B))$.*

PROOF. Suppose I has a winning strategy S in $G(A, T(B))$. We describe a winning strategy S^* of II in $G^*(B)$. This is contrary to costationarity of B. Suppose $\beta_0 < \beta_1 < \cdots < \beta_n$ have been played in $G^*(B)$. The strategy S^* is based on playing also $G(A, T(B))$. Suppose the game $G(A, T(B))$ has proceeded to move (δ, t) of I, with $\sup(\bigcup t) \geq \beta_n$, and II has not lost yet. Now S^* asks II to play in $G^*(B)$ some element $\beta_{n+1} > \sup(\bigcup t)$. Moreover, II continues the game $G(A, T(B))$ as follows: we know that II has a strategy by which he can continue $G(A, T(B))$ at least $\beta_{n+1} + 1$ moves without losing. After these moves the resulting move (δ^*, t^*) of I satisfies $\sup(\bigcup t^*) > \beta_{n+1}$ and II has not lost yet. The strategy S^* is now defined, and it remains to show that it is winning. After ω moves in this game a sequence $\beta_0 < \beta_1 < \cdots < \beta_n < \cdots$ is produced. Let $\delta = \sup_{n<\omega} \beta_n$. Notice that $\delta = \sup_{n<\omega} \sup(\bigcup t_n)$. Since II has not lost yet in $G(A, T(B))$, I has to be able to make more moves. This is possible only if $\delta \in B$. Thus II has won $G^*(B)$.

The second claim is proved similarly. This time II wins $G^*(\omega_1 - (A - B))$, contrary to stationarity of $A - B$. (If the δ produced is not in $\omega_1 - (A - B)$, then II can go on playing $G(A, T(B))$, but I has only one more node in $\sigma T(B)$ to go to.) QED.

The tree B_∞ has the property that any $T \ll B_\infty$ also satisfies $\sigma T \ll B_\infty$. In fact any tree of height $\gamma = \bigcup \gamma$ with a branch of length γ has the same property. This gives rise to the following notion: A tree T is a *limit tree* if $T' \ll T$ implies $\sigma T' \ll T$ for all T'. Notice that σT is never a limit tree.

PROPOSITION 4.5. *Suppose $A \subseteq \omega_1$ is unbounded. Then $T(A)$ is a limit tree.*

PROOF. Easy. QED.

PROPOSITION 4.6. *Let A and B be subsets of ω_1 such that $A \subseteq B$ and A, $B - A$ and $\omega_1 - B$ are stationary. Then $T(A) < T(B)$, but not $T(A) \ll T(B)$.*

PROOF. Trivially, $T(A) \leq T(B)$. Suppose then that f is an order-preserving mapping $T(B) \to T(A)$. Let C be the cub set of countable ordinals α such that if $t \in T(B)$ with $\sup(t) < \alpha$, then $\sup(f(t)) < \alpha$. Let $\alpha \in C \cap (B - A)$. We may assume that the heights of $t \in T(B)$ with $\sup(t) < \alpha$ are unbounded below α. Let (t_n) be an ascending sequence in $T(B)$ with $\sup\{\bigcup t_n \mid n < \omega\} = \alpha$. Then $\sup\{\bigcup f(t_n) \mid n < \omega\} = \alpha$. We get a contradiction, since $\cup\{t_n \mid n < \omega\} \cup \{\alpha\} \in T(B)$ whereas $\bigcup\{f(t_n) \mid n < \omega\}$ has no extension in $T(A)$.

To prove the failure of $T(A) \ll T(B)$, let S be a winning strategy of I in $G(T(A), T(B))$. We show that II has a winning strategy in $G^*(B)$. Whenever I plays α_n in $G^*(B)$, II plays α_n more moves against S in $G(T(A), T(B))$, and then lets his move in $G^*(B)$ be the last element of the last move of I in $G(T(A), T(B))$. After ω moves are made in $G^*(B)$, the supremum δ of the moves is equal to the supremum of the moves of I in $G(T(A), T(B))$. Since I is following S in $G(T(A), T(B))$, he must be able to make at least one more move in $T(B)$, whence $\delta \in B$. This ends the description of the winning strategy of II in $G^*(B)$. Since $\omega_1 - B$ is stationary, we have a contradiction. QED.

An inspection of the above proof reveals that if $A \subseteq \omega_1$ is bistationary and every node of a tree T has extensions of all countable heights, then not $T \ll T(A)$. Note that such a tree T is easily constructed and even, so that it is special. Then $T < T(A)$, so here again $G(T, T(A))$ is nondetermined.

We close this section with some observations on the sizes of trees T such that I has a winning strategy in $G(A, T)$. We know from 4.2 that such T are uncountable.

PROPOSITION 4.7 (T. HUUSKONEN). *If $A \subseteq \omega_1$ is bistationary and I has a winning strategy in $G(A, T)$, where T has no uncountable branches, then $|T| \geq 2^\omega$.*

PROOF. Suppose I has a winning strategy S in $G(A, T)$. Let B be the tree of all possible sequences of triples $s_\alpha = (x_\alpha, t_\alpha, y_\alpha)$ of moves of I and II in $G(A, T)$ when I follows S and II has not lost yet. It easily follows from 4.2 that every node of B has extensions $(x_\alpha, t_\alpha, y_\alpha)$ with t_α of arbitrary height $< \omega_1$. So above any node of B

there is a node s_α after which the t_α branches. Thus we can build a full binary tree inside B in such a way that it reflects a full binary tree in T. Any branch of this binary tree has an extension because $\forall$ is following a winning strategy. Hence T has at least 2^ω elements. QED.

PROPOSITION 4.8. *If $A \subseteq \omega_1$ is bistationary and* I *has a winning strategy in $G(A, T)$, then $T(A) \ll T$.*

PROOF. Suppose I has a winning strategy S in $H = G(A, T)$. Let us denote the game $G_1(T(A), T)$ by G. We describe a winning strategy of $\forall$ in G. While we play G we let I play H according to his winning strategy. Let the first move of I in H consist of $\alpha_0 \in \omega_1$ and $t_0 \in T$. The first move of $\forall$ in G is t_0. Let the first move of $\exists$ in G be s_0. Suppose then I has played $\alpha_\xi \in \omega_1$ and $t_\xi \in T$ in H, $\forall$ has played t_ξ in G and $\exists$ has played s_ξ in G. If $\beta_\xi = \max(s_\xi) > \alpha_\xi$, we let II move β_ξ in H, after which I answers with $\alpha_{\xi+1} \in \omega_1$ and $t_{\xi+1} \in T$, and $\forall$ has a chance of playing $t_{\xi+1}$ in G. Otherwise we let II play $\alpha_\xi + 1$ more moves in H without losing. This is possible in view of the remark following Lemma 4.2. The corresponding moves of I give rise to a chain C_ξ in T. During the next $\alpha_\xi + 1$ moves of G player $\forall$ follows the chain C_ξ. The point of doing this is that the maximum of the next move of $\exists$ in G will exceed α_ξ, which will be relevant when we come to a limit. Indeed, at limit points of this strategy the supremum of the maxima of the moves of $\exists$ in G coincides with the supremum of the moves I has made in ω_1 in the game H. So if $\exists$ can go on playing G after $\forall$ has made the canonical limit moves in G and H, player II has not lost H yet, whence I has to be able to go on playing H in order to demonstrate his victory. Therefore the above strategy enables $\forall$ to play G and win. QED.

It follows from the above proposition and Theorem 3.4 of [Tod] that if $A \subseteq \omega_1$ is bistationary and I has a winning strategy in $G(A, \sigma T)$, then T is nonspecial.

§5. Examples of Scott trees. In this section we apply the results of §4 to Scott and Karp tree computations. Our models will be dense ω_1-like linear orderings, and we will show that in the field of these uncountable models the notions of Karp trees and Scott trees are very different from each other.

Let $A \subseteq \omega_1$ be bistationary. Let η denote the order-type of the rationals. Let $\Phi(A)$ denote the dense ω_1-like ordering $\sum_{\alpha<\omega_1} \tau_\alpha$, where

$$\tau_\alpha = \begin{cases} 1 + \eta & \text{if } \alpha \in A \text{ or } \alpha = 0, \\ \eta & \text{if } \alpha \notin A \text{ and } \alpha > 0. \end{cases}$$

For $x \in \Phi(A)$, let $r(x)$ be the unique α for which $x \in \tau_\alpha$. Models like $\Phi(A)$ have been studied e.g. in [Nad] and [Hyt1]. If T is a tree, we denote by T' the tree obtained from T by adding one element to the end of each maximal branch of T.

PROPOSITION 5.1. *Suppose $A \subseteq \omega_1$ is bistationary.*

(1) *Player $\forall$ has a winning strategy in $G(\Phi(A), \Phi(\omega_1), T')$ if and only if* I *has a winning strategy in $G(A, T)$.*

(2) *Player $\exists$ has a winning strategy in $G(\Phi(A), \Phi(\omega_1), T')$ if and only if* II *has a winning strategy in $G(A, T)$.*

PROOF. (1) Suppose $\forall$ has a winning strategy S in $G(\Phi(A), \Phi(\omega_1), T')$. Then I has the following winning strategy in $G(A, T)$. First I plays (x_0, t_0) according to S. The first move of $\forall$ in $G(A, T)$ is $\alpha_0 = r(x_0)$. Then $\exists$ plays some $\beta_0 > \alpha_0$. Suppose then that I has played (α_ξ, t_ξ), $\xi < \gamma$, and II has played β_ξ, $\xi < \gamma$, in $G(A, T)$. Suppose we have at the same time played (x_ξ, t_ξ), $\xi < \gamma$, and y_ξ, $\xi < \gamma$, in $G(\Phi(A), \Phi(\omega_1), T')$. If γ is limit, then I moves (x_γ, t_γ) according to S and lets $(\alpha_\gamma, t_\gamma)$ be $(\sup\{\alpha_\xi \mid \xi < \gamma\}, t_\gamma)$. If $\alpha_\gamma \notin A$, I has won already. If $\gamma = \zeta + 1$ we define $(\alpha_\gamma, t_\gamma)$ as follows. Let us look at the game $G(\Phi(A), \Phi(\omega_1), T')$. Unless I has won $G(A, T)$ already, player $\exists$ has a trivial strategy in $G(\Phi(A), \Phi(\omega_1), T')$ which keeps him in the game until $\forall$ has moved, following S, a pair (x, t) with $r(x) > \beta_\zeta$. Now I lets $\alpha_\gamma = r(x)$ and $t_\gamma = t$ in $G(A, T)$. We have described a strategy of I in $G(A, T)$. This is a winning strategy for the following reason. While I plays $G(A, T)$ according to the above strategy, the game $G(\Phi(A), \Phi(\omega_1), T')$ is played as well and $\forall$ wins it. The only way for $\forall$ to win is that there be some limit γ with $\alpha_\gamma = r(x_\gamma) \notin A$. Then I has won $G(A, T)$.

Suppose then that I has a winning strategy S in $G(A, T)$. We describe a winning strategy of $\forall$ in $G(\Phi(A), \Phi(\omega_1), T')$. Suppose S gives I the pair (α_0, t_0) as the opening move in $G(A, T)$. Then $\forall$ moves some $x_0 \in \Phi(\omega_1)$ with $r(x_0) = \alpha_0$. Now $\exists$ plays y_0 in $\Phi(A)$. Let II play the maximum of $r(y_0)$ and $\alpha_0 + 1$ in $G(A, T)$. Suppose S gives α_2 to I. We let $\forall$ move some $x_1 \in \Phi(A)$ with $r(x_1) = \alpha_2$, etc. At limits $\forall$ plays the same way he played x_0. We know I wins $G(A, T)$ at some limit point which is not in A. At this point $\forall$ easily settles the game $G(\Phi(A), \Phi(\omega_1), T')$ as well, making use of the extra move he has in T' over T.

The proof of (2) is based on the same ideas as that of (1). QED.

REMARK. If we combine Proposition 5.1 with Lemma 4.2, we get the result that $G(\Phi(A), \Phi(\omega_1), T)$ is nondetermined whenever T is a tree of height $\alpha, \omega + 1 < \alpha < \omega_1$. In particular, the pair $(\Phi(A), \Phi(\omega_1))$ has B_∞ as Karp tree but no Karp trees of height $> \omega + 1$. It also follows that $(\Phi(A), \Phi(\omega_1))$ has no determined Scott trees.

PROPOSITION 5.2. *Suppose $A \subseteq \omega_1$ is bistationary. Then $T'(A)$ is a smallest Scott tree of $(\Phi(A), \Phi(\omega_1))$. If $B \supseteq A$ is costationary, then $T'(B)$ is a Scott tree of $(\Phi(A), \Phi(\omega_1))$.*

PROOF. By Lemmas 4.3 and 4.4, I has a winning strategy in $G(A, \sigma T(B))$ but not in $G(A, T(B))$. Now the second claim follows from Proposition 5.1. The proof of Proposition 4.8 can be modified to give $T'(A) \ll T$ whenever $\forall$ has a winning strategy in $G(\Phi(A), \Phi(\omega_1), T)$. This implies the first claim. QED.

THEOREM 5.3. *There are nonisomorphic models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality ω_1 such that the pair $(\mathfrak{A}, \mathfrak{B})$ has 2^{ω_1} Scott trees which are mutually noncomparable by $\leq$.*

PROOF. Let $A \subseteq \omega_1$ be bistationary and let A_α, $\alpha < \omega_1$, be mutually disjoint stationary sets in the complement of A. Let $\{X_\alpha \mid \alpha < 2^{\omega_1}\}$ be a list of all nonempty subsets of ω_1, and let $B_\alpha = \bigcup\{A_\beta \mid \beta \in X_\alpha\} \cup A$. The trees $T(B_\alpha)$ and hence the trees $T'(B_\alpha)$ are noncomparable. This is proved similarly to the proof that $T(A) < T(B)$ in Proposition 4.6. By Lemma 5.2, $T'(B_\alpha)$ is a Scott tree of $(\Phi(A), \Phi(\omega_1))$ for each $\alpha < 2^{\omega_1}$. QED.

REMARK. L. Hella and T. Huuskonen have in their recent work constructed models with large Karp trees.

REFERENCES

[Bar] J. BARWISE, ***Admissible sets and structures***, Springer-Verlag, Berlin, 1975.

[Bau] J. BAUMGARTNER, L. HARRINGTON and E. KLEINBERG, *Adding a closed unbounded set*, this JOURNAL, vol. 41 (1976), pp. 481–482.

[Hin] J. HINTIKKA amd V. RANTALA, *A new approach to infinitary languages*, ***Annals of Mathematical Logic***, vol. 10 (1976), pp. 95–115.

[Hyt1] T. HYTTINEN, *Games and infinitary languages*, ***Annales Academiae Scientiarum Fennicae Series A I: Mathematica Dissertationes***, vol. 64 (1987).

[Hyt2] ———, *Model theory for infinite quantifier languages*, ***Fundamenta Mathematicae*** (to appear).

[Jech] T. JECH, ***Set theory***, Academic Press, New York, 1978.

[Kar] M. KARTTUNEN, *Syntax and semantics of infinitely deep languages*, ***Annales Academiae Scientiarum Fennicae Series A I: Mathematica Dissertationes***, vol. 50 (1984).

[Kue] D. KUEKER, *Countable approximations and Löwenheim-Skolem theorems*, ***Annals of Mathematical Logic***, vol. 11 (1977), pp. 57–103.

[Nad] M. NADEL and J. STAVI, *$L_{\infty\lambda}$-equivalence, isomorphism and potential isomorphism*, ***Transactions of the American Mathematical Society***, vol. 236 (1978), pp. 51–74.

[Tod] S. TODORČEVIĆ, *Stationary sets, trees and continuums*, ***Publications de l'Institut Mathématique, Nouvelle Série***, vol. 29 (43) (1981), pp. 249–262.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HELSINKI
00100 HELSINKI, FINLAND