In F. Jensen, B. Mayoh and K. Møller, editors, Proceedings from 5th Scandinavian Logic Symposium, pages 309-328, Aalborg University Press, 1979.

ON HANF-NUMBERS OF UNBOUNDED LOGICS

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Abstract *An unbounded logic is one which is capable of characterizing the notion of well-ordering. Well-known examples of such logics are* $L_{\omega_1\omega_1}$ *and second order logic* L^{II}. *This paper surveys some known results concerning these logics and announces new results, including solutions to Question* 6.3 *of Silver's thesis and to number 9 of Friedman's 102 Problems. An introduction to the subject is provided by the first three chapters.*

Chapter 1. Minima and maxima of spectra

Let us consider first order logic $L_{\omega\omega}$. For any $\phi \in L_{\omega\omega}$ we can define the *spectrum* of ϕ

$$Sp(\phi) = \{|A| \mid A \models \phi\},$$

where $|A|$ denotes the cardinality of the domain of the structure A. The finite parts of spectra of sentences of $L_{\omega\omega}$ form an interesting subclass of the class of computable sets of natural numbers. For example, the following open problem is widely known:

Finite spectrum probelm: If S is a spectrum, is there a spectrum S' such that $\omega \cap S = \omega - S'$?

On the other hand, the infinite part of the spectrum of an $L_{\omega\omega}$-sentence is trivial: either empty or contains every infinite number.

If we consider spectra of $L_{\omega_1\omega}$-sentences, a different situation emerges: every set of natural numbers is a spectrum. Also some spectra have a non-trivial infinite part, like $\{\kappa \mid \kappa \leq 2^{\omega}\}$ which is the spectrum of

$$\forall\forall xy(\forall z(zEx \longleftrightarrow zEy) \to x=y) \wedge \forall xy(xEy \to N(x)) \wedge \bigwedge_{n<\omega} N(c_n) \wedge \forall x(N(x) \to \bigvee_{n<\omega} x=c_n).$$

However, it is still true that the infinite part of any spectrum of an $L_{\omega_1\omega}$-sentence is either of the form $\{\kappa \mid \omega \leq \kappa < \lambda\}$ for some $\lambda < \beth_{\omega_1}$ or of the form $\{\kappa \mid \omega \leq \kappa\}$.

Let us consider then second order logic L^{II}. The spectra of L^{II} may be extremely complicated. If $\phi(x)$ is any Σ_1- or Π_1-formula of set theory (for example, "x is weakly inaccessible"), then the class

$$\{\kappa \mid \phi(\kappa)\}$$

is a spectrum of a second order sentence. For a complete characterization of spectra of L^{II} see [23]. Even for L^{II} we can still find a cardinal κ such that if S is any spectrum of L^{II}, then either S is a proper class or $\sup S < \kappa$. This κ is called the Hanf-number of L^{II}.

After these preliminary examples we shall define two characteristic numbers for any (abstract) logic L^*. We are assuming that the class of all L^*-sentences is a set. Let

$l(L^*)$ = sup{min S | S is an L^*-spectrum}
= the *Löwenheim-number* of L^*.
$h(L^*)$ = sup{sup S | S is an L^*-spectrum but not a proper class}.
= the *Hanf-number* of L^*.

To put it in other words, $l(L^*)$ is the least κ such that if any $\phi \in L^*$ has a model at all, then ϕ has a model of power $\leq \kappa$, and $h(L^*)$ is the least κ such that if any $\phi \in L^*$ has a model of power $\geq \kappa$ then ϕ has arbitrarily large models.

These numbers can be explicitly computed for some logics. The following diagram gives some examples:

L^*	$L_{\omega\omega}$	$L_{\omega_1\omega}$	LQ_1	LQ_0	$L_{\omega_1\omega_1}$	L^{II}
$l(L^*)$	ω	ω	ω_1	ω	2^{ω}	?
$h(L^*)$	ω	$\beth_{\omega_1}$	$\beth_{\omega}$	$\beth_{\omega_1^{ck}}$	?	?
reference		[5]	[19]	[4],[1]	[3]	

When computing the Hanf- or Löwenheim-number of a logic it is often useful to know that L^* can be, one way or other, embedded into another logic L^+. For example, the fact that the Hanf-number of ω-logic is $\beth_{\omega_1^{ck}}$, follows easily from the fact that ω-logic and LQ_0 can be embedded into each other in a sufficiently regular way. This procedure can be made precise by introducing the following notions:

Suppose L^* is an abstract logic. We allow the signatures to be many-sorted. A class of models K is $\Sigma(L^*)$ if there is a sentence ψ of L^* such that the class of reducts of models of ψ to the signature of K is the class of all models of K. In symbols: Let L be the signature of K. Then for any A of type L,

$$(*) \qquad A \in K \text{ if and only if } \exists B(B \models \psi \wedge B|_L = A).$$

The family of all $\Sigma(L^*)$ model classes can be regarded as a logic itself and we denote this logic by $\Sigma(L^*)$ (it is sometimes denoted by $RPC(L^*)$ or $\Sigma^{RPC}(L^*)$).

It is easily seen that Σ preserves Löwenheim-numbers, that is, $l(L^*) = l(\Sigma(L^*))$. However, the same is not true of Hanf-numbers (see [22]). An operation which resembles Σ but respects Hanf-numbers, is obtained if the following condition is added to $(*)$:

(**) There is a cardinal κ such that $\forall B[(B \models \psi \wedge B|_L = A) \to B < \kappa]$.

Let us denote this restricted operation by $\Sigma^B(L^*)$. It is proved in [22] that $\Sigma(L^*) = \Sigma^B(L^*)$ for most logics L^*, and that Σ^B preserves Hanf-numbers.

The logics $\Sigma(L^*)$ and $\Sigma^B(L^*)$ are not closed under negation. Their largest sublogics which are closed under negation are denoted by $\Delta(L^*)$ and $\Delta^B(L^*)$ respectively. For an account of Δ, see [13].

As a simple application, we may use the fact that $\Delta^B(LQ_o) = L_A$, A = the smallest admissible set containing ω, to conclude that $h(L_A) = h(LQ_o)$. Other examples will follow.

We end this chapter with a few remarks on other possible definitions of Hanf-numbers.

If A is a many-sorted structure, we let $card_*(A)$ denote the least of the cardinalities of the sorts of A. Paulos [15] considered the following number: Let $h^+(L^*)$ be the least κ such that if $\phi \in L^*$ and ϕ has a model of $card_* > \kappa$ then ϕ has models of arbitraily large $card_*$. He proved that Δ preserves h^+. In fact (see [22])

$$h^+(L^*) = h(\Sigma(L^*)).$$

Also the following variant occurs in the literature: Let $h^-(L^*)$ be the least κ such that if $\phi \in L^*$ has a model of power κ then ϕ has arbitrarily large models. In [22] it is proved that

$$h(L^*) = h^-(\Sigma^B(L^*)).$$

Chapter 2. Unbounded logics

Consider the quantifier

$WxyA(x,y)$ if and only if $A(\cdot,\cdot)$ well-orders its field and let LW denote the logic $L_{\omega\omega}$ endowed with the quantifier W. The model theory of LW corresponds to the model theory of $L_{\omega\omega}$ with well-ordered models. Therefore it is obvious that $l(LW) = \omega$. Of the very few other facts known about LW one should mention that its *decision problem* (the set of valid sentences) is the complete Π^1_2-set of integers (see [6]).

The logic LW is an example of what we call an unbounded logic:

An abstract logic L^* is *unbounded* if the quantifier W is $\Sigma^B(L^*)$-definable. It is *weakly unbounded* if W is only $\Sigma(L^*)$-definable. It is not known to the author whether these two notions are actually one and the same. Probably not. Note that $\sim W$ is always $\Sigma^B(L^*)$, so that it is really question of W being $\Delta(L^*)$ or $\Delta^B(L^*)$.

<u>Examples</u>

A. The infinitary logic $L_{\kappa\lambda}$ is unbounded whenever $\kappa \geq \lambda > \omega$. Indeed

$$WxyA(x,y) \text{ if and only if } \forall x_0 x_1 \ldots x_n \ldots \bigvee_{n<\omega} \sim A(x_{n+1}, x_n).$$

The usual definition of $L_{\kappa\lambda}$ becomes vacuous if $\lambda > \kappa$. Therefore the following redefinition may be considered: For $\lambda > \kappa$, $L_{\kappa\lambda}$ is $L_{\kappa\kappa}$ enriched with the weak second order quantifiers

$$\exists_\alpha X\phi(X),$$

that is,

$$\exists X(|X| \leq \alpha \wedge \phi(X))$$

for all $\alpha < \lambda$. With this definition, $L_{\kappa\lambda}$ $(\lambda \geq \kappa)$ is always a sublogic of $L_{\lambda\lambda}$, and already $L_{\omega\omega_1}$ is unbounded:

$$WxyA(x,y) \text{ if and only if } \forall_\omega X(\exists yX(y) \rightarrow \exists y(X(y) \wedge \forall z(X(z) \rightarrow \sim A(z,y)))).$$

Note that $L_{\omega\omega_1} < \Delta^B(L^{II})$ but $L_{\omega\omega_1} \not< L^{II}$, because the monadic fragment of L^{II} is compact.

We let $L^{II}_{\kappa\kappa}$ denote the result of adding full second order quantification to $L_{\kappa\kappa}$. Thus $L^{II}_{\omega\omega}$ is just L^{II}.

B. The logic LI, that is, $L_{\omega\omega}$ with the *Härtig-quantifier*:

$$IxyA(x)B(y) \text{ if and only if } |A(\cdot)| = |B(\cdot)|$$

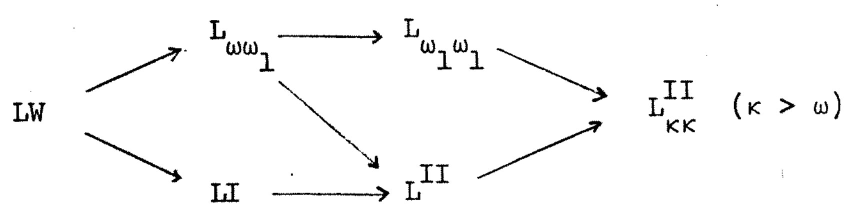
is unbounded ([9]). The proof is based on the following observation: $\langle A, \lessdot \rangle$ is well-founded if and only if there are sets $X_a, a \in A$, such that for $a,b \in A$: $a < b \leftrightarrow |X_a| < |X_b|$. Clearly $LI < \Delta^B(L^{II})$.

C. The logic LS, where S is the *similarity-quantifier*:

SxyuvA(x,y)B(u,v) if and only if there is a bijection f such that $A(x,y) \leftrightarrow B(f(x),f(y))$ for all x and y,

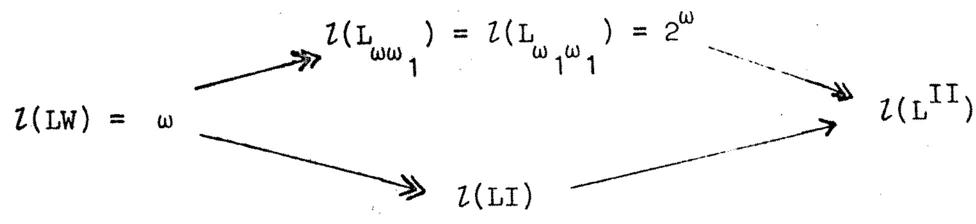
is unbounded. In fact, $\Delta^B(LS) = \Delta^B(L^{II})$ (see [21]).

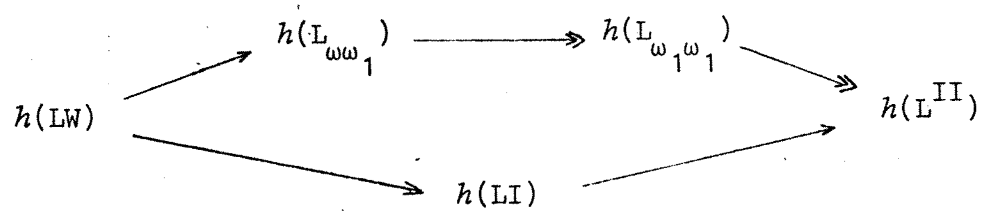
If we write $L^* \rightarrow L^+$ for $L^* < \Delta^B(L^+)$, we obtain the following diagram:



The subdiagram consisting of $L_{\omega\omega_1}$, LI and L^{II} deserves some comments. Fairly trivially, $LI \not\rightarrow L_{\omega\omega_1}$. However, if $V = L$, then $L_{\omega\omega_1} \rightarrow L^{II} \rightarrow LI$. In [20] a method is described which yields an ω_1-closed Boolean algebra B such that $V^B \models l(LI) < 2^{\omega_1}$. Thus, starting with V=L, we obtain a model in which $L_{\omega\omega_1} \rightarrow LI$ but $L^{II} \not\rightarrow LI$. Finally, if $l(LI) < 2^{\omega}$ (see [20]), then $L_{\omega\omega_1} \not\rightarrow LI$.

Using the preservation of Löwenheim- and Hanf-numbers under Δ^B, the following two diagrams obtain: (We write, for any terms t and u of set theory, $t \longrightarrow u$, if $ZFC \vdash t \leq u$, and $t \twoheadrightarrow u$, if $ZFC \vdash t < u$.)





The diagram of Löwenheim numbers presents no problems. Note that $l(LI)$ and $l(L^{II})$ coincide if $V = L$, but differ drastically if $l(LI) < 2^\omega$. This explains the absence of arrows between 2^ω and $l(LI)$.

The diagram of Hanf numbers is more subtle. Again, if $V = L$, then $h(LI)$ and $h(L^{II})$ coincide, but they can also differ (see later). Whether the other two improper inequalities are really improper, is not known the author. The only non-trivial arrow follows from:

Lemma 1 $h(L_{\omega_1\omega_1}) < h(L^{II})$.

Proof. Let ϕ be an L^{II}-sentence saying "I am an $R_\lambda \omega+$ and $\lambda = h(L_{\omega_1\omega_1})$". Suppose this sentence has a model M of power $> \beth_\kappa \omega+$, where $\kappa = h(L_{\omega_1\omega_1})$. We may assume that $M = R_\lambda \omega+$, where $\lambda > \kappa$. Now $\kappa < h(L_{\omega_1\omega_1})$ in M, whence there is a $\psi \in L_{\omega_1\omega_1}$ such that "ψ has a model A of power $>\kappa$ but none of power λ^ω" holds in M. Clearly, $A \models \psi$. Therefore there is a $B \models \psi$ such that $|B| = \lambda^\omega$, a contradiction. Thus ϕ has a model of power $\beth_\kappa \omega+$, but none larger. □

We shall return to these diagrams later. Let us have a glance forward, however, with the following curious lemma:

Lemma 2 (i) $h(L_{\omega\omega_1}) \leq l(L^{II})$.

(ii) If V = L, then $h(LW) = h(L_{\omega\omega_1}) = l(L^{II})$.

Proof. (i): Suppose $\phi \in L_{\omega\omega_1}$ and $\kappa = \sup Sp(\phi)$. Let $\lambda = (\kappa^+)^\omega$. It is true in R_{λ^+} that ϕ has no models of power λ. On the other hand, if $\mu = (\nu^+)^\omega$ and it is true in R_{μ^+} that ϕ has no models of power μ, then $\mu \geq \kappa$. Indeed, if such a μ were $< \kappa$, we could use the downward Löwenheim-Skolem theorem of $L_{\omega\omega_1}$ to find a model $A \in R_{\mu^+}$ of ϕ of power μ. Thus the L^{II}-sentence "I am an R_{μ^+} such that $\mu = (\nu^+)^\omega$ and ϕ has no models of power μ" has a model but no models of power $<\kappa$. Therefore $\kappa < l(L^{II})$.

(ii): Suppose $\phi \in L^{II}$ and $\kappa = \min Sp(\phi)$. Let $\psi \in LW$ say "I am a well-founded model of a certain finite part of ZF + V=L and ϕ has no models what so ever". The sentence ψ is consistent because $L_\kappa \models \psi$. Suppose then L_α is a model of ψ of power $> \kappa^+$. Then $\alpha > \kappa^+$ whence $\kappa^+ \in L_\alpha$ and $P(L_\kappa) \subset L_\alpha$. Therefore ϕ has a model in L_α, a contradiction. Hence $\kappa < h(LW)$. □

The main motivation for singling out the family of unbounded logics here is that unboundedness seems to be just the crucial factor which makes $h(L_{\omega_1\omega_1})$ and $h(L^{II})$ so awkward. As a mild indication of the effect of unboundedness, consider the following simple and well-known fact:

Lemma 3 Suppose that L* is an unbounded abstract logic. If there are inaccessible cardinals, then $h(L^*)$ exceeds the first inaccessible.

Proof. Suppose κ is the first inaccessible and ϕ the $\Sigma^B(L^*)$-sentence which says that "I am a well-founded model of ZFC without inaccessible cardinals". Now $R_\kappa \models \phi$. Suppose then $M \models \phi$ and $|M| > \kappa$. We may assume that M is a transitive ε-model of ZFC and $\kappa \in M$. As the predicate "α is regular" is Π_1, we have $M \models$ "κ is inaccessible", a contradiction. Thus ϕ has no models of power $> \kappa$. □

The only fact we needed to know of inaccessibility in the above proof was that it is inherited by transitive submodels (with the same ordinals). Thus the same proof yields the result: If there is a subtle cardinal, then $h(L^*)$ exceeds the first subtle (and therefore the first totally indescribable). The lemma is also true if "inaccessible" be replaced by "hyperinaccessible" , "Mahlo" or by "hyper-Mahlo".

Chapter 3. Measuring $h(L^*)$: Large cardinals

The purpose of this chapter is to give a short review of the various large cardinal notions needed in the subsequent chapters.

Let us note at first, however, that $h(L^*)$ (for unbounded L^*) can be characterized in terms of definable ordinals of set theory: Suppose D is a class of formulae of set theory. We say that an ordinal α is D-*definable* if $\alpha = \{\beta \mid \phi(\beta)\}$ for some $\phi(x) \in D$. Examples of characterizations of this kind are

$$h(LW) = \sup\{\alpha \mid \alpha \text{ is } \Sigma_1\text{-definable}\}$$

$$l(L^{II}) = \sup\{\alpha \mid \alpha \text{ is } \Pi_2\text{-definable}\}$$

$$h(L^{II}) = \sup\{\alpha \mid \alpha \text{ is } \Sigma_2\text{-definable}\}.$$

For the proofs of these, and related results, we refer the reader to [7] and [23].

As Lemma 3 indicated, $h(L^*)$ (for unbounded L^*) exceeds the smallest large cardinals. This observation manifests the possibility of measuring $h(L^*)$ by trying to locate it in the scale of all, small and large, large cardinals.

The following presentation of some large cardinals is slightly non-standard, but being in a sense purely model-theoretic, fits well into the spirit of this paper.

Consider the following two important properties of a regular cardinal κ and an abstract logic L^*:

LST(κ): If A is a model for a language of power $< \kappa$, $X \subset |A|$ has power $< \kappa$ and $\phi \in L^*$, then there is a $B \subset A$ such that, $X \subset |B|$, $\mathrm{card}(|B|) < \kappa$ and $B \models \phi$.

CMP(κ): If T is a set of L^*-sentences and every subset of T of power $< \kappa$ has a model, then T has a model.

First order logic $L_{\omega\omega}$ has the properties LST(ω_1) and CMP(ω). Indeed, these two properties characterize $L_{\omega\omega}$ (see [10]). Already long ago the question was raised, whether there are $\kappa > \omega$ such that $L_{\kappa\kappa}$ has CMP(κ) or LST(κ). We have the following facts:

(1) κ is inaccessible iff $L_{\kappa\kappa}$ has LST(κ).

(2) κ is strongly compact iff $L_{\kappa\kappa}$ has CMP(κ).

Fact (1) is easily proved and (2) is often taken as a definition of strong compactness.

Among various weaker forms of CMP(κ) the following two seem particularly pertaining:

MCMP(κ): If $T_0 \subset \ldots \subset T_\alpha \subset \ldots$, $\alpha < \kappa$, is a sequence of sets of L^*-sentences such that each T_α has a model, then $\bigcup_{\alpha<\kappa} T_\alpha$ has a model. ("Medium compactness")

WCMP(κ): The cardinal κ is strongly inaccessible, and if T is a set of L^*-sentences, $|T| = \kappa$ and every subset of T of power $< \kappa$ has a model, then T has a model. ("Weak compactness")

We have the following facts:

(3) κ is measurable iff $L_{\kappa\kappa}$ has MCMP(κ).

(4) κ is weakly compact iff $L_{\kappa\kappa}$ has WCMP(κ).

Fact (3) is from [2] page 198, and fact (4) is often taken as the definition of weak compactness.

So we may consider the sequence *inaccessible, weakly compact, measurable, strongly compact*, as arising from certain properties of increasing strength of $L_{\omega\omega}$, generalized to $L_{\kappa\kappa}$. What if these properties are imposed on some other logic, like $L^{II}_{\kappa\kappa}$ for example ? Magidor[11] proved that

(5) κ is extendible iff $L^{II}_{\kappa\kappa}$ has CMP(κ).

(6) The first supercompact is the first κ such that $L^{II}_{\kappa\kappa}$ has LST(κ)

Examination of the proof of (5) reveals that also

(7) κ is extendible iff $L^{II}_{\kappa\kappa}$ has MCMP(κ).

The cardinals κ for which $L^{II}_{\kappa\kappa}$ satisfies WCMP(κ) lie somewhere between the first subtle (see[18]) and the first totally indescribable.

Finally, we shall need some Erdös-cardinals, which unfortunately still lack model-theoretic definitions. The notation $\kappa \to (\lambda)^{<\omega}$ means: if $f:A \to \lambda$, where A is the set of finite subsets of κ, then there is a subset H of κ of ordertype λ such that for any two subsets x and y of H of the same finite power, $f(x) = f(y)$. The least κ such that

$\kappa \to (\lambda)^{<\omega}$ is denoted by $\kappa(\lambda)$.

To sum up, we have the following diagram:

On	
extendible	$L^{II}_{\kappa\kappa}$ has $CMP(\kappa)$
1st supercompact	$L^{II}_{\kappa\kappa}$ has $LST(\kappa)$
strongly compact	$L_{\kappa\kappa}$ has $CMP(\kappa)$
measurable	$L_{\kappa\kappa}$ has $MCMP(\kappa)$
$\kappa(\omega_1)$	?
$\kappa(\omega)$	?
?	$L^{II}_{\kappa\kappa}$ has $WCMP(\kappa)$
weakly compact	$L_{\kappa\kappa}$ has $WCMP(\kappa)$
inaccessible	$L_{\kappa\kappa}$ has $LST(\kappa)$
0	

Chapter 4. Basic results

In his thesis [16] J. Silver made an important contribution to the theory of Hanf-numbers of unbounded logics by establishing the following results:

<u>Theorem 4</u> (Silver [16])

(1) If $\kappa(\omega)$ exists, then $\kappa(\omega) < h(LW)$.

(2) If $\kappa(\alpha)$ exists for every $\alpha < \omega_1$, then $\sup\{\kappa(\alpha) \mid \alpha < \omega_1\} < h(LW)$.

(3) If $\kappa(\omega_1)$ exists, then $h(LW) < \kappa(\omega_1)$.

Claims (1) and (2) are proved as Lemma 3, using the fact that the definition of $\kappa(\alpha)$ is sufficiently absolute downwards. To prove (3), suppose $A \models \phi$, $\phi \in LW$ and $|A| = \kappa(\omega_1)$. Using the partition property of $\kappa(\omega_1)$, we can find an uncountable set H of order indiscernibles for A relative to formulae of LW. There is a standard procedure for generating arbitrarily large elementary extensions for models with indiscernibles (see e.g. the Streching Theorem in [2]). These extensions will be elementary also with respect to the logic LW, because well-foundedness depends on countable information only, and that is all contained in H.

The above theorem essentially solves the problem of the size of $h(LW)$ in the presence of large cardinals. Silver asks in his thesis (Question 6.3) whether it is possible for $h(LW)$ to be, in the absence of $\kappa(\omega_1)$, below the first weakly compact. In the next chapter we shall answer this question in the affirmative.

Let us now consider the logics $L_{\omega\omega_1}$ and $L_{\omega_1\omega_1}$. K.Kunen proved the following result for $L_{\omega_1\omega_1}$ but his proof actually uses $L_{\omega\omega_1}$ only.

<u>Theorem 5</u> (Kunen [8]) If $V = L^\mu$, then $h(L_{\omega\omega_1})$ exceeds the measurable cardinal.

The proof is based on the following idea. Suppose $V = L^\mu$, where μ is the normal measure on κ. Let T be a suitable finite part of ZF + $V = L^\mu$. There are, we can assume, transitive models M of T such that ${}^\omega M \subset M$. But suppose such an M has a normal measure on $\sigma > i_{0\omega}(\kappa)$. Then a large enough iterated ultrapower of the universe contains M. But the set $a = \{i_{0n}(\kappa) \mid n < \omega\}$ is in M, whence a already occurs after ω iterations, a contradiction, because then $i_{0\omega}(\kappa)$ would be a measurable cardinal of cofinality ω (in the ultrapower).

If $V = L^\mu$, then $\Delta^B(LI) = \Delta^B(L^{II})$ (see [24]) whence $h(LI)$ and $l(LI)$ exceed the measurable cardinal.

Concerning upper bounds for $h(L_{\omega_1\omega_1})$ we have the trivial:

<u>Remark 5</u> If there are strongly compact cardinals, then $h(L_{\omega_1\omega_1})$ is below the first of them.

M.Magidor proved that the first strongly compact can be the first measurable, establishing:

<u>Theorem 7</u> (Magidor [12]) If Con(ZFC + there is a strongly compact cardinal), then Con(ZFC + $h(L_{\omega_1\omega_1})$ < the first measurable).

The method used in the next chapter allows the words "strongly compact cardinal" above be replaced by "proper class of measurable cardinals".

Let us now consider L^{II}. Here we have the following important result implicit in Magidor's work:

<u>Theorem 8</u> (Magidor [11])

(1) If there is a supercompact cardinal, then

$$\text{1st measurable} < l(L^{II}) < \text{1st supercompact}.$$

(2) If there is an extendible cardinal, then

$$\text{1st supercompact} < h(L^{II}) < \text{1st extendible}.$$

Note at first that measurability of κ is second order definable over R_κ, whence there are lots and lots of measurables below $l(L^{II})$ (if they exist at all). That $l(L^{II})$ is below the first supercompact follows from the much stronger fact ([11]) that $L^{II}_{\kappa\kappa}$ has LST(κ). Similarly, $h(L^{II})$ < 1st extendible follows from the fact that $L^{II}_{\kappa\kappa}$ has CMP(κ). Finally, κ is the first supercompact iff for every $\alpha > \kappa$ there is a $\beta < \alpha$ such that R_β is elementarily embeddable into R_α. From this it follows that such a κ is below $h(L^{II})$.

Chapter 5. New results

Our first result provides a solution to Question 6.3 of [16]:

Theorem 9 Let L^* be any of the logics LW, $L_{\omega\omega_1}$, $L_{\omega_1\omega_1}$, LI. If Con(ZFC + there is a proper class of weakly compact cardinals), then Con(ZFC + $h(L^*)$ < 1st weakly compact).

As an indication of the rather long proof of Theorem 9, consider the following assumption: There is a $\phi \in$ LW such that

ZFC $\vdash$ If κ is the first weakly compact, then sup Sp(ϕ) exists and exceeds κ.

We shall derive a contradiction from this hypothesis on the assumption that there is a proper class κ_α, $\alpha \in$ On, of weakly compact cardinals and V = L. For simplicity, we also assume $\alpha < \kappa_\alpha$ for all $\alpha \in$ On. Using certain forcing techniques we can construct a sequence M_α, $\alpha \in$ On, of Boolean extensions such that

(i) $M_\alpha \subset M_\beta \subset M_0$ if $0 < \alpha < \beta$.
(ii) $M_\alpha \models \kappa_\alpha$ is the least weakly compact cardinal , $\alpha \geq 0$.

It follows from our hypothesis, that for some $\kappa \geq \kappa_0$, $M_0 \models$ sup Sp(ϕ) = κ. Pick $\alpha > 0$ such that $\kappa < \kappa_\alpha$. Then $M_\alpha \models$ sup Sp(ϕ) > κ_α, whence there is an $A \in M_\alpha$ such that $M_0 \models (|A| > \kappa \;\&\; A \models \phi)$, a contradiction with the choice of κ.

To obtain the full Theorem 9, an iteration of the above argument is used. Theorem 9 permits, in fact, a more general formulation: the logic L^* can be any logic, the syntax and semantics of which is absolute with respect to ω_1-closed Boolean extensions which preserve cofinalities.

The Theorem remains true if "weakly compact" be replaced by "Ramsey" or by "measurable". Thus we have a solution to number 9 of H.Friedman's 102 Problems in Mathematical Logic [4].

<u>Theorem 10</u> If L^* extends LI and Con(ZFC + there is a supercompact cardinal), then Con(ZFC + $h(L^*) >$ the first supercompact).

The desired property of L^* takes place in a model with the following properties:

(i) There <u>is</u> a supercompact cardinal κ.
(ii) There is an unbounded set of singular cardinals λ below κ such that $2^\lambda = \lambda^{+++}$.
(iii) Every singular cardinal $> \kappa$ is strong limit.

If $\phi \in L$ is a sentence having a model of power λ iff λ is singular and $2^\lambda \geq \lambda^{+++}$, then ϕ has arbitrarily large models of power $< \kappa$ (by (ii)), but none of power $> \kappa$ (by (iii) and Solovay's result [17] : GCH holds at every singular strong limit above a strongly compact).

In our third result we touch upon the propblem of measuring the Löwenheim number of a logic between LI and L^{II}. We have remarked already that $l(LI)$ can be below 2^ω even with MA, but on the other hand, if $V = L^\mu$, then $l(LI)$ exceeds the measurable cardinal. The following result demonstrates more strikingly the independence of $l(LI)$ from the points of our scale:

<u>Theorem 11</u> Suppose L^* is a sublogic of L^{II} but extends LI. Furthermore, suppose that the syntax and semantics of L^* are absolute with respect to Boolean extensions which preserve cardinals (e.g. $L^* = LI$). Assuming the consistency of sufficiently large cardinals, ZFC is consistent with any of the following conditions:

(1) $l(L^*) <$ 1st real-valued measurable $< 2^\omega$.
(2) 1st real-valued measurable $< l(L^*) < 2^\omega$.
(3) $l(L^*) <$ 1st inaccessible.
(4) 1st inaccessible $< l(L^*) <$ 1st Mahlo.
(5) 1st weakly inaccessible $< l(L^*) <$ 1st weakly Mahlo.
(6) 1st Mahlo $< l(L^*) <$ 1st weakly compact.
(7) 1st weakly compact $< l(L^*) <$ 1st Ramsey.
(8) 1st Ramsey $< l(L^*) <$ 1st measurable.
(9) 1st measurable $< l(L^*) <$ 1st supercompact.

The proofs of these independence results use various forms of iterated forcing. To indicate the general idea, we sketch the proof of the consistency of (7).

Suppose $(\kappa_\alpha)_{\alpha \in On}$ is a proper class of Ramsey cardinals in ascending order. We may assume the GCH and that $\alpha < \kappa_\alpha$ for $\alpha \in On$. Our first Boolean extension M_0 destroys the GCH at κ^+ where κ is the first weakly compact cardinal. Now in M_0, $\kappa < l(L^*)$. The next task is to kill all Ramsey cardinals below $l(L^*)$. The simplest way of doing this is to add, for every Ramsey λ below $l(L^*)$, λ new subsets to κ^+. In this procedure $l(L^*)$ may go up giving rise to a feeling of a never-ending iteration. However, if we let μ denote the number of all sentences of L^*, and we iterate μ^+ times the procedure of killing all Ramseys below $l(L^*)$, there will be a stage in which $l(L^*)$ does not go up, yielding the desired Boolean extension.

We have put a rather heavy absoluteness assumption on L^* in Theorem 11. In many of the cases (1) - (9) much less is needed.

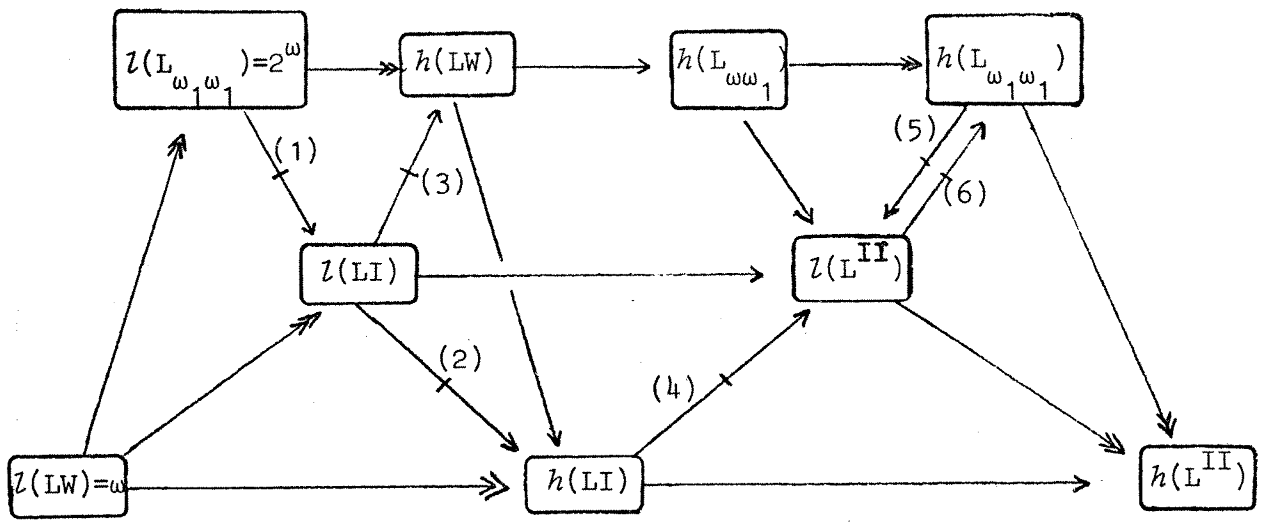
A major open problem in connection with LI is the following:

Problem Can $l(LI)$ be below the first weakly inaccessible cardinal ?

We have only been able to prove that one cannot obtain an affirmative answer by (set)forcing over L, $L[0^\#]$, or L^μ. This makes it look like a difficult problem.

Chapter 6. Conclusion

Let us now return to the diagrams of Löwenheim- and Hanf-numbers. A combined diagram looks like this:



Explanations:

(1) The consistency of $l(LI) < 2^\omega$ is proved in [20].

(2) The consistency of $h(LI) < l(LI)$ is proved in [24].

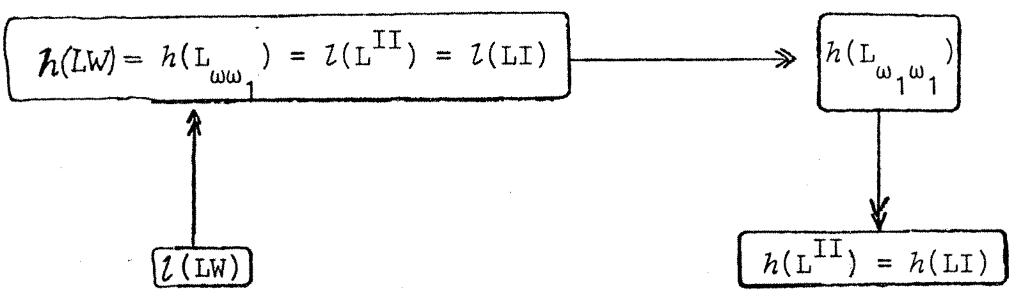
(3) If $V = L$ and $\exists\kappa(\omega)$, then $h(LW) < \kappa(\omega) < l(LI)$.

(4) If $V = L$, then $l(L^{II}) < h(L^{II}) = h(LI)$.

(5) If $V = L$, then $l(L^{II}) = h(L_{\omega\omega_1}) < h(L_{\omega_1\omega_1})$.

(6) If $h(L_{\omega_1\omega_1}) <$ 1st weakly compact, then $h(L_{\omega_1\omega_1}) < l(L^{II})$.

In the inner model L we have the following collapsed diagram:



Consideration of these diagrams raises, among others, the following questions:

Problems Decide the consistency of the following statements:

(1) $h(\mathrm{LW}) = h(\mathrm{LI})$.

(2) $h(\mathrm{LI}) \leq h(L_{\omega\omega_1})$.

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