Models and Games

Jouko Väänänen

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Preface

When I was a beginning mathematics student a friend gave me a set of lecture notes for a course on infinitary logic given by Ronald Jensen. On the first page was the definition of a partial isomorphism: a set of partial mappings between two structures with the back-and-forth property. I became immediately interested and now—37 years later—I have written a book on this very concept.

This book can be used as a text for a course in model theory with a game- and set-theoretic bent.

I am indebted to the students who have given numerous comments and corrections during the courses I have given on the material of this book both in Amsterdam and in Helsinki. I am also indebted to members of the Helsinki Logic Group, especially Tapani Hyttinen and Juha Oikkonen, for discussions, criticisms and new ideas over the years on Ehrenfeucht-Fraïssé Games in uncountable structures. I am grateful to Fan Yang for reading and commenting on parts of the manuscript.

I am extremely grateful to my wife Juliette Kennedy for encouraging me to finish this book, for reading and commenting on the manuscript pointing out necessary corrections, and for supporting me in every possible way during the writing process.

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1
Introduction

A recurrent theme in this book is the concept of a game. There are essentially three kinds of games in logic. One is the Semantic Game, also called the Evaluation Game, where the *truth* of a given sentence in a given model is at issue. Another is the Model Existence Game, where the *consistency* in the sense of having a model, or equivalently in the sense of impossibility to derive a contradiction, is at issue. Finally there is the Ehrenfeucht-Fraïssé Game, where *separation* of a model from another by finding a property that is true in one given model but false in another is the goal. The three games are closely linked to each other and one can even say they are essentially variants of just one basic game. This basic game arises from our understanding of the quantifiers. The purpose of this book is to make this strategic aspect of logic perfectly transparent and to show that it underlies not only first order logic but infinitary logic and logic with generalized quantifiers alike.

We call the close link between the three games the *Strategic Balance of Logic* (Figure 1.1). This balance is perfectly commutative, in the sense that winning strategies can be transferred from one game to another. This mere fact is testimony to the close connection between logic and games, or, thinking semantically, between games and models. This connection arises from the nature of quantifiers. Introducing infinite disjunctions and conjunctions does not upset the balance, barring some set theoretic issues that may surface. In the last chapter of this book we consider generalized quantifiers and show that the Strategic Balance of Logic persists even in the presence of generalized quantifiers.

The purpose of this book is to present the Strategic Balance of Logic in all its glory.

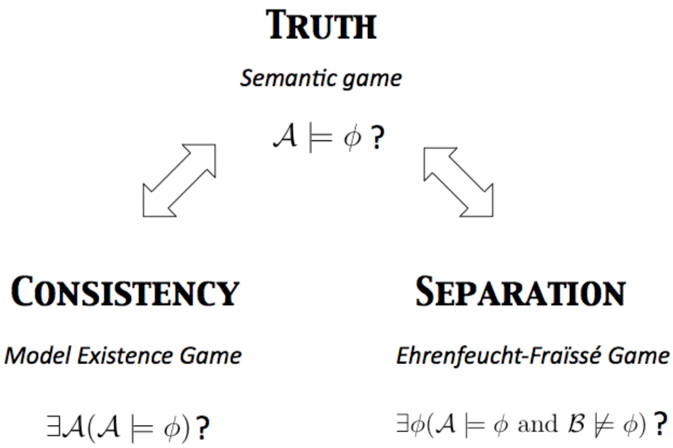

Figure 1.1 The Strategic Balance of Logic.

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Games

3.1 Introduction

In this first part we march through the mathematical details of zero-sum two-person games of perfect information in order to be well prepared for the introduction of the three games of the Strategic Balance of Logic (see Figure 1.1) in the subsequent parts of the book. Games are useful as intuitive guides in proofs and constructions but it is also important to know how to make the intuitive arguments and concepts mathematically exact.

3.2 Two-Person Games of Perfect Information

Two-person games of perfect information are like chess: two players set their wits against each other with no role for chance. One wins and the other loses. Everything is out in the open, and the winner wins simply by having a better strategy than the loser.

A Preliminary Example: Nim

In the game of Nim, if it is simplified to the extreme, there are two players **I** and **II** and a pile of six identical tokens. During each round of the game player **I** first removes one or two tokens from the top of the pile and then player **II** does the same, if any tokens are left. Obviously there can be at most three rounds. The player who removes the last token wins and the other one loses.

The game of Figure 3.1 is an example of a zero-sum two-person game of perfect information. It is zero-sum because the victory of one player is the loss of the other. It is of perfect information because both players know what the other player has played. A moment's reflection reveals that player **II** has a way

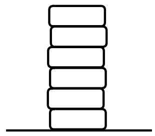

Figure 3.1 The game of Nim.

Play	Winner
111111	**II**
11112	**I**
11121	**I**
11211	**I**
1122	**II**
12111	**I**
1212	**II**
1221	**II**
21111	**I**
2112	**II**
2121	**II**
2211	**II**
222	**I**

Figure 3.2 Plays of Nim.

of playing which guarantees that she[1] wins: During the first round she takes away one token if player **I** takes away two tokens, and two tokens if player **I** takes away one token. Then we are left with three tokens. During the second round she does the same: she takes away the last token if player **I** takes away two tokens, and the last two tokens if player **I** takes away one token. We say that player **II** has a winning strategy in this game.

If we denote the move of a player by a symbol – 1 or 2 –we can form a list of all sequences of ones and twos that represent a play of the game. (See Figure 3.2.)

The set of finite sequences displayed in Figure 3.2 has the structure of a tree, as Figure 3.3 demonstrates. The tree reveals easily the winning strategy of player **II**. Whatever player **I** plays during the first round, player **II** has an option which leaves her in such a position (node 12 or 21 in the tree) that whether the opponent continues with 1 or 2, she has a winning move (1212, 1221, 2112 or 2121).

We can express the existence of a winning strategy for player **II** in the above game by means of first order logic as follows: Let us consider a vocabulary

[1] We adopt the practice of referring to the first player by "he" and the second player by "she".

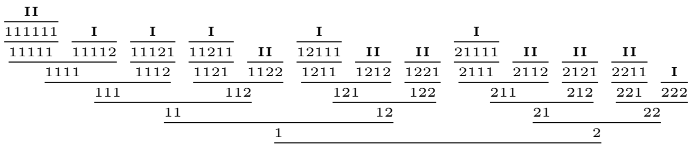

Figure 3.3

$L = \{W\}$, where W is a 4-place predicate symbol. Let $\mathcal{M}$ be an L-structure[2] with $M = \{1, 2\}$ and

$$W^{\mathcal{M}} = \{(a_0, b_0, a_1, b_1) \in M^4 : a_0 + b_0 + a_1 + b_1 = 6\}.$$

Now we have just proved

$$\mathcal{M} \models \forall x_0 \exists y_0 \forall x_1 \exists y_1 W(x_0, y_0, x_1, y_1). \tag{3.1}$$

Conversely, if $\mathcal{M}$ is an arbitrary L-structure, condition (3.1) defines *some* game, maybe not a very interesting one but a game nonetheless: Player **I** picks an element $a_0 \in M$, then player **II** picks an element $b_0 \in M$. Then the same is repeated: player **I** picks an element $a_1 \in M$, then player **II** picks an element $b_1 \in M$. After this player **II** is declared the winner if $(a_0, b_0, a_1, b_1) \in W^{\mathcal{M}}$, and otherwise player **I** is the winner. By varying the structure $\mathcal{M}$ we can model in this way various two-person two-round games of perfect information. This gives a first hint of the connection between games and logic.

Games—a more general formulation

Above we saw an example of a two-person game of perfect information. This concept is fundamental in this book. In general, the simplest formulation of such a game is as follows (see Table 3.2): There are two players[3] **I** and **II**, a domain A, and a natural number n representing the length of the game. Player **I** starts the game by choosing some element $x_0 \in A$. Then player **II** chooses $y_0 \in A$. After x_i and y_i have been played, and $i + 1 < n$, player **I** chooses $x_{i+1} \in A$ and then player **II** chooses $y_{i+1} \in A$. After n rounds the game ends. To decide who wins we fix beforehand a set $W \subseteq A^{2n}$ of sequences

$$(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \tag{3.2}$$

[2] For the definition of an L-structure see Definition 5.1.
[3] There are various names in the literature for player **I** and **II**, such as player **I** and player **II**, spoiler and duplicator, Nature and myself, or Abelard and Eloise.

I	**II**
x_0	
	y_0
x_1	
	y_1
$\vdots$	
	$\vdots$
x_{n-1}	
	y_{n-1}

Table 3.1 *A game.*

and declare that player **II** wins the game if the sequence formed during the game is in W; otherwise player **I** wins. We denote this game by $\mathcal{G}_n(A, W)$. For example, if $W = \emptyset$, player **II** cannot possibly win, and if $W = A^{2n}$, player **I** cannot possibly win. If W is a set of sequences $(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ where $x_0 = x_1$ and if moreover A has at least two elements, then **II** could not possibly win, as she cannot prevent player **I** from playing x_0 and x_1 differently. On the other hand, W could be the set of all sequences (3.2) such that $y_0 = y_1$. Then $\exists$ can always win because all she has to do during the game is make sure that she chooses y_0 and y_1 to be the same element.

If player **II** has a way of playing that guarantees a sure win, i.e. the opponent **I** loses whatever moves he makes, we say that player **II** has a winning strategy in the game. Likewise, if player **I** has a way of playing that guarantees a sure win, i.e. player **II** loses whatever moves she makes, we say that player **I** has a winning strategy in the game. To make intuitive concepts, such as "way of playing" more exact in the next chapter we define the basic concepts of game-theory in a purely mathematical way.

Example 3.1 The game of Nim presented in the previous chapter is in the present notation $\mathcal{G}_3(\{1, 2\}, W)$, where

$$W = \{(a_0, b_0, a_1, b_1, a_2, b_2) \in \{1, 2\}^6 : \sum_{i=0}^{n}(a_i + b_i) = 6 \text{ for some } n \leq 2\}.$$

We allow three rounds as theoretically the players could play three rounds even if player **II** can force a win in two rounds.

Example 3.2 Consider the following game on a set A of integers:

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Models

5.1 Introduction

The concept of a model (or structure) is one of the most fundamental in logic. In brief, while the meaning of logical symbols $\wedge$, $\vee$, $\exists$, ... is always fixed, models give meaning to non-logical symbols such as constant, predicate and function symbols. When we have agreed about the meaning of the logical and non-logical symbols of logic, we can then define the meaning of arbitrary formulas.

Depending on context and preference, models appear in logic in two roles. They can serve the auxiliary role of clarifying logical derivation. For example, one quick way to tell what it means for φ to be a logical consequence of ψ is to say that in every model where ψ is true also φ is true. It is then an almost trivial matter to understand why for example $\forall x \exists y \varphi$ is a logical consequence of $\exists y \forall x \varphi$ but $\forall y \exists x \varphi$ is in general not.

Alternatively models can be the prime objects of investigation and it is the logical derivation that is in an auxiliary role of throwing light on properties of models. This is manifestly demonstrated by the Completeness Theorem which says that any set T of first order sentences has a model unless a contradiction can be logically derived from T, which entails that the two alternative perspectives of models are really equivalent. Since derivations are finite, this implies the important Compactness Theorem: If a set of first order sentences is such that each of its finite subsets has a model it itself has a model. The Compactness Theorem has led to an abundance of non-isomorphic models of first order theories, and constitutes the origin of the whole subject of Model Theory. In this chapter models are indeed the prime objects of investigation and we introduce auxiliary concepts such as the Ehrenfeucht-Fraïssé Game that help us understand models.

We use the words "model" and "structure" as synonyms. We have a slight preference for the word "structure" in a context where absolute generality prevails and the structures are not assumed to satisfy any particular axioms. Respectively, our preference is to call a structure that satisfies some given axioms a model, so a structure satisfying a theory is called a model of the theory.

5.2 Basic Concepts

A *vocabulary* is any set L of predicate symbols $P, Q, R, \ldots$, function symbols $f, g, h, \ldots$, and constant symbols $c, d, e, \ldots$. Each vocabulary has an *arity-function*

$$\#_L : L \to \mathbb{N}$$

which tells the arity of each symbol. Thus if $P \in L$, then P is a $\#_L(P)$-ary predicate symbol. If $f \in L$, then f is a $\#_L(f)$-ary function symbol. Finally, $\#_L(c)$ is assumed to be 0 for constants $c \in L$. Predicate or function symbols of arity 1 are called *unary* or *monadic*, and those of arity 2 are called *binary*. A vocabulary is called unary (or binary) if it contains only unary (respectively, binary) symbols. A vocabulary is called *relational* if it contains no function or constant symbols.

Definition 5.1 An L-*structure* (or L-*model*) is a pair $\mathcal{M} = (M, \text{Val}_{\mathcal{M}})$, where M is a non-empty set called the *universe (or the domain)* of $\mathcal{M}$, and $\text{Val}_{\mathcal{M}}$ is a function defined on L with the following properties:

1. If $R \in L$ is a relation symbol and $\#_L(R) = n$, then $\text{Val}_{\mathcal{M}}(R) \subseteq M^n$.
2. If $f \in L$ is a function symbol and $\#_L(f) = n$, then $\text{Val}_{\mathcal{M}}(f) : M^n \to M$.
3. If $c \in L$ is a constant symbol, then $\text{Val}_{\mathcal{M}}(c) \in M$.

We use $\text{Str}(L)$ to denote the class of all L-structures.

We usually shorten $\text{Val}_{\mathcal{M}}(R)$ to $R^{\mathcal{M}}$, $\text{Val}_{\mathcal{M}}(f)$ to $f^{\mathcal{M}}$ and $\text{Val}_{\mathcal{M}}(c)$ to $c^{\mathcal{M}}$. If no confusion arises, we use the notation

$$\mathcal{M} = (M, R_1^{\mathcal{M}}, \ldots, R_n^{\mathcal{M}}, f_1^{\mathcal{M}}, \ldots, f_m^{\mathcal{M}}, c_1^{\mathcal{M}}, \ldots, c_k^{\mathcal{M}})$$

for an L-structure $\mathcal{M}$, where $L = \{R_1, \ldots, R_n, f_1, \ldots, f_m, c_1 \ldots, c_k\}$.

Example 5.2 Graphs are L-structures for the relational vocabulary $L = \{E\}$, where E is a predicate symbol with $\#_L(E) = 2$. Groups are L-structures for $L = \{\circ\}$, where $\circ$ is a binary function symbol. Fields are L-structures for $L = \{+, \cdot, 0, 1\}$, where $+, \cdot$ are binary function symbols and $0, 1$ are constant symbols. Ordered sets (i.e. linear orders) are L-structures for the relational vocabulary $L = \{<\}$, where $<$ is a binary predicate symbol. If $L = \emptyset$, an L-structure (M) is a structure with just the universe and no structure in it.

If $\mathcal{M}$ is a structure and π maps M bijectively onto another set M', we can use π to copy the relations, functions and constants of $\mathcal{M}$ on M'. In this way we get a perfect copy $\mathcal{M}'$ of $\mathcal{M}$ which differs from $\mathcal{M}$ only in the respect that the underlying elements are different. We then say that $\mathcal{M}'$ is an isomorphic copy of $\mathcal{M}$. For all practical purposes we consider the structures $\mathcal{M}$ and $\mathcal{M}'$ as one and the same structure. However, they are not the same structure, just isomorphic. This may sound as if isomorphism was a rather trivial matter, but this is not true. In many cases it is a highly non-trivial enterprise to investigate whether two structures are isomorphic or not. In the realm of finite structures the question of deciding whether two given structures are isomorphic or not is a famous case of a complexity question which is between P (polynomial time) and NP (non-deterministic polynomial time) and about which we do not know whether it is NP-complete. In the light of present knowledge it is conceivable that this question is strictly between P and NP.

Definition 5.3 L–structures $\mathcal{M}$ and $\mathcal{M}'$ are *isomorphic* if there is a bijection

$$\pi : M \to M'$$

such that

1. For all $a_1, \ldots, a_{\#_L(R)} \in M$:

$$(a_1, \ldots, a_{\#_L(R)}) \in R^{\mathcal{M}} \iff (\pi(a_1), \ldots, \pi(a_{\#_L(R)})) \in R^{\mathcal{M}'}.$$

2. For all $a_1, \ldots, a_{\#_L(f)} \in M$:

$$f^{\mathcal{M}'}(\pi(a_1), \ldots, \pi(a_{\#_L(f)})) = \pi(f^{\mathcal{M}}(a_1, \ldots, a_{\#_L(f)})).$$

3. For all $c \in L$: $\pi(c^{\mathcal{M}}) = c^{\mathcal{M}'}$.

In this case we say that π is an *isomorphism* $\mathcal{M} \to \mathcal{M}'$, denoted

$$\pi : \mathcal{M} \cong \mathcal{M}'.$$

If also $\mathcal{M} = \mathcal{M}'$, we say that π an *automorphism* of $\mathcal{M}$.

Example 5.4 *Unary* (or *monadic*) structures, i.e. L-structures for unary L, are particularly simple and easy to deal with. Figure 5.1 depicts a unary structure. Suppose L consists of unary predicate symbols $R_1, \ldots, R_n$ and $\mathcal{A}$ is an L-structure. If $X \subseteq A$ and $d \in \{0, 1\}$, let $X^d = X$ if $d = 0$ and $X^d = A \setminus X$ otherwise. Suppose $\epsilon : \{1, \ldots, n\} \to \{0, 1\}$. The *$\epsilon$-constituent* of $\mathcal{A}$ is the set

$$C_\epsilon(\mathcal{A}) = \bigcap_{i=1}^{n} (R_i^{\mathcal{A}})^{\epsilon(i)}.$$

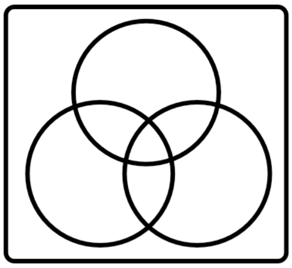

Figure 5.1 A unary structure.

A priori, the 2^n sets $C_\epsilon(\mathcal{A})$ can each have any cardinality whatsoever. It is the nature of unary structures that the constituents are totally independent from each other. If $\mathcal{A} \cong \mathcal{B}$, then

$$|C_\epsilon(\mathcal{A})| = |C_\epsilon(\mathcal{B})| \tag{5.1}$$

for every ϵ. Conversely, if two L-structures $\mathcal{A}$ and $\mathcal{B}$ satisfy equation (5.1) for every ϵ, then $\mathcal{A} \cong \mathcal{B}$ (see Exercise 5.6). We can say that the function $\epsilon \mapsto |C_\epsilon(\mathcal{A})|$ characterizes completely (i.e. up to isomorphism) the unary structure $\mathcal{A}$. There is nothing more we can say about $\mathcal{A}$ but this function.

Example 5.5 *Equivalence relations*, i.e. L-structures $\mathcal{M}$ for $L = \{\sim\}$ such that $\sim^{\mathcal{M}}$ is a symmetric ($x \sim y \Rightarrow y \sim x$), transitive ($x \sim y \sim z \Rightarrow x \sim z$) and reflexive ($x \sim x$) relation on M can be characterized almost as easily as unary structures. Let for every cardinal number $\kappa \leq |M|$ the number of equivalence classes of $\sim^{\mathcal{M}}$ of cardinality κ be denoted by $EC_\kappa(\mathcal{M})$. If $\mathcal{A} \cong \mathcal{B}$, then

$$EC_\kappa(\mathcal{A}) = EC_\kappa(\mathcal{B}) \tag{5.2}$$

for every $\kappa \leq |A|$. Conversely, if two L-structures $\mathcal{A}$ and $\mathcal{B}$ satisfy equation (5.2) for every $\kappa \leq |A \cup B|$, then $\mathcal{A} \cong \mathcal{B}$ (see Exercise 5.12). We can say that the function $\kappa \mapsto EC_\kappa(\mathcal{A})$ characterizes completely (i.e. up to isomorphism) the equivalence relation $\mathcal{A}$. There is nothing more we can say about $\mathcal{A}$ but this function. For equivalence relations on a finite universe of size n this function is a function $f : \{1, \ldots, n\} \to \{0, \ldots, n\}$ such that

$$\sum_{i=1}^{n} i f(i) = n.$$

The so-called Hardy-Ramanujan asymptotic formula says that the number of

Proof Let $P = \{f \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) : \mathrm{dom}(f) \text{ is finite}\}$. It turns out that this straightforward choice works. Clearly, $P \neq \emptyset$. Suppose then $f \in P$ and $a \in A$. Let us enumerate f as $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ where $a_1 < \ldots < a_n$. Since f is a partial isomorphism, also $b_1 < \ldots < b_n$. Now we consider different cases. If $a < a_1$, we choose $b < b_1$ and then $f \cup \{(a, b)\} \in P$. If $a_i < a < a_{i+1}$, we choose $b \in B$ so that $b_i < b < b_{i+1}$ and then $f \cup \{(a, b)\} \in P$. If $a_n < a$, we choose $b > b_n$ and again $f \cup \{(a, b)\} \in P$. Finally, if $a = a_i$, we let $b = b_i$ and then $f \cup \{(a, b)\} = f \in P$. We have proved (5.8). Condition (5.9) is proved similarly. □

Putting Proposition 5.16 and Proposition 5.17 together yields the famous result of Georg Cantor [Can95]: countable dense linear orders without endpoints are isomorphic. See Exercise 6.29 for a more general result.

5.5 The Ehrenfeucht-Fraïssé Game

In Section 4.3 we introduced the Ehrenfeucht-Fraïssé Game played on two graphs. This game was used to measure to what extent two graphs have similar properties, especially properties expressible in the first order language of graphs limited to a fixed quantifier rank. In this section we extend this game to the context of arbitrary structures, not just graphs.

Let us recall the basic idea behind the Ehrenfeucht-Fraïssé Game. Suppose $\mathcal{A}$ and $\mathcal{B}$ are L-structures for some relational L. We imagine a situation in which two mathematicians argue about whether $\mathcal{A}$ and $\mathcal{B}$ are isomorphic or not. The mathematician that we denote by **II** claims that they are isomorphic, while the other mathematician whom we call **I** claims the models have an intrinsic structural difference and they cannot possibly be isomorphic.

The matter would be quickly resolved if **II** was required to show the claimed isomorphism. But the rules of the game are different. The rules are such that **II** is required to show only small pieces of the claimed isomorphism.

More exactly, **I** asks what is the image of an element a_1 of A that he chooses at will. Then **II** is required to respond with some element b_1 of B so that

$$\{(a_1, b_1)\} \in \mathrm{Part}(\mathcal{A}, \mathcal{B}). \tag{5.10}$$

Alternatively, **I** might have chosen an element b_1 of B and then **II** would have been required to produce an element a_1 of A such that (5.10) holds. The one-element mapping $\{(a_1, b_1)\}$ is called the *position* in the game after the first move.

Now the game goes on. Again **I** asks what is the image of an element a_2 of

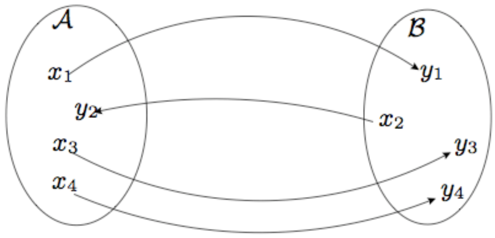

Figure 5.5 The Ehrenfeucht-Fraïssé Game

A (or alternatively he can ask what is the pre-image of an element b_2 of B). Then **II** produces an element b_2 of B (or in the alternative case an element a_2 of A). In either case the choice of **II** has to satisfy

$$\{(a_1, b_1), (a_2, b_2)\} \in \mathrm{Part}(\mathcal{A}, \mathcal{B}). \tag{5.11}$$

Again, $\{(a_1, b_1), (a_2, b_2)\}$ is called the position after the second move.

We continue until the position

$$\{(a_1, b_1), \ldots, (a_n, b_n)\} \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$$

after the nth move has been produced. If **II** has been able to play all the moves according to the rules she is declared the winner. Let us call this game $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$. Figure 5.5 pictures the situation after four moves. If **II** can win repeatedly whatever moves **I** plays, we say that **II** has a *winning strategy*.

Example 5.18 Suppose $\mathcal{A}$ and $\mathcal{B}$ are two L-structures and $L = \emptyset$. Thus the structures $\mathcal{A}$ and $\mathcal{B}$ consist merely of a universe with no structure on it. In this singular case any one-to-one mapping is a partial isomorphism. The only thing player **II** has to worry about, say in (5.11), is that $a_1 = a_2$ if and only if $b_1 = b_2$. Thus **II** has a winning strategy in $\mathrm{EF}_n(\mathcal{A}, \mathcal{B})$ if A and B both have at least n elements. So **II** can have a winning strategy even if A and B have different cardinality and there could be no isomorphism between them for the trivial reason that there is no bijection. The intuition here is that by playing a finite number of elements, or even $\aleph_0$ many, it is not possible to get hold of the cardinality of the universe if it is infinite.

Example 5.19 Let $\mathcal{A}$ be a linear order of length 3 and $\mathcal{B}$ a linear order of length 4. How many moves does **I** need to beat **II**? Suppose $A = \{a_1, a_2, a_3\}$ in increasing order and $B = \{b_1, b_2, b_3, b_4\}$ in increasing order. Clearly, if **I** plays at any point the smallest element, also **II** has to play the smallest element or face defeat on the next move. Also, if **I** plays at any point the smallest but one element, also **II** has to play the smallest but one element or face defeat in two moves. Now in $\mathcal{A}$ the smallest but one element is the same as the largest but one element, while in $\mathcal{B}$ they are different. So if **I** starts with a_2, **II** has to play b_2 or b_3, or else she loses in one move. Suppose she plays b_2. Now **I** plays b_3 and **II** has no good moves left. To obey the rules, she must play a_3. That is how long she can play, for now when **I** plays b_4, **II** cannot make a legal move anymore. In fact **II** has a winning strategy in $\mathrm{EF}_2(\mathcal{A}, \mathcal{B})$ but **I** has a winning strategy in $\mathrm{EF}_3(\mathcal{A}, \mathcal{B})$.

We now proceed to a more exact definition of the Ehrenfeucht-Fraïssé Game.

Definition 5.20 Suppose L is a vocabulary and $\mathcal{M}, \mathcal{M}'$ are L-structures such that $M \cap M' = \emptyset$. The *Ehrenfeucht-Fraïssé Game* $\mathrm{EF}_n(\mathcal{M}, \mathcal{M}')$ is the game $\mathcal{G}_n(M \cup M', W_n(\mathcal{M}, \mathcal{M}'))$, where $W_n(\mathcal{M}, \mathcal{M}') \subseteq (M \cup M')^{2n}$ is the set of $p = (x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ such that:

(G1) For all $i < n$: $x_i \in M \iff y_i \in M'$.
(G2) If we denote

$$v_i = \begin{cases} x_i & \text{if } x_i \in M \\ y_i & \text{if } y_i \in M \end{cases} \quad v_i' = \begin{cases} x_i & \text{if } x_i \in M' \\ y_i & \text{if } y_i \in M', \end{cases}$$

then

$$f_p = \{(v_0, v_0'), \ldots, (v_{n-1}, v_{n-1}')\}$$

is a partial isomorphism $\mathcal{M} \to \mathcal{M}'$.

We call v_i and v_i' *corresponding* elements. The infinite game $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ is defined quite similarly, that is, it is the game $\mathcal{G}_\omega(M \cup M', W_\omega(\mathcal{M}, \mathcal{M}'))$, where $W_\omega(\mathcal{M}, \mathcal{M}')$ is the set of $p = (x_0, y_0, x_1, y_1, \ldots)$ such that for all $n \in \mathbb{N}$ we have $(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in W_n(\mathcal{M}, \mathcal{M}')$.

Note that the game EF_ω is a closed game.

Proposition 5.21 *Suppose L is a vocabulary and $\mathcal{A}$ and $\mathcal{B}$ are L-structures. the following are equivalent:*

1. $\mathcal{A} \simeq_p \mathcal{B}$.
2. **II** *has a winning strategy in* $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$.

Proof Assume $A \cap B = \emptyset$. Let P be first a back-and-forth set for $\mathcal{A}$ and $\mathcal{B}$. We define a winning strategy $\tau = (\tau_i : i < \omega)$ for **II**. Since $P \neq \emptyset$ we can fix an element f of P. Condition (5.8) tells us that if $a_1 \in A$, then there are $b_1 \in B$ and g such that

$$f \cup \{(a_1, b_1)\} \subseteq g \in P. \tag{5.12}$$

Let $\tau_0(a_1)$ be one such b_1. Likewise, if $b_1 \in B$, then there are $a_1 \in A$ such that (5.12) holds and we can let $\tau_0(b_1)$ be some such a_1. We have defined $\tau_0(c_1)$ whatever c_1 is. To define $\tau_1(c_1, c_2)$, let us assume **I** played $c_1 = a_1 \in A$. Thus (5.12) holds with $b_1 = \tau_0(a_1)$. If $c_2 = a_2 \in A$ we can use (5.8) again to find $b_2 = \tau_1(a_1, a_2) \in B$ and h such that

$$f \cup \{(a_1, b_1), (a_2, b_2)\} \subseteq h \in P.$$

The pattern should now be clear. The back-and-forth set P guides **II** to always find a valid move. Let us then write the proof in more detail: Suppose we have defined τ_i for $i < j$ and we want to define τ_j. Suppose player **I** has played $x_0, \ldots, x_{j-1}$ and player **II** has followed τ_i during round $i < j$. During the inductive construction of τ_i we took care to define also a partial isomorphism $f_i \in P$ such that $\{v_0, \ldots, v_{i-1}\} \subseteq \mathrm{dom}(f_{i-1})$. Now player **I** plays x_j. By assumption there is $f_j \in P$ extending f_{j-1} such that if $x_j \in A$, then $x_j \in \mathrm{dom}(f_j)$ and if $x_j \in B$, then $x_j \in \mathrm{rng}(f_j)$. We let $\tau_j(x_0, \ldots, x_j) = f_j(x_j)$ if $x_j \in A$ and $\tau_j(x_0, \ldots, x_j) = f_j^{-1}(x_j)$ otherwise. This ends the construction of τ_j. This is a winning strategy because every f_p extends to a partial isomorphism $\mathcal{M} \to \mathcal{N}$.

For the converse, suppose $\tau = (\tau_n : n < \omega)$ is a winning strategy of **II**. Let Q consist of all plays of $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$ in which player **II** has used τ. Let P consist of all possible f_p where p is a position in the game $\mathrm{EF}_\omega(\mathcal{A}, \mathcal{B})$ with an extension in Q. It is clear that P is non-void and has the properties (5.8) and (5.9). □

To prove partial isomorphism of two structures we now have two alternative methods:

1. Construct a back-and-forth set.
2. Show that player **II** has a winning strategy in EF_ω.

By Proposition 5.21 these methods are equivalent. In practice one uses the game as a guide to intuition and then for a formal proof one usually uses a back-and-forth set.

5.6 Back-and-Forth Sequences

Back-and-forth sets and winning strategies of player **II** in the Ehrenfeucht-Fraïssé Game EF_ω correspond to each other. There is a more refined concept, called back-and-forth sequence, which corresponds to a winning strategy of player **II** in the finite game EF_n.

Definition 5.22 A *back-and-forth sequence* $(P_i : i \leq n)$ is defined by the conditions

$$\emptyset \neq P_n \subseteq \ldots \subseteq P_0 \subseteq \mathrm{Part}(\mathcal{A}, \mathcal{B}). \tag{5.13}$$

$$\forall f \in P_{i+1} \forall a \in A \exists b \in B \exists g \in P_i (f \cup \{(a, b)\} \subseteq g) \text{ for } i < n. \tag{5.14}$$

$$\forall f \in P_{i+1} \forall b \in B \exists a \in A \exists g \in P_i (f \cup \{(a, b)\} \subseteq g) \text{ for } i < n. \tag{5.15}$$

If P is a back-and-forth set, we can get back-and-forth sequences $(P_i : i \leq n)$ of any length by choosing $P_i = P$ for all $i \leq n$. But the converse is not true: the sets P_i need by no means be themselves back-and-forth sets. Indeed, pairs of countable models may have long back-and-forth sequences without having any back-and-forth sets. Let us write

$$\mathcal{A} \simeq_p^n \mathcal{B}$$

if there is a back-and-forth sequence of length n for $\mathcal{A}$ and $\mathcal{B}$.

Lemma 5.23 *The relation $\simeq_p^n$ is an equivalence relation on $Str(L)$.*

Proof Exactly as Lemma 5.15. □

Example 5.24 We use $(\mathbb{N} + \mathbb{N}, <)$ to denote the linear order obtained by putting two copies of $(\mathbb{N}, <)$ one after the other. (The ordinal of this order is $\omega + \omega$.) Now $(\mathbb{N}, <) \simeq_p^2 (\mathbb{N} + \mathbb{N}, <)$, for we may take

$$P_2 = \{\emptyset\}.$$
$$P_1 = \{\{(a, b)\} : 0 < a \in \mathbb{N},\ 0 < b \in \mathbb{N} + \mathbb{N}\} \cup \{(0, 0)\} \cup P_2.$$
$$P_0 = \{\{(a_0, b_0), (a_1, b_1)\} : a_0 < a_1 \in \mathbb{N},\ b_0 < b_1 \in \mathbb{N} + \mathbb{N}\} \cup P_1.$$

Note that $(\mathbb{N}, <) \not\simeq_p^3 (\mathbb{N} + \mathbb{N}, <)$.

Proposition 5.25 *Suppose $\mathcal{A}$ and $\mathcal{B}$ are discrete linear orders (i.e. every element with a successor has an immediate successor and every element with a predecessor has an immediate predecessor) with no endpoints, and $n \in \mathbb{N}$. Then $\mathcal{A} \simeq_p^n \mathcal{B}$.*

Proof Let P_i consist of $f \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$ with the following property: $f = \{(a_0, b_0), \ldots, (a_{n-i-1}, b_{n-i-1})\}$ where

$$a_0 \leq \ldots \leq a_{n-i-1},$$
$$b_0 \leq \ldots \leq b_{n-i-1},$$

and for all $0 \leq j < n - i - 1$ if $|(a_j, a_{j+1})| < 2^i$ or $|(b_j, b_{j+1})| < 2^i$, then $|(a_j, a_{j+1})| = |(b_j, b_{j+1})|$. □

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5.37 Show that there is a complete separable metric space (Polish space) $\mathcal{M} = (M, d, \mathbb{R}, <_{\mathbb{R}})$ and a non-complete separable metric space $\mathcal{M}' = (M', d', \mathbb{R}, <_{\mathbb{R}})$ such that $\mathcal{M} \simeq_p \mathcal{M}'$.

5.38 Suppose $\mathcal{A}$ and $\mathcal{B}$ are structures of the same relational vocabulary L and $A \cap B = \emptyset$. The *disjoint sum* of $\mathcal{A}$ and $\mathcal{B}$ is the L-structure

$$(A \cup B, (R^{\mathcal{A}} \cup R^{\mathcal{B}})_{R \in L}).$$

Show that partial isomorphism is preserved by disjoint sums of models.

5.39 Suppose $\mathcal{A}$ and $\mathcal{B}$ are structures of the same vocabulary L. The *direct product* of $\mathcal{A}$ and $\mathcal{B}$ is the L-structure

$$(A \times B, (R^{\mathcal{A}} \times R^{\mathcal{B}})_{R \in L},$$

$$(((a_0, b_0) ..., (a_n, b_n)) \mapsto (f^{\mathcal{A}}(a_0, ..., a_n), f^{\mathcal{B}}(b_0, ..., b_n)))_{f \in L},$$

$$((c^{\mathcal{A}}, c^{\mathcal{B}}))_{c \in L}).$$

Show that partial isomorphism is preserved by direct products of models.

5.40 Show that if two structures are partially isomorphic, then they are *potentially isomorphic*[2] i.e. there is a forcing extension in which they are isomorphic. Conversely, show that if two structures are potentially isomorphic, then they are partially isomorphic.

5.41 Consider $\mathrm{EF}_2(\mathcal{M}, \mathcal{N})$, where $\mathcal{M} = (\mathbb{R} \times \{0\}, f)$, $f(x, 0) = (x^2, 0)$ and $\mathcal{N} = (\mathbb{R} \times \{1\}, g)$, $g(x, 1) = (x^3, 1)$. Player **I** can win even without looking at the moves of **II**. How?

5.42 Consider $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{N})$, where $\mathcal{M} = (\mathbb{R} \times \{0\}, f)$, $f(x, 0) = (x^3, 0)$ and $\mathcal{N} = (\mathbb{R} \times \{1\}, g)$, $g(x, 1) = (x^5, 1)$. After a few moves player **I** resigns. Can you explain why?

5.43 Consider $\mathrm{EF}_2(\mathcal{M}, \mathcal{N})$, where $\mathcal{M} = (\mathbb{Z}, \{(a, b) : a - b = 10\})$ and $\mathcal{N} = (\mathbb{Q}, \{(a, b) : a - b = 2/3\})$. Suppose we are in position $(-8, -1/4)$ (i.e. $x_0 = -8$ and $y_0 = -1/4$). Then **I** plays $x_1 = 11/12$. What would be a good move for **II**?

5.44 Consider $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{N})$, where $\mathcal{M}$ and $\mathcal{N}$ are as in the previous exercise. Player **I** resigns before the game even starts. Can you explain why?

5.45 Suppose M and N are disjoint sets with 10 elements each. Let $c \in M$ and $d \in N$. Who has a winning strategy in $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{N})$ in the following cases:

1. $\mathcal{M} = (M, \{(a, b, c) : a = b\}), \mathcal{N} = (N, \{(a, b, d) : a = b\})$,
2. $\mathcal{M} = (M, \{(a, b, e) : a = b\}), \mathcal{N} = (N, \{(a, b, e) : b = e\})$.

[2] Some authors use the term potential isomorphism for partial isomorphism.

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6

First Order Logic

6.1 Introduction

We have already discussed the *first order language of graphs*. We now define the basic concepts of a more general first order language, denoted FO, one which applies to any vocabulary, not just the vocabulary of graphs. First order logic fits the Strategic Balance of Logic better than any other logic. It is arguably the most important of all logics. It has enough power to express interesting and important concept and facts, and still it is weak and flexible enough to permit powerful constructions as demonstrated e.g. by the Model Existence Theorem below.

6.2 Basic Concepts

Suppose L is a vocabulary. The *logical symbols* of the first order language (or logic) of the vocabulary L are $\approx, \neg, \wedge, \vee, \forall, \exists, (,), x_0, x_1, \ldots$. *Terms* are defined as follows: Constant symbols $c \in L$ are L-terms. Variables $x_0, x_1, ...$ are L-terms. If $f \in L$, $\#(f) = n$ and $t_1, \ldots, t_n$ are L-terms, then so is $ft_1 \ldots t_n$. *L-equations* are of the form $\approx tt'$ where t and t' are L-terms. *L-atomic formulas* are either L-equations or of the form $Rt_1 \ldots t_n$, where $R \in L$, $\#(R) = n$ and $t_1, ..., t_n$ are L-terms. A *basic formula* is an atomic formula or the negation of an atomic formula. *L-formulas* are of the form

$$\approx tt'$$
$$Rt_1 \ldots t_n$$
$$\neg\varphi$$
$$(\varphi \wedge \psi), (\varphi \vee \psi)$$
$$\forall x_n \varphi, \exists x_n \varphi$$

where $t, t', t_1, ..., t_n$ are L-terms, $R \in L$ with $\#(R) = n$, and φ and ψ are L-formulas.

Definition 6.1 An *assignment* for a set M is any function s with $\mathrm{dom}(s)$ a set of variables and $\mathrm{rng}(s) \subseteq \mathcal{M}$. The *value* $t^{\mathcal{M}}(s)$ of an L-term t in $\mathcal{M}$ under the assignment s is defined as follows: $c^{\mathcal{M}}(s) = \mathrm{Val}_{\mathcal{M}}(c)$, $x_n^{\mathcal{M}}(s) = s(x_n)$ and $(ft_1 \dots t_n)^{\mathcal{M}}(s) = \mathrm{Val}_{\mathcal{M}}(f)(t_1^{\mathcal{M}}(s), \dots, t_n^{\mathcal{M}}(s))$. The *truth* of L-formulas in $\mathcal{M}$ under s is defined as follows:

$$\begin{array}{lll}
\mathcal{M} \models_s Rt_1 \dots t_n & \text{iff} & (t_1^{\mathcal{M}}(s), \dots, t_n^{\mathcal{M}}(s)) \in \mathrm{Val}_{\mathcal{M}}(R) \\
\mathcal{M} \models_s \approx t_1 t_2 & \text{iff} & t_1^{\mathcal{M}}(s) = t_2^{\mathcal{M}}(s) \\
\mathcal{M} \models_s \neg\varphi & \text{iff} & \mathcal{M} \not\models_s \varphi \\
\mathcal{M} \models_s (\varphi \wedge \psi) & \text{iff} & \mathcal{M} \models_s \varphi \text{ and } \mathcal{M} \models_s \psi \\
\mathcal{M} \models_s (\varphi \vee \psi) & \text{iff} & \mathcal{M} \models_s \varphi \text{ or } \mathcal{M} \models_s \psi \\
\mathcal{M} \models_s \forall x_n \varphi & \text{iff} & \mathcal{M} \models_{s[a/x_n]} \varphi \text{ for all } a \in \mathcal{M} \\
\mathcal{M} \models_s \exists x_n \varphi & \text{iff} & \mathcal{M} \models_{s[a/x_n]} \varphi \text{ for some } a \in \mathcal{M}, \\
 & & \text{where } s[a/x_n](y) = \begin{cases} a & \text{if } y = x_n \\ s(y) & \text{otherwise.} \end{cases}
\end{array}$$

We assume the reader is familiar with such basic concepts as free variable, sentence, substitution of terms for variables etc. A standard property of first order (or any other) logic is that $\mathcal{M} \models_s \varphi$ depends only on $\mathcal{M}$ and the values of s on the variables that are free in φ. A *sentence* is a formula φ without free variables. Then $\mathcal{M} \models \varphi$ means $\mathcal{M} \models_{\emptyset} \varphi$. In this case we say that φ is *true* in $\mathcal{M}$.

Convention: If φ is an L-formula with the free variables $x_1, \dots, x_n$, we indicate this by writing φ as $\varphi(x_1, \dots, x_n)$. If $\mathcal{M}$ is an L-structure and s is an assignment for M such that $\mathcal{M} \models_s \varphi$, we write $\mathcal{M} \models \varphi(a_1, \dots, a_n)$, where $a_i = s(x_i)$ for $i = 1, \dots, n$.

Definition 6.2 The *quantifier rank* of a formula φ, denoted $\mathrm{qr}(\varphi)$, is defined as follows: $\mathrm{qr}(\approx tt') = \mathrm{qr}(Rt_1 \dots t_n) = 0$, $\mathrm{qr}(\neg\varphi) = \mathrm{qr}(\varphi)$, $\mathrm{qr}((\varphi \wedge \psi)) = \mathrm{qr}((\varphi \vee \psi)) = \max\{\mathrm{qr}(\varphi), \mathrm{qr}(\psi)\}$, $\mathrm{qr}(\exists x\varphi) = \mathrm{qr}(\forall x\varphi) = \mathrm{qr}(\varphi) + 1$. A formula φ is *quantifier free* if $\mathrm{qr}(\varphi) = 0$.

The quantifier rank is a measure of the longest sequence of "nested" quantifiers. In the first three of the following formulas the quantifiers $\forall x_n$ and $\exists x_n$ are nested but in the last unnested:

$$\forall x_0(P(x_0) \vee \exists x_1 R(x_0, x_1)) \tag{6.1}$$

$$\exists x_0(P(x_0) \wedge \forall x_1 R(x_0, x_1)) \tag{6.2}$$

$$\forall x_0(P(x_0) \vee \exists x_1 Q(x_1)) \tag{6.3}$$

$$(\forall x_0 P(x_0) \vee \exists x_1 Q(x_1)) \tag{6.4}$$

Note that formula (6.3) of quantifier rank 2 is logically equivalent to the formula (6.4) which has quantifier rank 1. So the nesting can sometimes be eliminated. In formulas (6.1) and (6.2) nesting cannot be so eliminated.

Proposition 6.3 *Suppose L is a finite vocabulary without function symbols. For every n and for every set $\{x_1, \ldots, x_n\}$ of variables, there are only finitely many logically non-equivalent first order L-formulas of quantifier rank $< n$ with the free variables $\{x_1, \ldots, x_n\}$.*

Proof The proof is exactly like that of Proposition 4.15. □

Note that Proposition 6.3 is not true for infinite vocabularies, as there would be infinitely many logically non-equivalent atomic formulas, and also not true for vocabularies with function symbols, as there would be infinitely many logically non-equivalent equations obtained by iterating the function symbols.

6.3 Characterizing Elementary Equivalence

We now show that the concept of a back-and-forth sequence provides an alternative characterization of elementary equivalence

$$\mathcal{A} \equiv \mathcal{B} \quad \text{i.e.} \quad \forall \varphi \in FO(\mathcal{A} \models \varphi \iff \mathcal{B} \models \varphi).$$

This is the original motivation for the concepts of a back-and-forth set, back-and-forth sequence and Ehrenfeucht-Fraïssé Game. To this end, let

$$\mathcal{A} \equiv_n \mathcal{B}$$

mean that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences of FO of quantifier rank $\leq n$.

We now prove an important leg of the Strategic Balance of Logic, namely the marriage of truth and separation:

Proposition 6.4 *Suppose L is an arbitrary vocabulary. Suppose $\mathcal{A}$ and $\mathcal{B}$ are L-structures and $n \in \mathbb{N}$. Consider the conditions:*

(i) $\mathcal{A} \equiv_n \mathcal{B}$.
(ii) $\mathcal{A}\restriction_{L'} \simeq_p^n \mathcal{B}\restriction_{L'}$ *for all finite* $L' \subseteq L$.

We have always $(ii) \to (i)$ and if L has no function symbols, then $(ii) \leftrightarrow (i)$.

Proof (ii)→(i). If $\mathcal{A} \not\equiv_n \mathcal{B}$, then there is a sentence φ of quantifier rank $\leq n$ such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$. Since φ has only finitely many symbols, there is a finite $L' \subseteq L$ such that $\mathcal{A}\restriction_{L'} \not\equiv_n \mathcal{B}\restriction_{L'}$. Suppose $(P_i : i \leq n)$ is a back-and-forth sequence for $\mathcal{A}\restriction_{L'}$ and $\mathcal{B}\restriction_{L'}$. We use induction on $i \leq n$ to prove the following

Claim If $f \in P_i$ and $a_1, \ldots, a_k \in \mathrm{dom}(f)$, then

$$(\mathcal{A}\restriction_{L'}, a_1, \ldots, a_k) \equiv_i (\mathcal{B}\restriction_{L'}, fa_1, \ldots, fa_k).$$

If $i = 0$, the claim follows from $P_0 \subseteq \mathrm{Part}(\mathcal{A}\restriction_{L'}, \mathcal{B}\restriction_{L'})$. Suppose then $f \in P_{i+1}$ and $a_1, \ldots, a_k \in \mathrm{dom}(f)$. Let $\varphi(x_0, x_1, \ldots, x_k)$ be an L'-formula of FO of quantifier rank $\leq i$ such that

$$\mathcal{A}\restriction_{L'} \models \exists x_0 \varphi(x_0, a_1, \ldots, a_k).$$

Let $a \in A$ so that $\mathcal{A}\restriction_{L'} \models \varphi(a, a_1, \ldots, a_k)$ and $g \in P_i$ such that $a \in \mathrm{dom}(g)$ and $f \subseteq g$. By the induction hypothesis, $\mathcal{B}\restriction_{L'} \models \varphi(ga, ga_1, \ldots, ga_k)$. Hence

$$\mathcal{B}\restriction_{L'} \models \exists x_0 \varphi(x_0, fa_1, \ldots, fa_k).$$

The claim is proved. Putting $i = n$ and using the assumption $P_n \neq \emptyset$, gives a contradiction with $\mathcal{A}\restriction_{L'} \not\equiv_n \mathcal{B}\restriction_{L'}$.

$(i) \rightarrow (ii)$. Assume L has no function symbols. Fix $L' \subseteq L$ finite. Let P_i consist of $f : A \rightarrow B$ such that $\mathrm{dom}(f) = \{a_0, \ldots, a_{n-i-1}\}$ and

$$(\mathcal{A}\restriction_{L'}, a_0, \ldots, a_{n-i-1}) \equiv_i (\mathcal{B}\restriction_{L'}, fa_0, \ldots, fa_{n-i-1}).$$

We show that $(P_i : i \leq n)$ is a back-and-forth sequence for $\mathcal{A}\restriction_{L'}$ and $\mathcal{B}\restriction_{L'}$. By (i), $\emptyset \in P_n$ so $P_n \neq \emptyset$. Suppose $f \in P_i, i > 0$, as above, and $a \in A$. By Proposition 6.3 there are only finitely many pairwise non-equivalent L'-formulas of quantifier rank $i - 1$ of the form $\varphi(x, x_0, \ldots, x_{n-i-1})$ in FO. Let them be $\varphi_j(x, x_0, \ldots, x_{n-i-1})$, $j \in J$. Let

$$J_0 = \{j \in J : \mathcal{A}\restriction_{L'} \models \varphi_j(a, a_0, \ldots, a_{n-i-1})\}.$$

Let

$$\psi(x, x_0, \ldots, x_{n-i-1}) = \bigwedge_{j \in J_0} \varphi_j(x, x_0, \ldots, x_{n-i-1}) \wedge \bigwedge_{j \in J \setminus J_0} \neg\varphi_j(x, x_0, \ldots, x_{n-i-1}).$$

Now $\mathcal{A}\restriction_{L'} \models \exists x \psi(x, a_0, \ldots, a_{n-i-1})$, so as we have assumed $f \in P_i$, we have $\mathcal{B}\restriction_{L'} \models \exists x \psi(x, fa_0, \ldots, fa_{n-i-1})$. Thus there is some $b \in B$ with $\mathcal{B}\restriction_{L'} \models \psi(b, fa_0, \ldots, fa_{n-i-1})$. Now $f \cup \{(a, b)\} \in P_{i-1}$. The other condition (5.15) is proved similarly. □

The above Proposition is the standard method for proving models elementary equivalent in FO. For example, Proposition 6.4 and Example 5.26 together give $(Z, <) \equiv (Z + Z, <)$. The exercises give more examples of partially isomorphic pairs—and hence elementary equivalent—structures. The restriction on function symbols can be circumvented by first using quantifiers to eliminate nesting of function symbols and then replacing the unnested equations $f(x_1, ..., x_{n-1}) = x_n$ by new predicate symbols $R(x_1, ..., x_n)$.

Let $\mathrm{Str}(L)$ denote the class of all L-structures. We can draw the following important conclusion from Proposition 6.4 (see Figure 6.1):

Corollary *Suppose L is a vocabulary without function symbols. Then for all $n \in \mathbb{N}$ the equivalence relation*

$$\mathcal{A} \equiv_n \mathcal{B}$$

divides $\mathrm{Str}(L)$ into finitely many equivalence classes C_i^n, $i = 1, \ldots, m_n$, such that for each C_i^n there is a sentence φ_i^n of FO with the properties:

1. *For all L-structures $\mathcal{A}$: $\mathcal{A} \in C_i^n \iff \mathcal{A} \models \varphi_i^n$.*
2. *If φ is an L-sentence of quantifier rank $\leq n$, then there are $i_1, \ldots, i_k$ such that $\models \varphi \leftrightarrow (\varphi_{i_1}^n \vee \ldots \vee \varphi_{i_k}^n)$*

Proof Let φ_i^n be the conjunction of all the finitely many L-sentences of quantifier rank $\leq n$ that are true in some (every) model in C_i^n (to make the conjunction finite we do not repeat logically equivalent formulas). For the second claim, let $\varphi_{i_1}^n, \ldots, \varphi_{i_k}^n$ be the finite set of all L-sentences of quantifier rank $\leq n$ that are consistent with φ. If now $\mathcal{A} \models \varphi$, and $\mathcal{A} \in C_i^n$, then $\mathcal{A} \models \varphi_i^n$. On the other hand, if $\mathcal{A} \models \varphi_i^n$ and there is $\mathcal{B} \models \varphi_i^n$ such that $\mathcal{B} \models \varphi$, then $\mathcal{A} \equiv_n \mathcal{B}$, whence $\mathcal{A} \models \varphi$. □

We can actually read from the proof of Proposition 6.4 a more accurate description for the sentences φ_i. This leads to the theory of so-called *Scott formulas* (see Section 7.4).

Theorem 6.5 *Suppose K is a class of L-structures. Then the following are equivalent (see Figure 6.2):*

1. *K is FO-definable, i.e. there is an L-sentence φ of FO such that for all Lstructures $\mathcal{M}$ we have $\mathcal{M} \in K \iff \mathcal{M} \models \varphi$.*
2. *There is $n \in \mathbb{N}$ such that K is closed under $\simeq_p^n$.*

As in the case of graphs, Theorem 6.5 can be used to demonstrate that certain properties of models are not definable in FO:

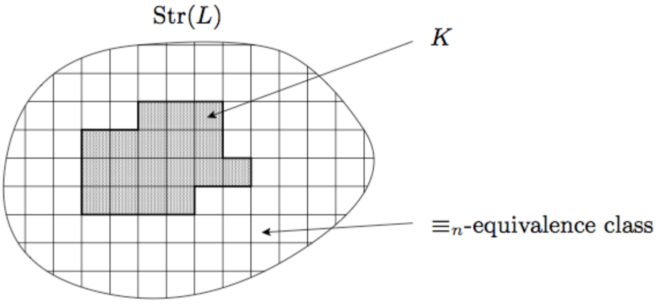

Figure 6.1 First order definable model class K

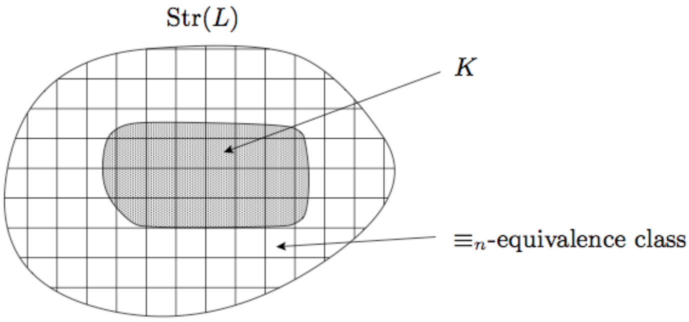

Figure 6.2 Not first order definable model class K

Example 6.6 Let $L = \emptyset$. The following properties of L-structures $\mathcal{M}$ are not expressible in FO:

1. M is infinite.
2. M is finite and even.

In both cases it is easy to find, for each $n \in \mathbb{N}$, two models $\mathcal{M}_n$ and $\mathcal{N}_n$ such that $\mathcal{M}_n \simeq_p^n \mathcal{N}_n$, $\mathcal{M}$ has the property, but $\mathcal{N}$ does not.

Example 6.7 Let $L = \{P\}$ be a unary vocabulary. The following properties of L-structures (M, A) are not expressible in FO:

1. $|A| = |M|$.
2. $|A| = |M \setminus A|$.

3. $|A| \leq |M \setminus A|$.

This is demonstrated by the models $(\mathbb{N}, \{1, \ldots, n\})$, $(\mathbb{N}, \mathbb{N} \setminus \{1, \ldots, n\})$ and $(\{1, \ldots, 2n\}, \{1, \ldots, n\})$.

Example 6.8 Let $L = \{<\}$ be a binary vocabulary. The following properties of L-structures $\mathcal{M} = (M, <)$ are not expressible in FO:

1. $\mathcal{M} \cong (\mathbb{Z}, <)$.
2. All closed intervals of $\mathcal{M}$ are finite.
3. Every bounded subset of $\mathcal{M}$ has a supremum.

This is demonstrated in the first two cases by the models $\mathcal{M}_n = (\mathbb{Z}, <)$ and $\mathcal{N}_n = (\mathbb{Z} + \mathbb{Z}, <)$ (see Example 5.26), and in the third case by the partially isomorphic models: $\mathcal{M} = (\mathbb{R}, <)$ and $\mathcal{N} = (\mathbb{R} \setminus \{0\}, <)$.

6.4 The Löwenheim-Skolem Theorem

In this section we show that if a first order sentence φ is true in a structure $\mathcal{M}$, it is true in a countable substructure of $\mathcal{M}$, and even more, there are countable substructures of $\mathcal{M}$ in a sense "everywhere" satisfying φ. To make this statement precise we introduce a new game due to D. Kueker [Kue77] called the cub game.

Definition 6.9 Suppose A is an arbitrary set. $\mathcal{P}_\omega(A)$ is defined as the set of all countable subsets of A.

The set $\mathcal{P}_\omega(A)$ is an auxiliary concept useful for the general investigation of countable substructures of a model with universe A. One should note that if A is infinite, the set $\mathcal{P}_\omega(A)$ is uncountable[1]. For example, $|\mathcal{P}_\omega(\mathbb{N})| = |\mathbb{R}|$. The set $\mathcal{P}_\omega(A)$ is closed under intersections and countable unions but not necessarily under complements, so it is a (distributive) lattice under the partial order $\subseteq$, but not a Boolean algebra. The sets in $\mathcal{P}_\omega(A)$ cover the set A entirely, but so do many proper subsets of $\mathcal{P}_\omega(A)$ such as the set of all singletons in $\mathcal{P}_\omega(A)$ and the set of all finite sets in $\mathcal{P}_\omega(A)$.

Definition 6.10 Suppose A is an arbitrary set and $\mathcal{C}$ a subset of $\mathcal{P}_\omega(A)$. The *cub game of* $\mathcal{C}$ is the game $G_{\text{cub}}(\mathcal{C}) = G_\omega(A, W)$, where W consists of sequences $(a_1, a_2, \ldots)$ with the property that $\{a_1, a_2, \ldots\} \in \mathcal{C}$.

[1] Its cardinality is $|A|^\omega$.

I	**II**
a_0	
	a_1
a_2	
	a_3
$\vdots$	$\vdots$

Figure 6.3 The game $G_{\text{cub}}(\mathcal{C})$.

In other words, during the game $G_{\text{cub}}(\mathcal{C})$ the players pick elements of the set A, player **I** being the one who starts. After all the infinitely many moves a set $X = \{a_1, a_2, \ldots\}$ has been formed. Player **II** tries to make sure that $X \in \mathcal{C}$ while player **I** tries to prevent this. If $\mathcal{C} = \emptyset$, player **II** has no chance. On the other hand, if $\mathcal{C} = \mathcal{P}_\omega(A)$, player **I** has no chance. When $\emptyset \neq \mathcal{C} \neq \mathcal{P}_\omega(A)$, there is a challenge for both players.

Example 6.11 Suppose $B \in \mathcal{P}_\omega(A)$ and $\mathcal{C} = \{X \in \mathcal{P}_\omega(A) : B \subseteq X\}$. Then player **II** has a winning strategy in $G_{\text{cub}}(\mathcal{C})$. Respectively, player **I** has a winning strategy in $G_{\text{cub}}(\mathcal{P}_\omega(A) \setminus \mathcal{C})$

Lemma 6.12 *Suppose $\mathcal{F}$ is a countable set of functions $f : A^{n_f} \to A$ and*

$$\mathcal{C} = \{X \in \mathcal{P}_\omega(A) : X \textit{ is closed under each } f \in \mathcal{F}\}.$$

Then player **II** *has a winning strategy in the game $G_{\text{cub}}(\mathcal{C})$.*

Proof We use the notation of Figure 6.3 for $G_{\text{cub}}(\mathcal{C})$, The strategy of player **II** is to make sure that the images of the elements a_m under the functions in $\mathcal{F}$ are eventually played. She cannot control player **I** 's moves, so she has to do it herself. On the other hand, she has nothing else to do in the game. Let $\mathcal{F} = \{f_i : i \in \mathbb{N}\}$. Let $b \in A$. If

$$m = \prod_{i=0}^{k} p_i^{m_i+1},$$

where $p_0, p_1, \ldots$ is the sequence of consecutive primes, and k is the arity of f_{m_0}, then player **II** plays

$$a_{2m+1} = f_{m_0}(a_{m_1}, \ldots, a_{m_k}).$$

Otherwise **II** plays $a_{2m+1} = b$. After all $a_0, a_1, \ldots$ have been played, the set $X = \{a_0, a_1, \ldots\}$ is closed under each f_i. Why? Suppose $f_{m_0} \in \mathcal{F}$ is k-ary

I	**II**
a_0^0	
	b_0^0
a_1^0	
	b_1^0
$\vdots$	$\vdots$

Figure 6.4 The game $G_{\text{cub}}(\bigcap_{n\in\mathbb{N}} \mathcal{C}_n)$.

and $a_{m_1}, \ldots, a_{m_k} \in X$. Let

$$m = \prod_{i=0}^{k} p_i^{m_i+1}.$$

Then $a_{2m+1} = f_{m_0}(a_{m_1}, \ldots, a_{m_k})$. Therefore $X \in \mathcal{C}$. For example, if $f_2 \in \mathcal{F}$ is binary, then

$$a_{2\cdot 2^3\cdot 3^6\cdot 5^7+1} = f_2(a_5, a_6).$$

□

In a countable vocabulary there are only countably many function symbols. On the other hand, the functions are the main concern in checking whether a subset of a structure is the universe of a substructure. This leads to the following application of Lemma 6.12:

Proposition 6.13 *Suppose L is a countable vocabulary and $\mathcal{M}$ is an L-structure. Let $\mathcal{C}$ be the set of domains of countable submodels of $\mathcal{M}$. Then player* **II** *has a winning strategy in $G_{\text{cub}}(\mathcal{C})$.*

Intuitively this means that the countable submodels of $\mathcal{M}$ extend everywhere in $\mathcal{M}$. We will improve this observation considerably below.

Let $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the bijection $\pi(x, y) = \frac{1}{2}((x + y)^2 + 3x + y)$ with the inverses ρ and σ such that $\rho(\pi(x, y)) = x$ and $\sigma(\pi(x, y)) = y$.

Lemma 6.14 *Suppose player* **II** *has a winning strategy in $G_{\text{cub}}(\mathcal{C}_n)$, where $\mathcal{C}_n \subseteq \mathcal{P}_\omega(A)$, for each $n \in \mathbb{N}$. Then she has one in $G_{\text{cub}}(\bigcap_{n\in\mathbb{N}} \mathcal{C}_n)$.*

Proof We use the notation of Figure 6.4 for $G_{\text{cub}}(\bigcap_{n=1}^{\infty} \mathcal{C}_n)$, and the notation of Figure 6.5 for $G_{\text{cub}}(\mathcal{C}_n)$. The idea is that while we play $G_{\text{cub}}(\bigcap_{n\in\mathbb{N}} \mathcal{C}_n)$, player **II** is playing the infinitely many games $G_{\text{cub}}(\mathcal{C}_n)$, using there her winning strategy. The strategy of player **II** is to choose

$$b^0_{\pi(n,k)} = b^{n+1}_k,$$

I	**II**
a_0^n	
	b_0^n
a_1^n	
	b_1^n
$\vdots$	$\vdots$

Figure 6.5 The game $G_{\text{cub}}(\mathcal{C}_n)$.

I	**II**
a_0	
	b_0
a_1	
	b_1
$\vdots$	$\vdots$

Figure 6.6 The game $G_{\text{cub}}(\triangle_{a\in A}\mathcal{C}_a)$.

where b_k^{n+1} is obtained from the the cub game of $\mathcal{C}_{n+1}$, where player **I** plays

$$a_{2j}^{n+1} = a_j^0, a_{2j+1}^{n+1} = b_j^0.$$

□

Lemma 6.15 *Suppose player* **II** *has a winning strategy in* $G_{cub}(\mathcal{C}_a)$*, where* $\mathcal{C}_a \subseteq \mathcal{P}_\omega(A)$ *for each* $a \in A$*. Then she has one in the cub game of the* diagonal intersection $\triangle_{a\in A}\mathcal{C}_a = \{X \in \mathcal{P}_\omega(A) : \forall a \in X(X \in \mathcal{C}_a)\}$.

Proof We use the notation of Figure 6.6 for $G_{\text{cub}}(\triangle_{a\in A}\mathcal{C}_a)$, the notation of Figure 6.7 for $G_{\text{cub}}(\mathcal{C}_{a_i})$, and the notation of Figure 6.8 for $G_{\text{cub}}(\mathcal{C}_{b_i})$. The idea is that while we play $G_{\text{cub}}(\triangle_{a\in A}\mathcal{C}_a)$, player **II** is playing the induced games

I	**II**
x_0^i	
	y_0^i
x_1^i	
	y_1^i
$\vdots$	$\vdots$

Figure 6.7 The game $G_{\text{cub}}(\mathcal{C}_{a_i})$.

I	**II**
u_0^i	
	v_0^i
u_1^i	
	v_1^i
$\vdots$	$\vdots$

Figure 6.8 The game $G_{\text{cub}}(\mathcal{C}_{b_i})$.

I	**II**
a_0	
	b_0
a_1	
	b_1
$\vdots$	$\vdots$

Figure 6.9 The game $G_{\text{cub}}(\triangledown_{a\in A}\mathcal{C}_a)$.

$G_{\text{cub}}(\mathcal{C}_{a_i})$ and $G_{\text{cub}}(\mathcal{C}_{b_i})$, using there her winning strategy. The strategy of player **II** is to choose

$$b_{2\pi(n,k)} = y_k^n, b_{2\pi(n,k)+1} = v_k^n,$$

where b_k^{n+1} is obtained from $G_{\text{cub}}(\mathcal{C}_{a_i})$, where player **I** plays

$$x_{2j}^{i+1} = a_j, x_{2j+1}^{i+1} = b_j,$$

and from $G_{\text{cub}}(\mathcal{C}_{b_i})$, where player **I** plays

$$u_{2j}^{i+1} = a_j, u_{2j+1}^{i+1} = b_j.$$

□

Lemma 6.16 *Suppose player* **II** *has a winning strategy in* $G_{\text{cub}}(\mathcal{C}_a)$*, where* $\mathcal{C}_a \subseteq \mathcal{P}_\omega(A)$*, for some* $a \in A$*. Then she has one in the cub game of the* diagonal union $\triangledown_{a\in A}\mathcal{C}_a = \{X \in \mathcal{P}_\omega(A) : \exists a \in X(X \in \mathcal{C}_a)\}$.

Proof We use the notation of Figure 6.9 for $G_{\text{cub}}(\triangledown_{a\in A}\mathcal{C}_a)$, and the notation of Figure 6.10 for $G_{\text{cub}}(\mathcal{C}_a)$.

The idea is that while we play $G_{\text{cub}}(\triangledown_{a\in A}\mathcal{C}_a)$, player **II** is playing the game $G_{\text{cub}}(\mathcal{C}_a)$ using there her winning strategy. The strategy of player **II** is to choose

$$b_0 = a, b_{n+1} = y_n,$$

I	II
x_0	
	y_0
x_1	
	y_1
$\vdots$	$\vdots$

Figure 6.10 The game $G_{\text{cub}}(\mathcal{C}_a)$.

where y_n is obtained from $G_{\text{cub}}(\mathcal{C}_a)$, where player **I** plays

$$x_0 = a, x_{i+1} = a_i.$$

□

The following new concept gives an alternative characterization of the cub game:

Definition 6.17 A subset $\mathcal{C}$ of $\mathcal{P}_\omega(A)$ is *unbounded* if for every $X \in \mathcal{P}_\omega(A)$ there is $X' \in \mathcal{C}$ with $X \subseteq X'$. A subset $\mathcal{C}$ of $\mathcal{P}_\omega(A)$ is *closed* if the union of any increasing sequence $X_0 \subseteq X_1 \subseteq \dots$ of elements of $\mathcal{C}$ is again an element of $\mathcal{C}$. A subset $\mathcal{C}$ of $\mathcal{P}_\omega(A)$ is *cub* if it is closed and unbounded.

A cub set of countable subsets of A covers A completely and permits the taking of unions of increasing sequences of sets.

Lemma 6.18 *Suppose $\mathcal{F}$ is a countable set of functions $f : A^{n_f} \to A$. Then the set*

$$\mathcal{C} = \{X \subseteq A : X \textit{ is closed under each } f \in \mathcal{F}\}$$

is a cub set in $\mathcal{P}_\omega(A)$.

Proof Let us first prove that $\mathcal{C}$ is unbounded. Suppose $B \in \mathcal{P}_\omega(A)$. Let

$$\begin{aligned} B^0 &= B, \\ B^{n+1} &= B^n \cup \{f(a_1, \dots, a_{n_f}) : a_1, \dots, a_{n_f} \in B^n\}, \\ B^* &= \bigcup_{n \in \mathbb{N}} B^n. \end{aligned}$$

As a countable union of countable sets, B^* is countable. Since clearly $B^* \in \mathcal{C}$, we have proved the unboundedness of $\mathcal{C}$. To prove that $\mathcal{C}$ is closed, let $X_0 \subseteq X_1 \subseteq \dots$ be elements of $\mathcal{C}$ and $X = \bigcup_{n \in \mathbb{N}} X_n$. If $f \in \mathcal{F}$ and $a_1, \dots, a_{n_f} \in X$, then there is $n \in \mathbb{N}$ such that $a_1, \dots, a_{n_f} \in X_n$. Since $X_n \in \mathcal{C}$, $f(a_1, \dots, a_{n_f}) \in X_n \subseteq X$. Thus $\mathcal{C}$ is indeed closed. □

Now we can prove a characterization of the cub game in terms of cub sets:

Proposition 6.19 *Suppose A is an arbitrary set and $\mathcal{C} \subseteq \mathcal{P}(A)$. Player* **II** *has a winning strategy in $G_{cub}(\mathcal{C})$ if and only if $\mathcal{C}$ contains a cub set.*

Proof Suppose first player **II** has a winning strategy $(\tau_0, \tau_1, \ldots)$ in $G_{\text{cub}}(\mathcal{C})$. Let $\mathcal{D}$ be the family of subsets of A that are closed under each τ_n, $n \in \mathbb{N}$. By Lemma 6.18 the set $\mathcal{D}$ is a cub set. To prove that $\mathcal{D} \subseteq \mathcal{C}$, let $X \in \mathcal{D}$. Let $X = \{a_0, a_1, \ldots\}$. Suppose player **I** plays $G_{\text{cub}}(\mathcal{C})$ by playing the elements $a_0, a_1, \ldots$ one at a time. If player **II** uses her strategy $(\tau_0, \tau_1, \ldots)$, her responses are all in X, the set X being closed under the functions τ_n. Thus at the end of the game we have the set X and since player **II** wins, $X \in \mathcal{C}$.

For the converse, suppose $\mathcal{C}$ contains a cub set $\mathcal{D}$. We need to show that player **II** has a winning strategy in $G_{\text{cub}}(\mathcal{C})$. She plays as follows: Suppose $a_0, b_0, \ldots, a_{n-1}, b_{n-1}, a_n$ have been played so far. Player **II** has as a part of her strategy produced elements $X_0 \subseteq \ldots \subseteq X_{n-1}$ of $\mathcal{D}$ such that $a_i \in X_i$ for each $i \leq n$. Let

$$X_i = \{x_0^i, x_1^i, \ldots\}.$$

The choice of player **II** for her next move is now

$$b_n = x_{\sigma(n)}^{\rho(n)}.$$

In the end, player **II** has listed all sets X_n, as after all, $x_j^i = b_{\pi(i,j)}$. Thus the set X that the players produce has to contain each set X_n, $n \in \mathbb{N}$. On the other hand, the players only play elements of A which are members of some of the sets X_n. Thus $X = \bigcup_{n \in \mathbb{N}} X_n$. Since $\mathcal{D}$ is closed, $X \in \mathcal{D} \subseteq \mathcal{C}$. □

If player **I** does not have a winning strategy in $G_{\text{cub}}(\mathcal{C})$, we call $\mathcal{C}$ a *stationary* subset of $\mathcal{P}_\omega(A)$. It is a non-trivial task to construct stationary sets which are not stationary for the trivial reason that they contain a cub (see Exercise 6.46).

Endowed with the powerful methods of the cub game and the cub sets, we can now return to the original problem of this section: how to find countable submodels satisfying a given sentence? We attack this problem by associating every first order sentence φ with a family $\mathcal{C}_\varphi$ of countable sets and showing that this set necessarily contains a cub set. Let us say that a formula of first order logic is in *negation normal form*, NNF in symbols, if it has negation symbols in front of atomic formulas only. Well-known equivalences show that every first order formula is logically equivalent to a formula in NNF.

Definition 6.20 Suppose L is a vocabulary and $\mathcal{M}$ an L-structure. Suppose φ is a first order formula in NNF and s is an assignment for the set M the domain of which includes the free variables of φ. We define the set $\mathcal{C}_{\varphi,s}$ of countable subsets of M as follows: If φ is atomic, $\mathcal{C}_{\varphi,s}$ contains as an element the domain A of a countable submodel $\mathcal{A}$ of $\mathcal{M}$ such that $\mathrm{rng}(s) \subseteq A$ and:

- If φ is $\approx tt'$, then $t^{\mathcal{A}}(s) = t'^{\mathcal{A}}(s)$.
- If φ is $\neg\approx tt'$, then $t^{\mathcal{A}}(s) \neq t'^{\mathcal{A}}(s)$.
- If φ is $Rt_1 \ldots t_n$, then $(t_1^{\mathcal{A}}(s), \ldots, t_n{}^{\mathcal{A}}(s)) \in R^{\mathcal{A}}$.
- If φ is $\neg Rt_1 \ldots t_n$, then $(t_1^{\mathcal{A}}(s), \ldots, t_n{}^{\mathcal{A}}(s)) \notin R^{\mathcal{A}}$.

For non-basic φ we define

- $\mathcal{C}_{\varphi\wedge\psi,s} = \mathcal{C}_{\varphi,s} \cap \mathcal{C}_{\psi,s}$.
- $\mathcal{C}_{\varphi\vee\psi,s} = \mathcal{C}_{\varphi,s} \cup \mathcal{C}_{\psi,s}$.
- $\mathcal{C}_{\exists x\varphi,s} = \bigtriangledown_{a\in M}\mathcal{C}_{\varphi,s(a/x)}$.
- $\mathcal{C}_{\forall x\varphi,s} = \triangle_{a\in M}\mathcal{C}_{\varphi,s[a/x]}$.

If φ is a sentence, we denote $\mathcal{C}_{\varphi,s}$ by $\mathcal{C}_{\varphi}$. If φ is not in NNF, we define $\mathcal{C}_{\varphi,s}$ and $\mathcal{C}_{\varphi}$ by first translating φ into a logically equivalent NNF formula.

The sets $\mathcal{C}_{\varphi}$ were defined with the following fact in mind:

Proposition 6.21 *Suppose $\mathcal{A}$ is an L-structure such that $A \in \mathcal{C}_{\varphi,s}$. Then $\mathcal{A} \models_s \varphi$.*

Proof This is trivial for atomic φ. The induction step is clear for $\varphi \wedge \psi$ and $\varphi \vee \psi$. Suppose $A \in \mathcal{C}_{\exists x\varphi,s}$. Thus $A \in \bigtriangledown_{a\in M}\mathcal{C}_{\varphi,s[a/x]}$. By the definition of diagonal union $A \in \mathcal{C}_{\varphi,s[a/x]}$ for some $a \in A$. By the induction hypothesis, $\mathcal{A} \models_{s[a/x]} \varphi$ for some $a \in A$. Thus $\mathcal{A} \models_s \exists x\varphi$. Finally, suppose $A \in \mathcal{C}_{\forall x\varphi,s}$. Thus $A \in \triangle_{a\in M}\mathcal{C}_{\varphi,s[a/x]}$. By the definition of diagonal intersection $A \in \mathcal{C}_{\varphi,s[a/x]}$ for all $a \in A$. By the induction hypothesis, $\mathcal{A} \models_{s[a/x]} \varphi$ for all $a \in A$. Thus $\mathcal{A} \models_s \forall x\varphi$. □

Proposition 6.22 *Suppose L is countable and $\mathcal{M}$ an L-structure such that $\mathcal{M} \models \varphi$. Then player* **II** *has a winning strategy in $G_{cub}(\mathcal{C}_{\varphi})$.*

Proof We use induction on φ to prove that if $\mathcal{M} \models_s \varphi$, then **II** has a winning strategy in $G_{\mathrm{cub}}(\mathcal{C}_{\varphi})$. For atomic formulas the claim follows from Proposition 6.13. The induction step is clear for $\varphi \vee \psi$. The induction step for $\varphi \wedge \psi$ follows from Lemma 6.14. The induction step for $\forall x\varphi$ and $\exists x\varphi$ follows from Lemma 6.16. Finally, the induction step for $\forall x\varphi$ follows from Lemma 6.15. □

Theorem 6.23 (Löwenheim-Skolem Theorem) *Suppose L is a countable vocabulary and T is a set of L-sentences. If $\mathcal{M}$ is a model of T, then player* **II** *has a winning strategy in*

$$G_{cub}(\{X \in \mathcal{P}_{\omega}(M) : [X]_{\mathcal{M}} \models T\}).$$

In particular, for every countable $X \subseteq M$ there is a countable submodel $\mathcal{N}$ of $\mathcal{M}$ such that $X \subseteq N$ and $\mathcal{N} \models T$.

Proof Let $T = \{\varphi_0, \varphi_1, \ldots\}$. By Proposition 6.22 player **II** has a winning strategy in $G_{\text{cub}}(\mathcal{C}_{\varphi_n})$. By Lemma 6.14, player **II** has a winning strategy in $G_{\text{cub}}(\bigcap_{n=0}^{\infty} \mathcal{C}_{\varphi_n})$. If $X \in \bigcap_{n=0}^{\infty} \mathcal{C}_{\varphi_n}$, then $[X]_{\mathcal{M}} \models T$. □

6.5 The Semantic Game

The truth of a first order sentence in a structure can be defined by means of a simple game called the Semantic Game. We examine this game in detail and give some applications of it.

Definition 6.24 Suppose L is a vocabulary, $\mathcal{M}$ is an L-structure, φ^* is an L-formula and s^* is an assignment for M. The game $\text{SG}^{\text{sym}}(\mathcal{M}, \varphi^*)$ is defined as follows. In the beginning player **II** holds (φ^*, s^*). The rules of the game are as follows:

1. If φ is atomic, and s satisfies it in $\mathcal{M}$, then the player who holds (φ, s) wins the game, otherwise the other player wins.
2. If $\varphi = \neg\psi$, then the player who holds (φ, s), gives (ψ, s) to the other player.
3. If $\varphi = \psi \wedge \theta$, then the player who holds (φ, s), switches to hold (ψ, s) or (θ, s), and the other player decides which.
4. If $\varphi = \psi \vee \theta$, then the player who holds (φ, s), switches to hold (ψ, s) or (θ, s), and can himself or herself decide which.
5. If $\varphi = \forall x\psi$, then the player who holds (φ, s), switches to hold $(\psi, s[a/x])$ for some a, and the other player decides for which.
6. If $\varphi = \exists x\psi$, then the player who holds (φ, s), switches to hold $(\psi, s[a/x])$ for some a, and can himself or herself decide for which.

As was pointed out in Section 4.2, $\mathcal{M} \models_s \varphi$ if and only if player **II** has a winning strategy in the above game, starting with (φ, s). Why? If $\mathcal{M} \models_s \varphi$, then the winning strategy of player **II** is to play so that if she holds (φ', s'), then $\mathcal{M} \models_{s'} \varphi'$, and if player **I** holds (φ', s'), then $\mathcal{M} \not\models_{s'} \varphi'$.

For practical purposes it is useful to consider a simpler game which presupposes that the formula is in negation normal form. In this game, as in the Ehrenfeucht-Fraïssé Game, player **I** assumes the role of a doubter and player **II** the role of confirmer. This makes the game easier to use than the full game $\text{SG}^{\text{sym}}(\mathcal{M}, \varphi)$.

I	**II**
x_0	
	y_0
x_1	
	y_1
$\vdots$	$\vdots$

Figure 6.11 The game $G_\omega(W)$.

x_n	y_n	Explanation	Rule
$(\varphi, \emptyset)$		**I** enquires about $\varphi \in T$.	
	$(\varphi, \emptyset)$	**II** confirms.	Axiom rule
(φ_i, s)		**I** tests a played $(\varphi_0 \wedge \varphi_1, s)$ by choosing $i \in \{0, 1\}$.	
	(φ_i, s)	**II** confirms.	$\wedge$-rule
$(\varphi_0 \vee \varphi_1, s)$		**I** enquires about a played disjunction.	
	(φ_i, s)	**II** makes a choice of $i \in \{0, 1\}$.	$\vee$-rule
$(\varphi, s[a/x])$		**I** tests a played $(\forall x\varphi, s)$ by choosing $a \in M$.	
	$(\varphi, s[a/x])$	**II** confirms.	$\forall$-rule
$(\exists x\varphi, s)$		**I** enquires about a played existential statement.	
	$(\varphi, s[a/x])$	**II** makes a choice of $a \in M$.	$\exists$-rule

Figure 6.12 The game SG$(\mathcal{M}, T)$.

Definition 6.25 The *Semantic Game* SG$(\mathcal{M}, T)$ of the set T of L-sentences in NNF is the game (see Figure 6.11) $G_\omega(W)$, where W consists of sequences $(x_0, y_0, x_1, y_1, \ldots)$ where player **II** has followed the rules of Figure 6.12 and if player **II** plays the pair (φ, s), where φ is a basic formula, then $\mathcal{M} \models_s \varphi$.

In the game SG$(\mathcal{M}, T)$ player **II** claims that every sentence of T is true in

$\mathcal{M}$. Player **I** doubts this and challenges player **II**. He may doubt whether a certain $\varphi \in T$ is true in $\mathcal{M}$, so he plays $x_0 = (\varphi, \emptyset)$. In this round, as in some other rounds too, player **II** just confirms and plays the same pair as player **I**. This may seem odd and unnecessary, but it is for book-keeping purposes only. Player **I** in a sense gathers a finite set of formulas confirmed by player **II** and tries to end up with a basic formula which cannot be true.

Theorem 6.26 *Suppose L is a vocabulary, T is a set of L-sentences, and $\mathcal{M}$ is an L-structure. Then the following are equivalent:*

1. $\mathcal{M} \models T$.
2. *Player* **II** *has a winning strategy in SG$(\mathcal{M}, T)$.*

Proof Suppose $\mathcal{M} \models T$. The winning strategy of player **II** in SG$(\mathcal{M}, T)$ is to maintain the condition $\mathcal{M} \models_{s_i} \psi_i$ for all $y_i = (\psi_i, s_i)$, $i \in \mathbb{N}$, played by her. It is easy to see that this is possible. On the other hand, suppose $\mathcal{M} \not\models T$, say $\mathcal{M} \not\models \varphi$, where $\varphi \in T$. The winning strategy of player **I** in SG$(\mathcal{M}, T)$ is to start with $x_0 = (\varphi, \emptyset)$, and then maintain the condition $\mathcal{M} \not\models_{s_i} \psi_i$ for all $y_i = (\psi_i, s_i)$, $i \in \mathbb{N}$, played by **II**:

1. If $y_i = (\psi_i, s_i)$, where ψ_i is basic, then player **I** has won the game, because $\mathcal{M} \not\models_{s_i} \psi_i$.
2. If $y_i = (\psi_i, s_i)$, where $\psi_i = \theta_0 \wedge \theta_1$, then player **I** can use the assumption $\mathcal{M} \not\models_{s_i} \psi_i$ to find $k < 2$ such that $\mathcal{M} \not\models_{s_i} \theta_k$. Then he plays $x_{i+1} = (\theta_k, s_i)$.
3. If $y_i = (\psi_i, s_i)$, where $\psi_i = \theta_0 \vee \theta_1$, then player **I** knows from the assumption $\mathcal{M} \not\models_{s_i} \psi_i$ that whether **II** plays (θ_k, s_i) for $k = 0$ or $k = 1$, the condition $\mathcal{M} \not\models_{s_i} \theta_k$ still holds. So player **I** can play $x_{i+1} = (\psi_i, s_i)$ and keep his winning criterion in force.
4. If $y_i = (\psi_i, s_i)$, where $\psi_i = \forall x\varphi$, then player **I** can use the assumption $\mathcal{M} \not\models_{s_i} \psi_i$ to find $a \in M$ such that $\mathcal{M} \not\models_{s_i[a/x]} \varphi$. Then he plays $x_{i+1} = (\varphi, s_i[a/x])$.
5. If $y_i = (\psi_i, s_i)$, where $\psi_i = \exists x\varphi$, then player **I** knows from the assumption $\mathcal{M} \not\models_{s_i} \psi_i$ that whatever $(\varphi, s_i[a/x])$ player **II** chooses to play, the condition $\mathcal{M} \not\models_{s_i[a/x]} \varphi$ still holds. So player **I** can play $(\exists x\varphi, s_i)$ and keep his winning criterion in force.

□

Example 6.27 Let $L = \{f\}$ and $\mathcal{M} = (\mathbb{N}, f^{\mathcal{M}})$, where $f(n) = n + 1$. Let

$$\varphi = \forall x \exists y \approx fxy.$$

I	II	Rule
$(\forall x\exists y{\approx}fxy, \emptyset)$		
	$(\forall x\exists y{\approx}fxy, \emptyset)$	Axiom rule
$(\exists y{\approx}fxy, \{(x, 25)\})$		
	$(\exists y{\approx}fxy, \{(x, 25)\})$	$\forall$-rule
$(\exists y{\approx}fxy, \{(x, 25)\})$		
	$({\approx}fxy, \{(x, 25), (y, 26)\})$	$\exists$-rule
$\vdots$	$\vdots$	

Figure 6.13 Player **II** has a winning strategy in SG$(\mathcal{M}, \{\varphi\})$.

I	II	Rule
$(\forall x\exists y{\approx}fyx, \emptyset)$		
	$(\forall x\exists y{\approx}fyx, \emptyset)$	Axiom rule
$(\exists y{\approx}fyx, \{(x, 0)\})$		
	$(\exists y{\approx}fyx, \{(x, 0)\})$	$\forall$-rule
$(\exists y{\approx}fyx, \{(x, 0)\})$		
	$({\approx}fyx, \{(x, 0), (y, 2)\})$	$\exists$-rule
	(**II** has no good move)	

Figure 6.14 Player **I** wins the game SG$(\mathcal{M}, \{\psi\})$.

Clearly, $\mathcal{M} \models \varphi$. Thus player **II** has, by Theorem 6.26, a winning strategy in the game SG$(\mathcal{M}, \{\varphi\})$. Figure 6.13 shows how the game might proceed.
On the other hand, suppose

$$\psi = \forall x\exists y{\approx}fyx.$$

Clearly, $\mathcal{M} \not\models \varphi$. Thus player **I** has, by Theorem 6.26 and Theorem 3.12, a winning strategy in the game SG$(\mathcal{M}, \{\varphi\})$. Figure 6.14 shows how the game might proceed:

Example 6.28 Let $\mathcal{M}$ be the graph of Figure 6.15.
and

$$\varphi = \forall x(\exists y\neg xEy \wedge \exists yxEy).$$

Clearly, $\mathcal{M} \models \varphi$. Thus player **II** has, by Theorem 6.26, a winning strategy in the game SG$(\mathcal{M}, \{\varphi\})$. Figure 6.16 shows how the game might proceed.
On the other hand, suppose

$$\psi = \exists x(\forall y\neg xEy \vee \forall yxEy).$$

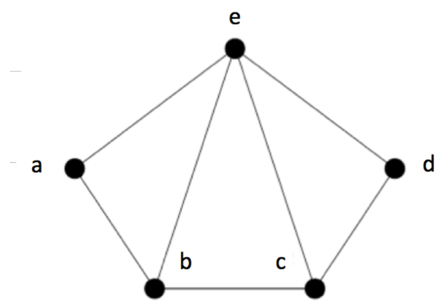

Figure 6.15 The graph $\mathcal{M}$.

I	II	Rule
$(\forall x(\exists y \neg xEy \wedge \exists y xEy), \emptyset)$		
	$(\forall x(\exists y \neg xEy \wedge \exists y xEy), \emptyset)$	Axiom rule
$(\exists y \neg xEy \wedge \exists y xEy, \{(x, d)\})$		
	$(\exists y \neg xEy \wedge \exists y xEy, \{(x, d)\})$	$\forall$-rule
$(\exists y xEy, \{(x, d)\})$		
	$(\exists y xEy, \{(x, d)\})$	$\wedge$-rule
$(\exists y xEy, \{(x, d)\})$		
	$(xEy, \{(x, d), (y, c)\})$	$\exists$-rule
$\vdots$	$\vdots$	

Figure 6.16 Player **II** has a winning strategy in SG$(\mathcal{M}, \{\varphi\})$.

Clearly, $\mathcal{M} \not\models \varphi$. Thus player **I** has, by Theorem 6.26 and Theorem 3.12, a winning strategy in the game SG$(\mathcal{M}, \{\varphi\})$. Figure 6.17 shows how the game might proceed.

6.6 The Model Existence Game

In this section we learn a new game associated with trying to construct a model for a sentence or a set of sentences. This is of fundamental importance in the sequel.

Let us first recall the game SG$(\mathcal{M}, T)$: The winning condition for **II** in the game SG$(\mathcal{M}, T)$ is the only place where the model $\mathcal{M}$ (rather than the set

I	II	Rule
$(\exists x(\forall y \neg xEy \vee \forall yxEy), \emptyset)$	$(\exists x(\forall y \neg xEy \vee \forall yxEy), \emptyset)$	Axiom rule
$(\exists x(\forall y \neg xEy \vee \forall yxEy), \emptyset)$	$(\forall y \neg xEy \vee \forall yxEy), \{(x, a)\})$	$\exists$-rule
$(\forall y \neg xEy \vee \forall yxEy, \{(x, a)\})$	$(\forall y \neg xEy, \{(x, a)\})$	$\vee$-rule
$(\neg xEy, \{(x, a), (y, d)\})$	$(\neg xEy, \{(x, a), (y, d)\})$	$\forall$-rule

Figure 6.17 Player **I** wins the game SG($\mathcal{M}, \{\psi\}$).

M) appears. If we do not start with a model $\mathcal{M}$ we can replace the winning condition with a slightly weaker one and get a very useful criterion for the existence of *some* $\mathcal{M}$ such that $\mathcal{M} \models T$:

Definition 6.29 The *Model Existence Game* $\mathrm{MEG}(T, L)$ of the set T of L-sentences in NNF is defined as follows. Let C be a countably infinite set of new constant symbols. $\mathrm{MEG}(T, L)$ is the game $G_\omega(W)$ (see Figure 6.11), where W consists of sequences $(x_0, y_0, x_1, y_1, \ldots)$ where player **II** has followed the rules of Figure 6.18 and for no atomic $L \cup C$-sentence φ both φ and $\neg\varphi$ are in $\{y_0, y_1, \ldots\}$.

The idea of the game $\mathrm{MEG}(T, L)$ is that player **I** does not doubt the truth of T (as there is no model around) but rather the mere consistency of T. So he picks those $\varphi \in T$ that he thinks constitute a contradiction and offers them to player **II** for confirmation. Then he runs through the subformulas of these sentences as if there was a model around in which they cannot all be true. He wins if he has made player **II** play contradictory basic sentences. It turns out it did not matter that we had no model around, as two contradictory sentences cannot hold in any model anyway.

Definition 6.30 Let L be a vocabulary with at least one constant symbol. A *Hintikka set (for first order logic)* is a set H of L-sentences in NNF such that:

1. $\approx tt \in H$ for every constant L-term t.
2. If $\varphi(x)$ is basic, $\varphi(c) \in H$ and $\approx tc \in H$, then $\varphi(t) \in H$.
3. If $\varphi \wedge \psi \in H$, then $\varphi \in H$ and $\psi \in H$.
4. If $\varphi \vee \psi \in H$, then $\varphi \in H$ or $\psi \in H$.

x_n	y_n	Explanation
φ		**I** enquires about $\varphi \in T$.
	φ	**II** confirms.
$\approx tt$		**I** enquires about an equation.
	$\approx tt$	**II** confirms.
$\varphi(t')$		**I** chooses played $\varphi(t)$ and $\approx tt'$ with φ basic and enquires about substituting t' for t in φ.
	$\varphi(t')$	**II** confirms.
φ_i		**I** tests a played $\varphi_0 \wedge \varphi_1$ by choosing $i \in \{0, 1\}$.
	φ_i	**II** confirms.
$\varphi_0 \vee \varphi_1$		**I** enquires about a played disjunction.
	φ_i	**II** makes a choice of $i \in \{0, 1\}$
$\varphi(c)$		**I** tests a played $\forall x\varphi(x)$ by choosing $c \in C$.
	$\varphi(c)$	**II** confirms.
$\exists x\varphi(x)$		**I** enquires about a played existential statement.
	$\varphi(c)$	**II** makes a choice of $c \in C$
t		**I** enquires about a constant $L \cup C$-term t.
	$\approx ct$	**II** makes a choice of $c \in C$

Figure 6.18 The game $\mathrm{MEG}(T, L)$.

5. If $\forall x\varphi(x) \in H$, then $\varphi(c) \in H$ for all $c \in L$
6. If $\exists x\varphi(x) \in H$, then $\varphi(c) \in H$ for some $c \in L$.
7. For every constant L-term t there is $c \in L$ such that $\approx ct \in H$.
8. There is no atomic sentence φ such that $\varphi \in H$ and $\neg\varphi \in H$.

Lemma 6.31 *Suppose L is a vocabulary and T is a set of L-sentences. If T has a model, then T can be extended to a Hintikka set.*

Proof Let us assume $\mathcal{M} \models T$. Let $L' \supseteq L$ such that L' has a constant symbol $c_a \notin L$ for each $a \in M$. Let $\mathcal{M}^*$ be an expansion of $\mathcal{M}$ obtained by interpreting c_a by a for each $a \in M$. Let H be the set of all L'-sentences true in $\mathcal{M}$. It is easy to verify that H is a Hintikka set.

□

Lemma 6.32 *Suppose L is a countable vocabulary and T is a set of L-sentences. If player* **II** *has a winning strategy in* $\mathrm{MEG}(T, L)$*, then the set T can be extended to a Hintikka set in a countable vocabulary extending L by constant symbols.*

Proof Suppose player **II** has a winning strategy in $\mathrm{MEG}(T, L)$. We first run through one carefully planned play of $\mathrm{MEG}(T, L)$. This will give rise to a model $\mathcal{M}$. Then we play again, this time providing a proof that $\mathcal{M} \models T$. To this end, let Trm be the set of all constant $L \cup C$-terms. Let

$$T = \{\varphi_n : n \in \mathbb{N}\},$$
$$C = \{c_n : n \in \mathbb{N}\},$$
$$Trm = \{t_n : n \in \mathbb{N}\}.$$

Let $(x_0, y_0, x_1, y_1, \ldots)$ be a play in which player **II** has used her winning strategy and player **I** has maintained the following conditions:

1. If $n = 3^i$, then $x_n = \varphi_i$.
2. If $n = 2 \cdot 3^i$, then x_n is $\approx c_i c_i$.
3. If $n = 4 \cdot 3^i \cdot 5^j \cdot 7^k \cdot 11^l$, y_i is $\approx t_j t_k$, and y_l is $\varphi(t_j)$, then x_n is $\varphi(t_k)$.
4. If $n = 8 \cdot 3^i \cdot 5^j$, y_i is $\theta_0 \wedge \theta_1$, and $j < 2$, then x_n is θ_j.
5. If $n = 16 \cdot 3^i$, and y_i is $\theta_0 \vee \theta_1$, then x_n is $\theta_0 \vee \theta_1$.
6. If $n = 32 \cdot 3^i \cdot 5^j$, y_i is $\forall x \varphi(x)$, then x_n is $\varphi(c_j)$.
7. If $n = 64 \cdot 3^i$, and y_i is $\exists x \varphi(x)$, then x_n is $\exists x \varphi(x)$.
8. If $n = 128 \cdot 3^i$, then x_n is t_i.

The idea of these conditions is that player **I** challenges player **II** in a maximal way. To guarantee this he makes a plan. The plan is, for example, that on round 3^i he always plays φ_i from the set T. Thus in an infinite game every element of T will be played. Also the plan involves the rule that if player **II** happens to play a conjunction $\theta_0 \wedge \theta_1$ on round i, then player **I** will necessarily play θ_0 on round $8 \cdot 3^i$ and θ_1 on round $8 \cdot 3^i \cdot 5$, etc. It is all just book-keeping—making sure that all possibilities will be scanned. This strategy of **I** is called the *enumeration strategy*. It is now routine to show that $H = \{y_0, y_1, \ldots\}$ is a Hintikka set. □

Lemma 6.33 *Every Hintikka set has a model in which every element is the interpretation of a constant symbol.*

Proof Let $c \sim c'$ if $\approx c'c \in H$. The relation $\sim$ is an equivalence relation on C (see Exercise 6.77). Let us define an $L \cup C$-structure $\mathcal{M}$ as follows. We let $M = \{[c] : c \in C\}$. For $c \in C$ we let $c^{\mathcal{M}} = [c]$. If $f \in L$ and $\#(f) = n$ we let $f^{\mathcal{M}}([c_{i_1}], \ldots, [c_{i_n}]) = [c]$ for some (any—see Exercise 6.78) $c \in C$ such that $\approx cfc_{i_1} \ldots c_{i_n} \in H$. For any constant term t there is a $c \in C$ such that $\approx ct \in H$. It is easy to see that $t^{\mathcal{M}} = [c]$. For the atomic sentence $\varphi = Rt_1 \ldots t_n$ we let $\mathcal{M} \models \varphi$ if and only if φ is in H. An easy induction on φ shows that if $\varphi(x_1, \ldots, x_n)$ is an L-formula and $\varphi(d_1, \ldots, d_n) \in H$ for some $d_1 \ldots, d_n$, then $\mathcal{M} \models \varphi(d_1, \ldots, d_n)$ (see Exercise 6.79). In particular, $\mathcal{M} \models T$. □

Lemma 6.34 *Suppose L is a countable vocabulary and T is a set of L-sentences. If T can be extended to a Hintikka set in a countable vocabulary extending L, then player* **II** *has a winning strategy in* $\mathrm{MEG}(T, L)$

Proof Suppose L^* is a countable vocabulary extending L such that some Hintikka set H in the vocabulary L^* extends T. Let $C = \{c_n : n \in \mathbb{N}\}$ be a new countable set of constant symbols to be used in $\mathrm{MEG}(T, L)$. Suppose $D = \{t_n : n \in \mathbb{N}\}$ is the set of constant terms of the vocabulary L^*. The winning strategy of player **II** in $\mathrm{MEG}(T, L)$ is to maintain the condition that if y_i is $\varphi(c_1, ..., c_n)$, then $\varphi(t_1, ..., t_n) \in H$. □

We can now prove the basic element of the Strategic Balance of Logic, namely the following equivalence between the Semantic Game and the Model Existence Game:

Theorem 6.35 (Model Existence Theorem) *Suppose L is a countable vocabulary and T is a set of L-sentences. The following are equivalent:*

1. *There is an L-structure $\mathcal{M}$ such that $\mathcal{M} \models T$.*
2. *Player* **II** *has a winning strategy in* $\mathrm{MEG}(T, L)$.

Proof If there is an L-structure $\mathcal{M}$ such that $\mathcal{M} \models T$, then by Lemma 6.31 there is a Hintikka set $H \supseteq T$. Then by Lemma 6.34 player **II** has a winning strategy in $\mathrm{MEG}(T, L)$. Suppose conversely that player **II** has a winning strategy in $\mathrm{MEG}(T, L)$. By Lemma 6.32 there is a Hintikka set $H \supseteq T$. Finally, this implies by Lemma 6.33 that T has a model. □

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set $\mathcal{C}_n \subseteq \mathcal{D}_n$. Let $\mathcal{C} = \bigcap \mathcal{C}_n$ and show that $\mathcal{C}$ can have only one element, which contradicts the fact that $\mathcal{C}$ is cub.)

6.47 Use the previous exercise to conclude that $\mathcal{CUB}_A$ is not an ultrafilter (i.e. a maximal filter) if A is infinite.

6.48 Show that the set NS^A of sets $\mathcal{C} \subseteq \mathcal{P}_\omega(A)$ which are non-stationary is a σ-ideal (i.e. (1) If $\mathcal{D} \in \mathrm{NS}^A$ and $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{P}_\omega(A)$, then $\mathcal{C} \in \mathrm{NS}^A$. (2) If $\mathcal{D}_n \in \mathrm{NS}^A$ for all $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} \mathcal{D}_n \in \mathrm{NS}^A$). In fact, NS^A is a normal ideal (i.e. if $\mathcal{D}_a \in \mathrm{NS}^A$ for all $a \in A$, then $\triangledown_{a\in A}\mathcal{D}_a \in \mathrm{NS}^A$).

6.49 Show that if a sentence is true in a stationary set of countable submodels of a model then it is true in the model itself. More exactly: Let L be a countable vocabulary, $\mathcal{M}$ an L-model and φ an L-sentence. Suppose $\{X \in \mathcal{P}_\omega(M) : [X]_\mathcal{M} \models \varphi\}$ is stationary. Show that $\mathcal{M} \models \varphi$.

6.50 In this and the following exercises we develop the theory of cub and stationary subsets of a regular cardinal $\kappa > \omega$. A set $C \subseteq \kappa$ is *closed* if it contains every non-zero limit ordinal $\delta < \kappa$ such that $C \cap \delta$ is unbounded in δ, and *unbounded* if it is unbounded as a subset of κ. We call $C \subseteq \kappa$ a *closed unbounded* (*cub*) set if C is both closed and unbounded. Show that the following sets are cub

(i) κ
(ii) $\{\alpha < \kappa : \alpha \text{ is a limit ordinal}\}$
(iii) $\{\alpha < \kappa : \alpha = \omega^\beta \text{ for some } \beta\}$
(iv) $\{\alpha < \kappa : \text{if } \beta < \alpha \text{ and } \gamma < \alpha, \text{ then } \beta + \gamma < \alpha\}$
(v) $\{\alpha < \kappa : \text{if } \alpha = \beta \cdot \gamma, \text{ then } \alpha = \beta \text{ or } \alpha = \gamma\}$.

6.51 Show that he following sets are not cub:

(i) $\emptyset$
(ii) $\{\alpha < \omega_1 : \alpha = \beta + 1 \text{ for some } \beta\}$
(iii) $\{\alpha < \omega_1 : \alpha = \omega^\beta + \omega \text{ for some } \beta\}$
(iv) $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$.

6.52 Show that a set C contains a cub subset of ω_1 if and only if player **II** wins the game $G_\omega(W_C)$, where

$$W_C = \{(x_0, x_1, x_2, \ldots) : \sup_n x_n \in C\}.$$

6.53 A filter $\mathcal{F}$ on M is λ-*closed* if $A_\alpha \in \mathcal{F}$ for $\alpha < \beta$, where $\beta < \lambda$, implies $\bigcap_\alpha A_\alpha \in \mathcal{F}$. A filter $\mathcal{F}$ on κ is *normal* if $A_\alpha \in \mathcal{F}$ for $\alpha < \kappa$ implies $\triangle_\alpha A_\alpha \in \mathcal{F}$, where

$$\triangle_\alpha A_\alpha = \{\alpha < \kappa : \alpha \in A_\beta \text{ for all } \beta < \alpha\}.$$

Note that normality implies κ-closure. Show that if $\kappa > \omega$ is regular, then the set $\mathcal{F}$ of subsets of κ that contain a cub set is a proper normal filter on κ. The filter $\mathcal{F}$ is called the *cub-filter* on κ.

6.54 A subset of κ which meets every cub set is called *stationary*. Equivalently, a subset S of κ is stationary if its complement is not in the cub-filter. A set which is not stationary, is *non-stationary*. Show that all sets in the cub-filter are stationary. Show that

$$\{\alpha < \omega_2 : \mathrm{cof}(\alpha) = \omega\}$$

is a stationary set which is not in the cub-filter on ω_2.

6.55 (Fodor's lemma, second formulation) Suppose $\kappa > \omega$ is a regular cardinal. If $S \subseteq \kappa$ is stationary and $f : S \to \kappa$ satisfies $f(\alpha) < \alpha$ for all $\alpha \in S$, then there is a stationary $S' \subseteq S$ such that f is constant on S'. (Hint: For each $\alpha < \kappa$ let $S_\alpha = \{\beta < \kappa : f(\beta) = \alpha\}$. Show that one of the sets S_α has to be stationary.)

6.56 Suppose κ is a regular cardinal $> \omega$. Show that there is a bistationary set $S \subseteq \kappa$ (i.e. both S and $\kappa \setminus S$ are stationary). (Hint: Note that $S = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ is always stationary. For $\alpha \in S$ let $\delta_\alpha : \omega \to \alpha$ be strictly increasing with $\sup_n \delta_\alpha(n) = \alpha$. By the previous exercise there is for each $n < \omega$ a stationary $A_n \subseteq S$ such that the regressive function $f_n(\alpha) = \delta_\alpha(n)$ is constant δ_n on A_n. Argue that some $\kappa \setminus A_n$ must be stationary.)

6.57 Suppose κ is a regular cardinal $> \omega$. Show that $\kappa = \bigcup_{\alpha<\kappa} S_\alpha$ where the sets S_α are disjoint stationary sets. (Hint: Proceed as in Exercise 6.56. Find $n < \omega$ such that for all $\beta < \kappa$ the set $S_\beta = \{\alpha < \kappa : \delta_\alpha(n) \geq \beta\}$ is stationary. Find stationary $S'_\beta \subseteq S_\beta$ such that $\delta_\alpha(n)$ is constant for $\alpha \in S'_\beta$. Argue that there are κ different sets S'_β.)

6.58 Show that $S \subseteq \omega_1$ is bistationary if and only if the game $G_\omega(W_S)$ is non-determined.

6.59 Suppose κ is regular $> \omega$. Show that $S \subseteq \kappa$ is stationary if and only if every regressive $f : S \to \kappa$ is constant on an unbounded set.

6.60 Prove that $C \subseteq \omega_1$ is in the cub filter if and only if almost all countable subsets of ω_1 have their sup in C.

6.61 Suppose $S \subseteq \omega_1$ is stationary. Show that for all $\alpha < \omega_1$ there is a closed subset of S of order-type $\geq \alpha$. (Hint: Prove a stronger claim by induction on α.)

6.62 Decide first which of the following are true and then show how the winner should play the game $\mathrm{SG}(\mathcal{M}, T)$:

1. $(\mathbb{R}, <, 0) \models \exists x \forall y (y < x \vee 0 < y)$
2. $(\mathbb{N}, <) \models \forall x \forall y (\neg y < x \vee \forall z (z < y \vee \neg z < x))$.

6.63 Prove directly that if **II** has a winning strategy in SG$(\mathcal{M}, T)$ and $\mathcal{M} \simeq_p \mathcal{N}$, then **II** has a winning strategy in SG$(\mathcal{N}, T)$.

6.64 The *Existential Semantic Game* SG$_\exists(\mathcal{M}, T)$ differs from SG$(\mathcal{M}, T)$ only in that the $\forall$-rule is omitted. Show that if **II** has a winning strategy in SG$_\exists(\mathcal{M}, T)$ and $\mathcal{M} \subseteq \mathcal{N}$, then **II** has a winning strategy in SG$_\exists(\mathcal{N}, T)$.

6.65 A formula in NNF is *existential* if it contains no universal quantifiers. (Then it is logically equivalent to one of the form $\exists x_1 \ldots \exists x_n \varphi$, where φ is quantifier free.) Show that if L is countable and T is a set of existential L-sentences, then $\mathcal{M} \models T$ if and only if player **II** has a winning strategy in the game SG$_\exists(\mathcal{M}, T)$.

6.66 The *Universal-Existential Semantic Game* SG$_{\forall\exists}(\mathcal{M}, T)$ differs from the game SG$(\mathcal{M}, T)$ only in that player **I** has to make all applications of the $\forall$-rule before all applications of the $\exists$-rule. Show that if $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \ldots$ and **II** has a winning strategy in each SG$_{\forall\exists}(\mathcal{M}_n, T)$, then **II** has a winning strategy in SG$_{\forall\exists}(\cup_{n=0}^{\infty} \mathcal{M}_n, T)$.

6.67 A formula in NNF is *universal-existential* if it is of the form

$$\forall y_1 \ldots \forall y_n \exists x_1 \ldots \exists x_m \varphi,$$

where φ is quantifier free. Show that if L is countable and T is a set of universal-existential L-sentences, then $\mathcal{M} \models T$ if and only if player **II** has a winning strategy in the game SG$_{\forall\exists}(\mathcal{M}, T)$.

6.68 The *Positive Semantic Game* SG$_{\text{pos}}(\mathcal{M}, T)$ differs from SG$(\mathcal{M}, T)$ only in that the winning condition "If player **II** plays the pair (φ, s), where φ is basic, then $\mathcal{M} \models_s \varphi$" is weakened to "If player **II** plays the pair (φ, s), where φ is atomic, then $\mathcal{M} \models_s \varphi$". Suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures. A surjection $h : M \to N$ is a *homomorphism* $\mathcal{M} \to \mathcal{N}$ if

$$\mathcal{M} \models \varphi(a_1, \ldots, a_n) \Rightarrow \mathcal{N} \models \varphi(f(a_1), \ldots, f(a_n))$$

for all atomic L-formulas φ and all $a_1, \ldots, a_n \in M$. Show that if **II** has a winning strategy in SG$_{\text{pos}}(\mathcal{M}, T)$ and $h : \mathcal{M} \to \mathcal{N}$ is a surjective homomorphism, then **II** has a winning strategy in SG$_{\text{pos}}(\mathcal{N}, T)$.

6.69 A formula in NNF is *positive* if it contains no negations. Show that if L is countable and T is a set of positive L-sentences, then $\mathcal{M} \models T$ if and only if player **II** has a winning strategy in the game SG$_{\text{pos}}(\mathcal{M}, T)$.

6.70 The game $\text{MEG}(T, L)$ is played with

$$T = \{Pc, \neg Qfc, \forall x_0(\neg Px_0 \vee Qx_0), \forall x_0(\neg Px_0 \vee Pfx_0)\}.$$

The game starts as in Figure 6.22. How does **I** play now and win?

I	**II**
$\neg Pc \vee Pfc$	
	Pfc

Figure 6.22

I	**II**
$\exists x_0 \forall x_1 Rx_0x_1$	
	$\forall x_1 Rc_0x_1$
$\exists x_1 \forall x_0 \neg Rx_0x_1$	
	$\forall x_0 \neg Rx_0c_1$

Figure 6.23

6.71 Consider $T = \{\exists x_0 \forall x_1 Rx_0x_1, \exists x_1 \forall x_0 \neg Rx_0x_1\}$. Now we start the game $\text{MEG}(T, L)$ as in Figure 6.23. How does **I** play now and win?

6.72 Consider $T = \{\forall x_0(\neg Px_0 \vee Qx_0), \exists x_0(Qx_0 \wedge \neg Px_0)\}$. The game $\text{MEG}(T, L)$ is played. Player **I** immediately resigns. Why?

6.73 The game $\text{MEG}(T, L)$ is played with

$$\begin{aligned} T &= \{\forall x_0 \neg x_0 E x_0, \forall x_0 \forall x_1(\neg x_0 E x_1 \vee x_1 E x_0), \\ &\quad \forall x_0 \exists x_1 x_0 E x_1, \forall x_0 \exists x_1 \neg x_0 E x_1\}. \end{aligned}$$

Player **I** immediately resigns. Why?

6.74 Use the game $\text{MEG}(T, L)$ to decide whether the following sets T have a model:

1. $\{\exists x Px, \forall y(\neg Py \vee Ry)\}$.
2. $\{\forall x Pxx, \exists y \forall x \neg Pxy\}$.

6.75 Prove the following by giving a winning strategy of player **I** in the appropriate game $\text{MEG}(T \cup \{\neg\varphi\}, L)$:

1. $\{\forall x(Px \to Qx), \exists x Px\} \models \exists x Qx$.
2. $\{\forall x Rxfx\} \models \forall x \exists y Rxy$.

6.76 Suppose T is the following theory

$$\begin{array}{l} \forall x_0 \neg x_0 < x_0 \\ \forall x_0 \forall x_1 \forall x_2(\neg(x_0 < x_1 \wedge x_1 < x_2) \vee x_0 < x_2) \\ \forall x_0 \forall x_1(x_0 < x_1 \vee x_1 < x_0 \vee x_0 \approx x_1) \\ \exists x_0(Px_0 \wedge \forall x_1(\neg Px_1 \vee x_0 \approx x_1 \vee x_1 < x_0) \\ \exists x_0(\neg Px_0 \wedge \forall x_1(Px_1 \vee x_0 \approx x_1 \vee x_1 < x_0) \end{array}$$

Give a winning strategy for player **I** in $\text{MEG}(T, L)$.

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7

Infinitary Logic

7.1 Introduction

As the name indicates, infinitary logic has infinite formulas. The oldest use of infinitary formulas is the elimination of quantifiers in number theory:

$$\exists x\varphi(x) \leftrightarrow \bigvee_{n\in\mathbb{N}} \varphi(n)$$

$$\forall x\varphi(x) \leftrightarrow \bigwedge_{n\in\mathbb{N}} \varphi(n).$$

Here we leave behind logic as a study of sentences humans can write down on paper. Infinitary formulas are merely mathematical objects used to study properties of structures and proofs. It turns out that games are particularly suitable for the study of infinitary logic. In a sense games replace the use of the Compactness Theorem which fails badly in infinitary logic.

7.2 Preliminary Examples

The games we have encountered so far have had a fixed length, which has been either a natural number or ω (an infinite game). Now we introduce a game which is "dynamic" in the sense that it is possible for player **I** to change the length of the game during the game. He may first claim he can win in five moves, but seeing what the first move of **II** is, he may decide he needs ten moves. In these games player **I** is not allowed to declare he will need infinitely many moves, although we shall study such games, too, later.

Before giving a rigorous definition of the Dynamic Ehrenfeucht-Fraïssé game we discuss some simple versions of it.

Definition 7.1 (Preliminary) Suppose $\mathcal{M}, \mathcal{M}'$ are L-structures such that L is a relational vocabulary and $M \cap M' = \emptyset$. The *Dynamic Ehrenfeucht-Fraïssé game*, denoted $\mathrm{EFD}_\omega(\mathcal{M}, \mathcal{M}')$ is defined as follows: First player **I** chooses a natural number n and then the game $\mathrm{EF}_n(\mathcal{M}, \mathcal{M}')$ is played.

Note that $\mathrm{EFD}_\omega(\mathcal{M}, \mathcal{M}')$ is *not* a game of length ω. Player **II** has a winning strategy in $\mathrm{EFD}_\omega(\mathcal{M}, \mathcal{M}')$ if she has one in each $\mathrm{EF}_n(\mathcal{M}, \mathcal{M}')$. On the other hand, player **I** has a winning strategy in $\mathrm{EFD}_\omega(\mathcal{M}, \mathcal{M}')$ if he can envisage a number n so that he has a winning strategy in $\mathrm{EF}_n(\mathcal{M}, \mathcal{M}')$.

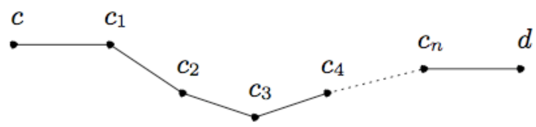
Example 7.2 If $\mathcal{M}$ and $\mathcal{M}'$ are L-structures such that M is finite and M' is infinite, then player **I** has a winning strategy in $\mathrm{EFD}_\omega(\mathcal{M}, \mathcal{M}')$. Suppose $|M| = n$. Player **I** has a winning strategy in $\mathrm{EF}_{n+1}(\mathcal{M}, \mathcal{M}')$. He first plays all n elements of M and then any unplayed element of M'. Player **II** is out of good moves, and loses the game.

Example 7.3 If $\mathcal{M}$ and $\mathcal{M}'$ are equivalence relations such that $\mathcal{M}$ has finitely many equivalence classes and $\mathcal{M}'$ infinitely many, then player **I** has a winning strategy in $\mathrm{EFD}_\omega(\mathcal{M}, \mathcal{M}')$. Suppose the equivalence classes of $\mathcal{M}$ are $[a_1], \ldots, [a_n]$. The strategy of **I** is to play first the elements $a_1, \ldots, a_n$. Then he plays an element from M' which is not equivalent to any element played so far. Player **II** is at a loss. She has to play an element of M equivalent to one of $a_1, \ldots, a_n$. She loses.

Definition 7.4 (Preliminary) Suppose $n \in \mathbb{N}$. The game $\mathrm{EFD}_{\omega+n}(\mathcal{M}, \mathcal{M}')$ is played as follows. First the game $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ is played for n moves. Then player **I** declares a natural number m and the game $\mathrm{EF}_\omega(\mathcal{M}, \mathcal{M}')$ is continued for m more moves. If **II** has not lost yet, she has won $\mathrm{EFD}_{\omega+n}(\mathcal{M}, \mathcal{M}')$. Otherwise player **I** has won.

Example 7.5 Suppose $\mathcal{G}$ and $\mathcal{G}'$ are graphs so that in $\mathcal{G}$ every vertex has a finite degree while in $\mathcal{G}'$ some vertex has infinite degree. Then player **I** has a winning strategy in $\mathrm{EFD}_{\omega+1}(\mathcal{G}, \mathcal{G}')$. Suppose $a \in G'$ has infinite degree. Player **I** plays first the element a. Let $b \in G$ be the response of player **II**. We know that every element of $\mathcal{G}$ has finite degree. Let the degree of b be n. Player **I** declares that we play $n+1$ more moves. Accordingly, he plays $n+1$ different neighbors of a. Player **II** cannot play $n+1$ different neighbors of b since b has degree n. She loses.

Example 7.6 Suppose $\mathcal{G}$ is a connected graph and $\mathcal{G}'$ a disconnected graph. Then player **I** has a winning strategy in $\mathrm{EFD}_{\omega+2}(\mathcal{G}, \mathcal{G}')$. Suppose a and b are elements of G' that are not connected by a path. Player **I** plays first elements a and b. Suppose the responses of player **II** are c and d. Since $\mathcal{G}$ is connected, there is a connected path $c = c_0, c_1, \ldots, c_n, c_{n+1} = d$ connecting c and d in $\mathcal{G}$.



Now player **I** declares that he needs n more moves. He plays the elements $c_1, \ldots, c_n$ one by one. Player **II** has to play a connected path $a_1, \ldots, a_n$ in $\mathcal{G}'$. Now d is a neighbor of c_n in $\mathcal{G}$ but b is not a neighbor of a_n in $\mathcal{G}'$ (see Figure 7.1).

Example 7.7 An *abelian group* is a structure $\mathcal{G} = (G, +)$ with $+_{\mathcal{G}} : G \times G \to G$ satisfying the conditions

(1) $x +_{\mathcal{G}} (y +_{\mathcal{G}} z) = (x +_{\mathcal{G}} y) +_{\mathcal{G}} z$ for x, y, z
(2) there is an element $0_{\mathcal{G}}$ such that $x +_{\mathcal{G}} 0_{\mathcal{G}} = 0_{\mathcal{G}} +_{\mathcal{G}} x = x$ for all x
(3) for all x there is $-x$ such that $x +_{\mathcal{G}} (-x) = 0_{\mathcal{G}}$
(4) for all x and y : $x +_{\mathcal{G}} y = y +_{\mathcal{G}} x$.

Examples of abelian groups are

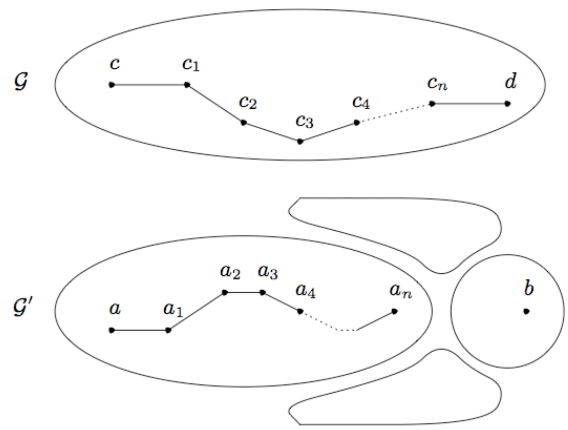


Figure 7.1

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8
Model Theory of Infinitary Logic

8.1 Introduction

The model theory of $L_{\omega_1\omega}$ is dominated by the Model Existence Theorem. It more or less takes the role of the Compactness Theorem which can be rightfully called the cornerstone of model theory of first order logic. The Model Existence Theorem is used to prove the Craig Interpolation Theorem and the important undefinability of the concept of well-order. When we move to the stronger logics $L_{\kappa^+\omega}$, $\kappa > \omega$, the Model Existence Theorem in general fails. However, we use a union of chains argument to prove the undefinability of well-order. In the final section we introduce game quantifiers. Here we cross the line to logics in which well-order is definable. Game quantifiers permit an approximation process which leads to the Covering Theorem, a kind of Interpolation Theorem.

8.2 Löwenheim-Skolem Theorem for $L_{\infty\omega}$

In Section 6.4 we saw that if a first order sentence is true in a model it is true in "almost" every countable approximation of that model. We now extend this to $L_{\infty\omega}$ but of course with some modification because $L_{\infty\omega}$ has consistent sentences without any countable models. We show that if a sentence φ of $L_{\infty\omega}$ is true in a structure $\mathcal{M}$, a countable "approximation" of φ is true in a countable "approximation" of $\mathcal{M}$, and even more, there are this kind of approximations of φ and $\mathcal{M}$ in a sense "everywhere". To make this statement precise we employ the cub game introduced in Definition 6.10. We say

$$\ldots X \ldots \text{ for almost all } X \in \mathcal{P}_\omega(A)$$

if

$$\text{player } \mathbf{II} \text{ has a winning strategy in } G_{\text{cub}}(\mathcal{P}_\omega(A)).$$

Recall the following facts:

1. If $X_0 \in \mathcal{P}_\omega(A)$, then $X_0 \subseteq X$ for almost all $X \in \mathcal{P}_\omega(A)$.
2. If $X \in \mathcal{C}$ for almost all $X \in \mathcal{P}_\omega(A)$ and $\mathcal{C} \subseteq \mathcal{C}'$, then $X \in \mathcal{C}'$ for almost all $X \in \mathcal{P}_\omega(A)$.
3. If for all $n \in \mathbb{N}$ we have $X \in \mathcal{C}_n$ for almost all $X \in \mathcal{P}_\omega(A)$, then $X \in \bigcap_{n\in\mathbb{N}} \mathcal{C}_n$ for almost all $X \in \mathcal{P}_\omega(A)$.
4. If for all $a \in A$ we have $X \in \mathcal{C}_a$ for almost all $X \in \mathcal{P}_\omega(A)$, then $X \in \triangle_{a\in A}\mathcal{C}_a$ for almost all $X \in \mathcal{P}_\omega(A)$.

In other words, the set of subsets of $\mathcal{P}_\omega(A)$ which contain almost all $X \in \mathcal{P}_\omega(A)$ is a countably complete filter.

Now that approximations extend not only to models but also to formulas we assume that models and formulas have a common universe V, which is supposed to be a transitive[1] set. As the following lemma demonstrates, the exact choice of this set V is not relevant:

Lemma 8.1 *Suppose $\emptyset \neq A \subseteq V$ and $\mathcal{C} \subseteq \mathcal{P}_\omega(A)$. Then the following are equivalent:*

1. *$X \in \mathcal{C}$ for almost all $X \in \mathcal{P}_\omega(A)$.*
2. *$X \cap A \in \mathcal{C}$ for almost all $X \in \mathcal{P}_\omega(V)$.*

Proof (1) implies (2): Let $a \in A$. Player **II** applies her winning strategy in $G_{\text{cub}}(\mathcal{C})$ in the game $G_{\text{cub}}(\{X \in \mathcal{P}_\omega(V) : X \cap A \in \mathcal{C}\})$ as follows: If **I** plays his element in A, player **II** interprets it as a move in $G_{\text{cub}}(\mathcal{C})$, where she has a winning strategy. If **I** plays x_n outside A, player **II** plays $y_n = a$. (2) implies (1): player **II** interprets all moves of **I** in A as his moves in V and then uses her winning strategy in $G_{\text{cub}}(\{X \in \mathcal{P}_\omega(V) : X \cap A \in \mathcal{C}\})$. □

Definition 8.2 Suppose $\varphi \in L_{\infty\omega}$ and X is a countable set. The approximation φ^X of φ is defined by induction as follows:

(1) $(\approx tt')^X = \approx tt'$
(2) $(Rt_1 \ldots t_n)^X = Rt_1 \ldots t_n$
(3) $(\neg\varphi)^X = \neg\varphi^X$
(4) $(\bigwedge \Phi)^X = \bigwedge\{\varphi^X : \varphi \in \Phi \cap X\}$
(5) $(\bigvee \Phi)^X = \bigvee\{\varphi^X : \varphi \in \Phi \cap X\}$
(6) $(\forall x_n\varphi)^X = \forall x_n(\varphi^X)$

[1] A set A is *transitive* if $y \in x \in A$ implies $y \in A$ for all x and y.

(7) $(\exists x_n\varphi)^X = \exists x_n(\varphi^X)$.

Note that φ^X is always in $L_{\omega_1\omega}$, whatever countable set X is.

Example 8.3 Suppose $X \cap \{\varphi_\alpha : \alpha < \omega_1\} = \{\varphi_{\alpha_0}, \varphi_{\alpha_1}, \ldots\}$. Then

$$(\forall x_0 \bigvee_{\alpha<\omega_1} \varphi_\alpha(x_0))^X = \forall x_0 \bigvee_n \varphi^X_{\alpha_n}(x_0)$$

Example 8.4 Suppose $X, \mathcal{M}, \theta_\delta \in V$, V transitive, and δ is the order type of $X \cap On$. Then for all $\alpha \geq \delta$ we have $\mathcal{M} \models \forall x_0(\theta^X_\alpha \leftrightarrow \theta_\delta)$ (Exercise 8.4).

Lemma 8.5 *If $\varphi \in L_{\omega_1\omega}$, then player* **II** *has a winning strategy in the game $G_{cub}(\{X \in \mathcal{P}_\omega(V) : \varphi^X = \varphi\})$. That is, almost all approximations of $\varphi \in L_{\omega_1\omega}$ are equal to φ.*

Proof We use induction on φ. If φ is atomic, the claim is trivial since $\varphi^X = \varphi$ holds for all X. Also negation and the cases of $\forall x_n\varphi$ and $\exists x_n\varphi$ are immediate. Let us then assume $\varphi = \bigwedge_{n\in\mathbb{N}} \varphi_n$ and the claim holds for each φ_n, that is, player **II** has a winning strategy in $G_{cub}(\{X \in \mathcal{P}_\omega(V) : \varphi^X_n = \varphi_n\})$ for each n. By Lemma 6.14 player **II** has a winning strategy in the cub game for the set

$$\bigcap_{n\in\mathbb{N}} \{X : \varphi^X_n = \varphi_n\} \cap \{X : \varphi_n \in X \text{ for all } n \in \mathbb{N}\}.$$

□

Definition 8.6 Suppose L is a vocabulary and $\mathcal{M}$ an L-structure. Suppose φ is a first order formula in NNF and s an assignment for the set M the domain of which includes the free variables of φ. We define the set $\mathcal{D}_{\varphi,s}$ of countable subsets of M as follows: If φ is basic, $\mathcal{D}_{\varphi,s}$ contains as an element any countable $X \subseteq V$ such that $X \cap M$ is the domain of a countable submodel $\mathcal{A}$ of $\mathcal{M}$ such that $\mathrm{rng}(s) \subseteq A$ and:

- If φ is $\approx tt'$, then $t^{\mathcal{A}}(s) = t'^{\mathcal{A}}(t)$.
- If φ is $\neg\approx tt'$, then $t^{\mathcal{A}}(s) \neq t'^{\mathcal{A}}(t)$.
- If φ is $Rt_1 \ldots t_n$, then $(t^{\mathcal{A}}_1(s), \ldots, t_n{}^{\mathcal{A}}(t)) \in R^{\mathcal{A}}$.
- If φ is $\neg Rt_1 \ldots t_n$, then $(t^{\mathcal{A}}_1(s), \ldots, t_n{}^{\mathcal{A}}(t)) \notin R^{\mathcal{A}}$.

For non-basic φ we define

- $\mathcal{D}_{\bigwedge\Phi,s} = \triangle_{\varphi\in\Phi}\mathcal{D}_{\varphi,s}$.
- $\mathcal{D}_{\bigvee\Phi,s} = \triangledown_{\varphi\in\Phi}\mathcal{D}_{\varphi,s}$
- $\mathcal{D}_{\forall x\varphi,s} = \triangle_{a\in M}\mathcal{D}_{\varphi,s[a/x]}$.
- $\mathcal{D}_{\exists x\varphi,s} = \triangledown_{a\in M}\mathcal{D}_{\varphi,s(a/x)}$.

If φ is a sentence, we denote $\mathcal{D}_{\varphi,s}$ by $\mathcal{D}_\varphi$. If φ is not in NNF, we define $\mathcal{D}_{\varphi,s}$ and $\mathcal{D}_\varphi$ by first translating φ into a logically equivalent NNF formula.

Intuitively, $\mathcal{D}_\varphi$ is the collection of countable sets X, which *simultaneously* give an $L_{\omega_1\omega}$-approximation φ^X of φ and a countable approximation $\mathcal{M}^X$ of $\mathcal{M}$ such that $\mathcal{M}^X \models \varphi^X$.

Proposition 8.7 *Suppose $\mathcal{A}$ is an L-structure and $X \in \mathcal{D}_{\varphi,s}$. Then $[X \cap A]_\mathcal{A} \models_t \varphi^X$.*

Proof This is trivial for basic φ. For the induction step for $\bigwedge \Phi$ suppose $X \in \mathcal{D}_{\bigwedge \Phi,s}$. Suppose $\varphi \in X \cap \Phi$. Then $X \in \mathcal{D}_{\varphi,s}$. By induction hypothesis $[X \cap A]_\mathcal{A} \models_t \varphi^X$. Thus $[X]_\mathcal{A} \models_t (\bigwedge \Phi)^X$. The other cases are as in the proof of Proposition 6.21. □

Proposition 8.8 *Suppose L is a countable vocabulary and $\mathcal{M}$ an L-structure such that $\mathcal{M} \models \varphi$. Then player* **II** *has a winning strategy in $G_{cub}(\mathcal{D}_\varphi)$.*

Proof We use induction on φ to prove that if $\mathcal{M} \models_s \varphi$, then **II** has a winning strategy in $G_{cub}(\mathcal{D}_{\varphi,s})$. Most steps are as in the proof of Proposition 6.22. Let us look at the induction step for $\bigwedge \Phi$. We assume $\mathcal{M} \models_s \bigwedge \varphi$. It suffices to prove that **II** has a winning strategy in $G_{cub}(\mathcal{D}_{\varphi,s})$ for each $\varphi \in \Phi$. But this follows from the induction hypothesis. □

Theorem 8.9 (Löwenheim-Skolem Theorem) *Suppose L is a countable vocabulary, $\mathcal{M}$ an arbitrary L-structure, and φ an $L_{\infty\omega}$-sentence of vocabulary L, and V a transitive set containing $\mathcal{M}$ and φ such that $M \cap TC(\varphi) = \emptyset$. Suppose $\mathcal{M} \models \varphi$. Let*

$$\mathcal{C} = \{X \in \mathcal{P}_\omega(V) : [X \cap M]_\mathcal{M} \models \varphi^X\}.$$

Then player **II** *has a winning strategy in the game $G_{cub}(\mathcal{C})$.*

Proof The claim follows from Propositions 8.7 and 8.8. □

Theorem 8.10 *1. $\mathcal{M} \equiv_{\infty\omega} \mathcal{N}$ if and only if $\mathcal{M}^X \cong \mathcal{N}^X$ for almost all X.*
2. $\mathcal{M} \not\equiv_{\infty\omega} \mathcal{N}$ if and only if $\mathcal{M}^X \not\cong \mathcal{N}^X$ for almost all X.

8.3 Model Theory of $L_{\omega_1\omega}$

The Model Existence Game $\mathrm{MEG}(T, L)$ of first order logic (Definition 6.35) can be easily modified to $L_{\omega_1\omega}$.

x_n	y_n	Explanation
φ		**I** enquires about φ.
	φ	**II** confirms.
$\approx tt$		**I** enquires about an equation.
	$\approx tt$	**II** confirms.
$\varphi(t)$		**I** chooses played $\varphi(c)$ and $\approx ct$ with φ basic and enquires about substituting t for c in φ.
	$\varphi(t)$	**II** confirms.
φ_i		**I** tests a played $\bigwedge_{i\in I}\varphi_i$ by choosing $i \in I$.
	φ_i	**II** confirms.
$\bigvee_{i\in I}\varphi_i$		**I** enquires about a played disjunction.
	φ_i	**II** makes a choice of $i \in I$.
$\varphi(c)$		**I** tests a played $\forall x\varphi(x)$ by choosing $c \in C$.
	$\varphi(c)$	**II** confirms.
$\exists x\varphi(x)$		**I** enquires about a played existential statement.
	$\varphi(c)$	**II** makes a choice of $c \in C$.
t		**I** enquires about a constant $L \cup C$-term t.
	$\approx ct$	**II** makes a choice of $c \in C$.

Figure 8.1 The game $\mathrm{MEG}(T, L)$.

Definition 8.11 The Model Existence Game $\mathrm{MEG}(\varphi, L)$ for a countable vocabulary L and a sentence φ of $L_{\omega_1\omega}$ is the game $G_\omega(W)$ where W consists of sequences $(x_0, y_0, x_1, y_1, \ldots)$ where player **II** has followed the rules of Figure 8.1 and for no atomic $L \cup C$-sentence ψ both ψ and $\neg\psi$ are in $\{y_0, y_1, \ldots\}$.

We now extend the first leg of the Strategic Balance of Logic, the equivalence between the Semantic Game and the Model Existence Game, from first order logic to infinitary logic:

Theorem 8.12 (Model Existence Theorem for $L_{\omega_1\omega}$) *Suppose L is a countable vocabulary and φ is an L-sentence of $L_{\omega_1\omega}$. the following are equivalent:*

(1) There is an L-structure $\mathcal{M}$ such that $\mathcal{M} \models \varphi$.
(2) Player **II** *has a winning strategy in* $\mathrm{MEG}(\varphi, L)$.

Proof The implication $(1) \rightarrow (2)$ is clear as **II** can keep playing sentences that are true in $\mathcal{M}$. For the other implication we proceed as in the proof of Theorem 6.35. Let $C = \{c_n : n \in \mathbb{N}\}$ and $Trm = \{t_n : n \in \mathbb{N}\}$. Let $(x_0, y_0, x_1, y_1, \ldots)$ be a play in which player **II** has used her winning strategy and player **I** has maintained the following conditions:

1. If $n = 0$, then $x_n = \varphi$.
2. If $n = 2 \cdot 3^i$, then x_n is $\approx c_i c_i$.
3. If $n = 4 \cdot 3^i \cdot 5^j \cdot 7^k \cdot 11^l$, y_i is $\approx c_j t_k$, and y_l is $\varphi(c_j)$, then x_n is $\varphi(c_i)$.
4. If $n = 8 \cdot 3^i \cdot 5^j$ and y_i is $\bigwedge_{m \in \mathbb{N}} \varphi_m$, then x_n is φ_j.
5. If $n = 16 \cdot 3^i$ and y_i is $\bigvee_{m \in \mathbb{N}} \varphi_m$, then x_n is $\bigvee_{m \in \mathbb{N}} \varphi_m$.
6. If $n = 32 \cdot 3^i \cdot 5^j$, y_i is $\forall x \varphi(x)$, then x_n is $\varphi(c_j)$.
7. etc.

The rest of the proof is exactly as in the proof of 6.35. □

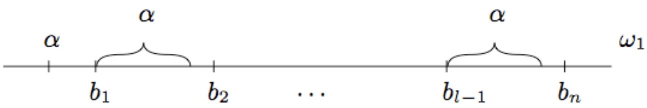
Our success in the above proof is based on the fact that even if we deal with infinitary formulas we can still manage to let player **I** list all possible formulas that are relevant for the consistency of the starting formula. If even one uncountable conjunction popped up, we would be in trouble.

It suffices to consider in $\mathrm{MEG}(\varphi, L)$ such constant terms t that are either constants or contain no other constants than those of C. Moreover, we may assume that if player **I** enquires about $\approx tt$, then $t = c_n$ for some $n \in \mathbb{N}$.

Corollary *Let L be a countable vocabulary. Suppose φ and ψ are sentences of $L_{\omega_1\omega}$. the following are equivalent:*

(1) $\varphi \models \psi$
(2) Player **I** *has a winning strategy in* $\mathrm{MEG}(\varphi \wedge \neg\psi, L)$.

The proof of Compactness Theorem does not go through, and should not, because there are obvious counter-examples to compactness in $\mathrm{L}_{\omega_1\omega}$. In many proofs where one would like to use the Compactness Theorem one can instead use the Model Existence Theorem. The non-definability of well-order in $L_{\infty\omega}$ was proved already in Theorem 7.26 but we will now prove a stronger version for $\mathrm{L}_{\omega_1\omega}$:



Theorem 8.13 (Undefinability of well-order) *Suppose L is a countable vocabulary containing a unary predicate symbol U and a binary predicate symbol $<$, and $\varphi \in L_{\omega_1\omega}$. Suppose that for all $\alpha < \omega_1$ there is a model $\mathcal{M}$ of φ such that $(\alpha, <) \subseteq (U^{\mathcal{M}}, <^{\mathcal{M}})$. Then φ has a model $\mathcal{N}$ such that $(\mathbb{Q}, <) \subseteq (U^{\mathcal{N}}, <^{\mathcal{N}})$.*

Proof Let $D = \{d_r : r \in \mathbb{Q}\}$ be a set of new constant symbols. Let us call them d-constants. Let $\theta = \bigwedge_{r<s}(d_r < d_s)$. We show that player **II** has a winning strategy in

$$\mathrm{MEG}(\varphi \wedge \theta, L \cup D).$$

This clearly suffices. The strategy of **II** is the following: Suppose she has played $\{y_0, \dots, y_{n-1}\}$ so far and $y_i = \theta$ or

$$y_i = \varphi_i(c_0, \dots, c_m, d_{r_1}, \dots, d_{r_l}),$$

where $d_{r_1}, \dots, d_{r_l}$ are the d-constants appearing in $\{y_0, \dots, y_{n-1}\}$ except in θ. She maintains the following condition:

($\star$) For all $\alpha < \omega_1$ there is a model $\mathcal{M}$ of φ and $b_1, \dots, b_l \in U^{\mathcal{M}} \subseteq \omega_1$ such that

$$\mathcal{M} \models \exists x_0 \dots \exists x_m \bigwedge_{i<n} \varphi_i(x_0, \dots, x_m, b_1, \dots, b_l)$$

and

$$\alpha \leq b_1, b_1 + \alpha \leq b_2, \dots, b_{l-1} + \alpha \leq b_l.$$

We show that player **II** can indeed maintain this condition.

For most moves of player **I** the move of **II** is predetermined and we just have to check that ($\star$) remains valid. For a start, if **I** plays φ, condition ($\star$) holds by assumption. If **I** enquires about substitution or plays a conjunct of a played conjunction, no new constants are introduced, so ($\star$) remains true. Also, if **I** tests a played $\forall x\varphi(x)$ or enquires about a played $\exists x\varphi(x)$, no new constants of D are introduced, so ($\star$) remains true. We may assume that **I** enquires about $\approx tt$ only if $t = c_n$ and so ($\star$) holds by the induction hypothesis. Let us then

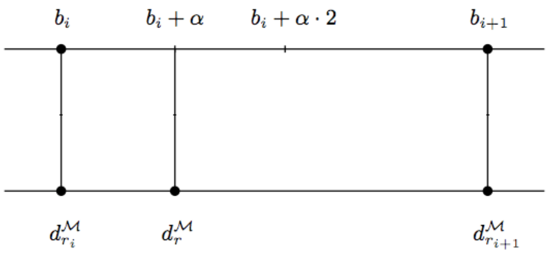


Figure 8.2

assume ($\star$) holds and **I** enquires about a played disjunction $\bigvee_{i \in I} \psi_i$. For each $\alpha < \omega_1$ we have a model $\mathcal{M}_\alpha$ as in ($\star$) and some $i_\alpha \in I$ such that $\mathcal{M}_\alpha \models \psi_{i_\alpha}$. Since **I** is countable, there is a fixed $i \in I$ such that for uncountably many $\alpha < \omega_1$: $\mathcal{M}_\alpha \models \psi_i$. If **II** plays this ψ_i, condition ($\star$) is still true.

The remaining case is that **I** enquires about a constant term t. We may assume $t = d_r$ as otherwise there is nothing to prove. The constants of D occurring so far in the game are $d_{r_1}, \ldots, d_{r_l}$. Let us assume $r_i < r < r_{i+1}$. To prove ($\star$), assume $\alpha < \omega_1$ and let $\beta = \alpha \cdot 2$. By the induction hypothesis there is $\mathcal{M}$ as in ($\star$) such that $b_i + \beta \leq b_{i+1}$. Let d_r be interpreted in $\mathcal{M}$ as $b_i + \alpha$. Now $\mathcal{M}$ satisfies the condition ($\star$) (see Figure 8.3).

□

The following Corollary is due to Lopez-Escobar [LE66b].

Corollary *If φ is a sentence of $L_{\omega_1\omega}$ in a vocabulary which contains the unary predicate U and the binary predicate $<$, and $(U^{\mathcal{M}}, <^{\mathcal{M}})$ is well-ordered in every model of φ, then there is $\alpha < \omega_1$ such that the order type of $(U^{\mathcal{M}}, <^{\mathcal{M}})$ is $< \alpha$ for every model $\mathcal{M}$ of φ.*

Corollary *The class of well-orderings is not a PC-class of $L_{\omega_1\omega}$.*

The undefinability of well-ordering as a PC-class of $L_{\infty\omega}$ will be established later. We now prove the Craig Interpolation Theorem for $L_{\omega_1\omega}$. There are several different proofs of this theorem, some of which employ the above Corollary directly. Our proof is like the original proof by Lopez-Escobar, except that we operate with model existence games instead of Gentzen systems.

Theorem 8.14 (Separation Theorem) *Suppose L_1 and L_2 are vocabularies. Suppose φ is an L_1-sentence of $L_{\omega_1\omega}$ and ψ is an L_2-sentence of $L_{\omega_1\omega}$ such*

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I	**II**	Winning condition
$a_0 \in A$		
	$b_0 \in A$	$\varphi_0(a_0, b_0)$
$a_1 \in A$		
	$b_1 \in A$	$\varphi_1(a_0, b_0, a_1, b_1)$
$\vdots$	$\vdots$	

Figure 8.7 The game quantifier.

8.6 Game Logic

In this section we sketch the basic properties and applications of the so-called closed game quantifier of length ω

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots \bigwedge_{n<\omega} \varphi_n(x_0, y_0 \ldots, x_n, y_n) \tag{8.18}$$

and its generalization, the so-called closed Vaught-formula of length ω

$$\forall x_0 \bigvee_{i_0 \in I_0} \bigwedge_{j_0 \in J_0} \exists y_0 \forall x_1 \bigvee_{i_1 \in I_1} \bigwedge_{j_1 \in J_1} \exists y_1 \ldots \bigwedge_{n<\omega} \varphi^{i_0 j_0 \ldots i_n j_n}(x_0, y_0 \ldots, x_n, y_n) \tag{8.19}$$

as well as their open counterparts. We use the general term *game quantification* to cover expressions of the above type. The semantics of these expressions is defined below by reference to a proper version of a *Semantic Game*.

The first application of game quantifiers in model theory was Svenonius's Theorem [Sve65] to the effect that every PC-definable class of models is recursively axiomatizable in countable models. Moschovakis [Mos72] introduced the game quantifier to descriptive set theory showing that inductive relations on countable acceptable structures are definable by the game quantifier. Vaught [Vau73] applied game expressions to develop a general definability theory for $L_{\omega_1\omega}$ including a Covering Theorem. Subsequently game quantifiers have become a standard tool in model theory.

Closed game formulas

We shall first discuss the simpler case (8.18) and show how it can be used as technical tool for an analysis of PC-definability in first order logic.

Definition 8.36 (Game quantifier) The truth of a game expression (8.18) in a model $\mathcal{A}$ means the existence of a winning strategy of player **II** in the game of length ω of Figure 8.7. Player **II** wins this game if

$$\mathcal{A} \models \varphi_n(a_0, b_0, \ldots, a_n, b_n)$$

for all $n \in \mathbb{N}$. A winning strategy of **II** is a sequence $\{\tau_n : n < \omega\}$ of functions on A such that

$$\mathcal{A} \models \varphi_n(a_0, \tau_0(a_0), a_1, \tau_1(a_0, a_1), \ldots, a_n, \tau_n(a_0, \ldots, a_n))$$

for all $a_0, a_1, \ldots$ in A.

Example 8.37 Examples of formulas of the form (8.18) are

1. $\exists x_0 \exists x_1 \exists x_2 \ldots \bigwedge_n x_{n+1} < x_n$, which in a linearly ordered model $(A, <)$ says that the linear order is not a well-order.
2. $\exists y \exists z \forall x_0 \forall x_1 \forall x_2 \ldots \bigwedge_n ((yEx_0 \wedge x_0Ex_1 \wedge \ldots x_{n-1}Ex_n) \rightarrow \neg{\approx} x_n z)$, which in a graph says that the graph is not connected.
3. $\forall x_0 \forall x_1 \forall x_2 \ldots \bigwedge_{n>2} ((x_0Ex_1 \wedge \ldots \wedge x_{n-1}Ex_n \wedge \bigwedge_{0 \leq i < j < n} \neg{\approx} x_i x_j) \rightarrow \neg{\approx} x_0 x_n)$, which in a graph says that the graph is cycle-free.

As the above examples show, game expressions are more powerful than $L_{\infty\omega}$. In fact, we shall see below that they can express even things that go beyond $L_{\infty\infty}$. Therefore we cannot expect the model theory of game expressions to be as nice as that of $L_{\omega_1\omega}$. However, the game expressions permit one very useful technique. This is the method of approximations, originally due in model theory to Keisler and then extensively used by Makkai, Vaught and others.

We use $\bar{x}_i, \bar{y}_i$ or just $\bar{x}, \bar{y}$, when the length of the sequences is clear from the context, to denote $x_0, y_0, \ldots, x_{i-1}, y_{i-1}$.

Definition 8.38 Suppose Φ is the closed game formula (8.18). We shall associate Φ with a sequence Φ^n_γ,$\gamma \in On$, of $L_{\infty\omega}$-formulas, called *approximations*, as follows:

$$\begin{array}{lcl}
\Phi^n_0(\bar{x}_n, \bar{y}_n) & = & \bigwedge_{j<n} \varphi_{j-1}(\bar{x}_j, \bar{y}_j) \\
\Phi^n_{\gamma+1}(\bar{x}_n, \bar{y}_n) & = & \forall x_n \exists y_n \Phi^{n+1}_\gamma(\bar{x}_{n+1}, \bar{y}_{n+1}) \\
\Phi^n_\nu(\bar{x}, \bar{y}) & = & \bigwedge_{\gamma<\nu} \Phi^n_\gamma(\bar{x}, \bar{y}) \text{ for limit } \nu.
\end{array}$$

The trivial properties of these approximations are proved easily by transfinite induction:

- $\Phi^n_\gamma(\bar{x}, \bar{y}) \in L_{\kappa\omega}$ for $\gamma < \kappa$
- $\mathrm{qr}(\Phi^n_\nu(\bar{x}, \bar{y})) = \nu$ for limit $\nu > 0$
- $\models \Phi \rightarrow \Phi^0_\gamma$ for all γ
- $\models \Phi^n_\gamma(\bar{x}, \bar{y}) \rightarrow \Phi^{\bar{n}}_\beta(\bar{x}, \bar{y})$ for $\beta \leq \gamma$.

Less trivial is the following important and characteristic property of the approximations:

Proposition 8.39 *If* $|A| = \kappa$ *and* $\mathcal{A} \models \Phi^0_\alpha$ *for all* $\alpha < \kappa^+$*, then* $\mathcal{A} \models \Phi$.

Proof We define a winning strategy $\{\tau_n : n \in \mathbb{N}\}$ of **II** in the game (8.7) as follows: Suppose $a_0, b_0, ..., a_{n-1}, b_{n-1}$ have been played. The strategy of **II** is to maintain the property

$$\text{For all } \alpha < \kappa^+\text{: } \mathcal{A} \models \Phi^n_\alpha(a_0, b_0, ..., a_{n-1}, b_{n-1}). \tag{8.20}$$

In the beginning this condition holds by assumption. Suppose the condition holds after $a_0, b_0, ..., a_{n-1}, b_{n-1}$ have been played. Now **I** plays a_n. If there is no b_n such that

$$\text{for all } \alpha < \kappa^+ : \mathcal{A} \models \Phi^{n+1}_\alpha(a_0, b_0, ..., a_n, b_n), \tag{8.21}$$

then for every $b_n \in A$ there is $\alpha(b_n) < \kappa^+$ such that

$$\mathcal{A} \not\models \Phi^{n+1}_{\alpha(b_n)}(a_0, b_0, ..., a_n, b_n).$$

Let $\delta = \sup_{b_n \in A} \alpha(b_n)$. Note that $\delta < \kappa^+$. Hence by assumption $\mathcal{A} \models \Phi^n_{\delta+1}(a_0, b_0, ..., a_{n-1}, b_{n-1})$. We obtain immediately a contradiction. Thus there must be a b_n such that (8.21). □

Corollary *In countable models the game formula* Φ *and the* $L_{\omega_2\omega}$*-sentence* $\bigwedge_{\alpha<\omega_1} \Phi^0_\alpha$ *are logically equivalent.*

Thus as far as countable models are concerned, the only thing that the closed game formulas (8.18) add to $L_{\omega_1\omega}$ is an uncountable conjunction. When we move to bigger models, longer and longer conjunctions are needed, but that is all.

Definition 8.40 A structure in a countable recursive vocabulary is *recursively saturated* if it satisfies

$$\forall x_1...x_n((\bigwedge_{n<\omega} \exists y \bigwedge_{m<n} \varphi_m(x_1, ..., x_n, y)) \to \exists y \bigwedge_{n<\omega} \varphi_n(x_1, ..., x_n, y))$$

for all recursive sequences $\{\varphi_m(x_1, ..., x_n, y) : m < \omega\}$ of first order formulas.

Examples of recursively saturated structures are the dense linear order $(\mathbb{Q}, <)$, and the field $(\mathbb{C}, +, \cdot)$ of complex numbers. Every infinite model of a recursive vocabulary has a countable recursively saturated elementary extension (see [CK90, Section 2.4]).

Proposition 8.41 *Suppose* $\mathcal{A}$ *is recursively saturated. Then* $\mathcal{A} \models \Phi \leftrightarrow \bigwedge_{n<\omega} \Phi^0_n$.

Proof We proceed as in the proof of Proposition 8.39. Suppose $\mathcal{A} \models \bigwedge_{n<\omega} \Phi_n^0$. We define a winning strategy $\{\tau_n : n \in \mathbb{N}\}$ of **II** in the game (8.7) as follows. Suppose $a_0, b_0, ..., a_{n-1}, b_{n-1}$ have been played. The strategy of **II** is to maintain the property

$$\text{For all } m < \omega\text{: } \mathcal{A} \models \Phi_m^n(a_0, b_0, ..., a_{n-1}, b_{n-1}). \tag{8.22}$$

In the beginning this condition holds by assumption. Suppose the condition holds after $a_0, b_0, ..., a_{n-1}, b_{n-1}$ have been played. Now **I** plays a_n. We look for b_n such that

$$\mathcal{A} \models \bigwedge_{m<\omega} \Phi_m^{n+1}(a_0, b_0, ..., a_n, b_n), \tag{8.23}$$

i.e. we want to show

$$\mathcal{A} \models \exists y_n \bigwedge_{m<\omega} \Phi_m^{n+1}(a_0, b_0, ..., a_n, y_n). \tag{8.24}$$

Since $\mathcal{A}$ is ω-saturated, it suffices to prove

$$\mathcal{A} \models \bigwedge_{m<\omega} \exists y_n \bigwedge_{k<m} \Phi_k^{n+1}(a_0, b_0, ..., a_n, y_n). \tag{8.25}$$

Suppose $m < \omega$ is given. By (8.22) we have

$$\mathcal{A} \models \forall x_n \exists y_n \Phi_k^{n+1}(a_0, b_0, ..., x_n, y_n).$$

By choosing the value of x_n to be a_n we get b such that

$$\mathcal{A} \models \Phi_k^{n+1}(a_0, b_0, ..., a_n, b).$$

We have proved (8.25). □

What about structures that are not recursively saturated? To conclude $\mathcal{A} \models \Phi$ we have to assume $\mathcal{A} \models \Phi_\alpha^0$ for some infinite ordinals α, too. We refer to Barwise [Bar75] for details.

Barwise [Bar76] observed that game formulas can be used to "straighten" partially ordered quantifiers. Consider the so called *Henkin quantifier*

$$\begin{pmatrix} \forall x & \exists y \\ \forall u & \exists v \end{pmatrix} \varphi(x, y, u, v, \bar{z}) \tag{8.26}$$

the meaning of which is

There are f and g such that for all a and b $\varphi(a, f(a), b, g(b), \bar{z})$. (8.27)

We call formulas of the form (8.26), with $\varphi(x, y, u, v)$ first order, *Henkin-formulas*. Indeed, they were introduced by Henkin [Hen61]. An alternative notation for Henkin-formulas is offered by *dependence logic* [Vää07]:

$$\forall x \exists y \forall u \exists v (=(u, \bar{z}, v) \wedge \varphi(x, y, u, v, \bar{z})),$$

where the intuitive interpretation of $=(u, \bar{z}, v)$ is "v depends only on u and $\bar{z}$". Let us compare (8.26) with the game formula

$$\begin{array}{l} \forall x_0 \exists y_0 \forall u_0 \exists v_0 \forall x_1 \exists y_1 \forall u_1 \exists v_1 \ldots \\ \bigwedge_{i,j,k,l}((\approx x_i x_j \wedge \approx u_k u_l) \to (\approx y_i y_j \wedge \approx v_k v_l \wedge \varphi(x_i, y_i, u_i, v_i, \bar{z}))). \end{array} \tag{8.28}$$

Clearly, (8.26), or rather (8.27), implies (8.28) as **II** can let $y_n = f(x_n)$ and $v_n = g(u_n)$. In a countable model the converse is true: Suppose $\mathcal{A}$ is a countable model and s is an assignment. Let (a_n, b_n), $n \in \mathbb{N}$, list all pairs of elements of A. Let us play the game associated with the formula (8.28) in $\mathcal{A}$ so that **I** plays $x_n = a_n$ and $u_n = b_n$. Let the responses of **II** be $y_n = a_n^*$ and $v_n = b_n^*$. Let $f(a_n) = a_n^*$ and $g(b_n) = b_n^*$. It is easy to see that $\mathcal{A} \models_s \varphi(a_n, f(a_n), b_m, g(b_m), \bar{z})$ for all n and m. Thus (8.26) holds in $\mathcal{A}$ under the assignment s. We have proved:

Proposition 8.42 *The formulas (8.26) and (8.28) are equivalent in all countable models.*

In consequence, in a countable recursively saturated model (8.26) is, by Proposition 8.41, equivalent to $\bigwedge_{n<\omega} \Phi_n^0$.

Let us say that $\{\varphi_n : n < \omega\}$ is a *first order axiomatization* of a class K of models, if the following are equivalent for all first order ψ:

1. ψ is true in all models in K.
2. $\{\varphi_n : n < \omega\} \models \psi$.

Proposition 8.43 $\{\Phi_n^0 : n < \omega\}$ *is a first order axiomatization of the class of models of (8.26).*

Proof Suppose a first order sentence ψ follows from (8.26). We show that it follows from $\{\Phi_n^0 : n < \omega\}$. If not, then there is a model $\mathcal{A}$ of $\{\Phi_n^0 : n < \omega\}$ which satisfies $\neg\psi$. Take a countable recursively saturated model of the first order theory $\{\Phi_n^0 : n < \omega\} \cup \{\neg\psi\}$. We get a contradiction. □

Example 8.44 (Models with an involution) Suppose L is the vocabulary $\{R\}$, where R is (for simplicity) binary. The class of L-models with an involution (non-trivial automorphism of order two) can be axiomatized by the Henkin-sentence

$$\Phi = \exists z \begin{pmatrix} \forall x & \exists y \\ \forall u & \exists v \end{pmatrix} \varphi(x, y, u, v, z),$$

where $\varphi(x, y, u, v, z)$ is the conjunction of $(\approx xu \to \approx yv)$, $(\approx xv \to \approx yu)$, $(Rxu \leftrightarrow Ryv)$, and $\approx xz \to \neg\approx xy$. In countable models this Henkin-sentence is equivalent to

$$\exists z \forall x_0 \exists y_0 \forall u_0 \exists v_0 \forall x_1 \exists y_1 \forall u_1 \exists v_1 \ldots \\ \bigwedge_{i,j,k,l}((\approx x_i x_j \wedge \approx u_k u_l) \to (\approx y_i y_j \wedge \approx v_l v_k \wedge \varphi(x_i, y_i, u_i, v_i, z))).$$

By inspecting the approximations Φ_n^0 of Φ we see that a first order sentence has a model with an involution if and only if it is consistent with the set of the first order sentences

$$\exists z \forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots \forall x_m \exists y_m \\ \bigwedge_{i,j,k,l \leq m}((\approx x_i x_j \to \approx y_i y_j) \wedge (\approx x_i y_j \to \approx x_j y_i) \wedge \\ (\approx x_i z \to \neg\approx x_i y_i) \wedge (R x_i x_j \leftrightarrow R y_i y_j)),$$

where $m \in \mathbb{N}$.

The above result about Henkin-formulas are not limited to the particular form of (8.26). The meaning of the more general formula

$$\begin{pmatrix} \forall x_{i_1^1} \ldots \forall x_{i_{m_1}^1} \exists y_1 \\ \vdots \quad \vdots \quad \vdots \\ \forall x_{i_1^n} \ldots \forall x_{i_{m_n}^n} \exists y_n \end{pmatrix} \varphi(x_{i_1^1}, \ldots, x_{i_{m_1}^1}, y_1, \ldots, x_{i_1^n}, \ldots, x_{i_{m_n}^n}, y_n, \bar{z}) \quad (8.29)$$

is simply: There are $f_1, ..., f_n$ such that for all $a_{i_1^1}, \ldots, a_{i_{m_1}^1}$ $(i = 1, ..., n)$,

$$\varphi(a_{i_1^1}, \ldots, a_{i_{m_1}^1}, b_1, \ldots, a_{i_1^n}, \ldots, a_{i_{m_n}^n}, b_n, \bar{z}),$$

where

$$b_j = f_j(a_{i_1^j}, \ldots, a_{i_{m_j}^j}), \text{ for } j = 1, ..., n.$$

Note that (8.29) makes perfect sense even if the rows of the quantifier prefix are of different lengths, as in

$$\begin{pmatrix} \forall x_1 \forall x_2 & \exists y \\ \forall u & \exists v \end{pmatrix} \varphi(x_1, x_2, y, u, v, \bar{z}). \quad (8.30)$$

We call all formulas of the form (8.29) *Henkin-formulas*. Let $\bar{\Phi}$ be obtained from (8.29) as (8.28) was obtained from (8.26). The following proposition is proved mutatis mutandis as in Proposition 8.42:

Proposition 8.45 *The formulas (8.29) and $\bar{\Phi}$ are equivalent in all countable models.*

In consequence, in a countable recursively saturated model (8.29) is, by Proposition 8.41, equivalent to $\bigwedge_{n<\omega} \bar{\Phi}^0_n$.

Enderton [End70] and Walkoe [Wal70] observed that any PC-class can be defined by a Henkin-formula:

Theorem 8.46 *For every PC-class K there is a Henkin-sentence $\bar{\Phi}$ such that for all $\mathcal{M}$:*

$$\mathcal{M} \in K \iff \mathcal{M} \models \bar{\Phi}.$$

Proof Suppose K is the class of reducts of a first order sentence φ. We may assume that φ is of the form

$$\forall x_1 \ldots \forall x_m \psi, \tag{8.31}$$

where ψ is quantifier free but contains new function symbols $f_1, ..., f_n$. (This is the so called *Skolem Normal Form* of φ). We will perform some reductions on (8.31) in order to make it more suitable for the construction of φ.

Step 1: If ψ contains nesting of the function symbols $f_1, \ldots, f_n$ or of the function symbols of the vocabulary, we can remove them one by one by using the equivalence of

$$\models \theta(f_i(t_1, \ldots, t_m))$$

and

$$\forall x_1 \ldots \forall x_m((t_1 = x_1 \ \wedge \ldots \wedge \ t_m = x_m) \to \theta(f_i(x_1, \ldots, x_m)))$$

for any first order θ. Thus we may assume that all terms occurring in ψ are of the form x_i or $f_i(x_{i_1}, \ldots, x_{i_k})$.

Step 2: If ψ contains an occurrence of a function symbol $f_i(x_{i_1}, \ldots, x_{i_k})$ with the same variable occurring twice, e.g. $i_s = i_r$, $1 < r < k$, we can remove such by means of a new variable x_l and the equivalence

$$\begin{aligned}&\models \forall x_1 \ldots \forall x_m \theta(f_i(x_{i_1}, \ldots, x_{i_k})) \leftrightarrow \\ &\quad \forall x_1 \ldots \forall x_m \forall x_l(x_l = x_r \to \theta(f_i(x_{i_1}, \ldots, x_{i_{r-1}}, x_l, x_{i_{r+1}}, \ldots, x_{i_k})))\end{aligned}$$

for any first order θ. Thus we may assume that if a term such as $f_i(x_{i_1}, \ldots, x_{i_k})$ occurs in ψ, its variables are all distinct.

Step 3: If ψ contains two occurrences of the same function symbol but with different variables or with the same variables in different order, we can remove such by using appropriate equivalences. If $\{i_1, ..., i_k\} \cap \{j_1, ..., j_k\} = \emptyset$, we have the equivalence

$$\models \forall x_1 \ldots \forall x_m \theta(f_i(x_{i_1}, \ldots, x_{i_k}), f_i(x_{j_1}, \ldots, x_{j_k})) \leftrightarrow$$

$$\exists f'_i \forall x_1 \dots \forall x_m (\theta(f_i(x_{i_1}, \dots, x_{i_k}), f'_i(x_{j_1}, \dots, x_{j_k})) \wedge$$
$$((x_{i_1} = x_{j_1} \wedge \dots \wedge x_{i_k} = x_{j_k}) \to$$
$$f_i(x_{i_1}, \dots, x_{i_k}) = f'_i(x_{j_1}, \dots, x_{j_k})))$$

for any first order θ. We can reduce the more general case, where $\{i_1, ..., i_k\} \cap \{j_1, ..., j_k\} \neq \emptyset$, to this case by introducing new variables, as in Step 2. Thus we may assume that for each function symbol f_i occurring in ψ there are $j^i_1, \dots, j^i_{n_i}$ such that *all* occurrences of f_i are of the form $f_i(x_{j^i_1}, \dots, x_{j^i_{m_i}})$ and $j^i_1, \dots, j^i_{m_i}$ are all different from each other.

In sum we may assume the function terms that occur in ψ are of the form $f_i(x_{j^i_1}, \dots, x_{j^i_{m_i}})$ and for each i the variables $x_{j^i_1}, \dots, x_{j^i_{m_i}}$ and their order is the same. Let N be greater than all the $x_{j^i_k}$. Let $\bar{\Phi}$ be the Henkin-sentence

$$\begin{pmatrix} \forall x_{j^1_1} & \dots & \forall x_{j^1_{m_1}} & \exists x_{N+1} \\ \vdots & & \vdots & \vdots \\ \forall x_{j^n_1} & \dots & \forall x_{j^n_{m_n}} & \exists x_{N+n} \end{pmatrix} \psi'$$

where ψ' is obtained from ψ by replacing $f_i(x_{j^i_1}, \dots, x_{j^i_{m_i}})$ everywhere by x_{N+i}. This is clearly the desired Henkin-sentence. In the notation of dependence logic ([Vää07]) this would look like:

$$\begin{aligned} \forall x_1 \dots \forall x_m \exists x_{N+1} \dots \exists x_{N+n} \ \ &(=(x_{j^1_1}, \dots, x_{j^1_{m_1}}, x_{N+1}) \wedge \\ &\dots \\ &=(x_{j^n_1}, \dots, x_{j^n_{m_n}}, x_{N+n}) \wedge \psi'). \end{aligned}$$

□

By combining the above observations we get the following result of Svenonius [Sve65]:

Theorem 8.47 *For every PC-class K there is a closed game sentence Φ and a sequence φ_n of first order sentences such that for all structures $\mathcal{M}$:*

1. *If $\mathcal{M} \in K$, then $\mathcal{M} \models \Phi$ and $\mathcal{M} \models \varphi_n$ for all $n \in \mathbb{N}$.*
2. *If $\mathcal{M} \models \Phi$ and M is countable, then $\mathcal{M} \in K$.*
3. *If $\mathcal{M} \models \bigwedge_n \varphi_n$ and $\mathcal{M}$ is countable recursively saturated, then $\mathcal{M} \in K$.*
4. *If ψ is any first order sentence, then ψ has a model in K if and only if ψ is consistent with $\{\varphi_n : n < \omega\}$.*

Moreover, the sequence $\{\varphi_n : n \in \mathbb{N}\}$ (or rather the set of Gödel-numbers of the φ_n) is recursive.

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9
Stronger Infinitary Logics

9.1 Introduction

The infinitary logics $L_{\kappa\omega}$, $L_{\infty\omega}$ and $L_{\infty G}$ of the previous chapter had one important feature in common with first order logic: the truth predicate of these logics is absolute[1] in set theory. We now move on to logics which do not have this property. We lose something but we also gain something else. For example, we lose the last remnants of the Completeness Theorem of first order logic. On the other hand, we can express deeper properties of models, such as uncountability, completeness of a separable order, and other properties, too. Perhaps surprisingly, some methods, such as the method of Ehrenfeucht-Fraïssé games, still work perfectly even with these strong logics.

9.2 Infinite Quantifier Logic

First order logic and the infinitary logic $L_{\infty\omega}$ are able to express

$$\exists V\varphi \text{ and } \forall V\varphi$$

when V is any finite set of variables. In the infinite quantifier logics of this section we can express this even when V is an infinite set of variables.

Before actually defining the infinite quantifier logics, we first define the appropriate version of the Ehrenfeucht-Fraïssé game. In this game the players play sequences of a given length. Each round consists of a choice of a sequence by **I** followed by a choice of a sequence by **II**. The goal of **II** is to make sure the played sequences form, element by element, a partial isomorphism. Thus

[1] More exactly, if M is a transitive model of ZFC containing $\mathcal{A}$ and φ as elements, then $\mathcal{A}$ is a model of φ if and only if the set-theoretical statement "$\mathcal{A} \models \varphi$" holds in the model M.

if **I** plays a sequence

$$x_0 = (x_0(0), \ldots, x_0(n), \ldots)$$

which is a descending sequence relative to a linear order $<$ in one of the models, player **II** tries to play likewise a sequence

$$y_0 = (y_0(0), \ldots, y_0(n), \ldots)$$

which constitutes a descending sequence relative to $<$ in the other model. If that other model is well-ordered by $<$, she loses right away. Note that players have made so far just one move each, albeit a move with infinitely many components.

For another example, suppose one of the models is countable while the other is uncountable. If player **I** is allowed to play countable sequences he can immediately let x_0 enumerate the countable model. Whichever countable sequence **II** plays, **I** wins during the next round by playing an element from the uncountable model which is different from all the elements played by **II**.

To define the new game more exactly, we fix some notation. A function $s : \alpha \to M$ is called a *sequence* of *length* $\mathrm{len}(s) = \alpha$. The set of all sequences of length α of elements of M is denoted by M^α. We define

$$M^{<\alpha} = \bigcup_{\beta<\alpha} M^\beta$$

and

$$\mathrm{Part}_\kappa(\mathcal{A}, \mathcal{B}) = \{p \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) : |p| < \kappa\}.$$

Now we can define the new Ehrenfeucht-Fraïssé Game:

Definition 9.1 Suppose κ is a cardinal. The *Ehrenfeucht-Fraïssé game with moves of size* $< \kappa$ on $\mathcal{M}$ and $\mathcal{M}'$, denoted $\mathrm{EF}^\kappa_\omega(\mathcal{M}, \mathcal{M}')$, is the game in which player **I** plays

$$x_n \in M^{<\kappa} \cup (M')^{<\kappa}$$

and **II** responds with

$$y_n \in M^{<\kappa} \cup (M')^{<\kappa}$$

for all $n \in \mathbb{N}$. Player **II** wins if for all n,

(1) $\mathrm{len}(x_n) = \mathrm{len}(y_n)$
(2) $x_n \in M^{<\kappa} \leftrightarrow y_n \in (M')^{<\kappa}$

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9.3 The Transfinite Ehrenfeucht-Fraïssé Game

All our games up to now have had at most ω rounds. There is no difficulty in imagining what a game of, say, length $\omega + \omega$ would look like: it would be like playing two games of length ω one after the other. For example, it is by now well known to the reader that the second player has a winning strategy in the Ehrenfeucht-Fraïssé game of length ω on $(\mathbb{R}, <)$ and $(\mathbb{R} \setminus \{0\}, <)$. But if player **I** is allowed one more move after the ω moves, he wins.

For a more enlightening example, suppose $\mathcal{M}$ and $\mathcal{N}$ are equivalence relations such that $\mathcal{M}$ has $\aleph_1$ countable classes and $\aleph_0$ uncountable classes while $\mathcal{N}$ has $\aleph_1$ countable classes and $\aleph_1$ uncountable classes. Does **II** have a winning strategy in EF_ω. Yes! She just keeps matching different equivalence classes with different equivalence classes. But she can actually win the game of length $\omega + \omega$, too! During the first ω moves she matches countable equivalence classes with countable ones and uncountable equivalence classes with uncountable ones. After the first ω moves she may have to match a countable equivalence class with an uncountable class, but **I** will not be able to call **II**'s bluff. It is only when **I** has $\omega + \omega + 1$ moves that he has a winning strategy: During the first ω moves **I** plays one element from each uncountable class of $\mathcal{M}$. Then **I** plays one element b from an unused uncountable equivalence class of $\mathcal{N}$. Player **II** will match this element with an element c from a countable equivalence class of $\mathcal{M}$. During the next ω rounds player **I** enumerates the countable equivalence class of c. Finally he plays an unplayed element equivalence to b. Player **II** loses as all elements equivalent to c have been played already.

Let L be a vocabulary and $\mathcal{A}_0$ and $\mathcal{A}_1$ two L-structures. We give a rigorous definition of a transfinite version of the Ehrenfeucht-Fraïssé Game on the two models $\mathcal{A}_0$ and $\mathcal{A}_1$. We let the number of rounds of this game be an arbitrary ordinal δ.

In the sequel we allow the domains of $\mathcal{M}_0$ and $\mathcal{M}_1$ to intersect and incorporate a mechanism to account for this..

We shall all the time refer to sequences

$$\bar{z} = \langle z_\alpha : \alpha < \delta \rangle, \text{where } z_\alpha = (c_\alpha, x_\alpha),$$

of elements of $\{0, 1\} \times (A_0 \cup A_1)$. If $\bar{y} = \langle y_\alpha : \alpha < \delta \rangle$ is a sequence of elements of $A_0 \cup A_1$, the relation

$$p_{\bar{z},\bar{y}} \subseteq (A_0 \cup A_1)^2$$

is defined as follows:

$$p_{\bar{z},\bar{y}} = \{(a_\alpha, b_\alpha) : \alpha < \delta\}$$

I	z_0		z_1		$\dots$		z_α	$\dots$	$(\alpha < \delta)$	
II		y_0		y_1		$\dots$		y_α	$\dots$	$(\alpha < \delta)$

Figure 9.6 The Ehrenfeucht-Fraïssé Game

where

$$a_\alpha = \begin{cases} x_\alpha & \text{if } c_\alpha = 0 \\ y_\alpha & \text{if } c_\alpha = 1 \end{cases} \quad b_\alpha = \begin{cases} y_\alpha & \text{if } c_\alpha = 0 \\ x_\alpha & \text{if } c_\alpha = 1. \end{cases}$$

Remark We shall often use the fact that $p_{\bar{z},\bar{y}} = \cup_{\sigma<\delta} p_{\bar{z}\restriction_\sigma,\bar{y}\restriction_\sigma}$ if δ is a limit ordinal.

We are interested in the question whether

$$p_{\bar{z},\bar{y}} \in \mathrm{Part}(\mathcal{A}_0, \mathcal{A}_1) \tag{9.5}$$

or not. In the Ehrenfeucht-Fraïssé Game one player chooses $\bar{z}$ trying to make (9.5) false, and the other player chooses $\bar{y}$ trying to make (9.5) true. Let

$$\mathrm{Seq}_\delta(A_0, A_1)$$

be the set of all sequences $\langle (c_\alpha, x_\alpha) : \alpha < \delta \rangle$ where $c_\alpha \in \{0, 1\}$ and $x_\alpha \in A_{c_\alpha}$.

Definition 9.46 Let $\delta \in On$. The Ehrenfeucht-Fraïssé game of length δ on $\mathcal{A}_0$ and $\mathcal{A}_1$, in symbols

$$\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$$

is defined as follows. There are two players **I** and **II** . During one round of the game player **I** chooses an element c_α of $\{0, 1\}$ and an element x_α of A_{c_α}, and then player **II** chooses an element y_α of A_{1-c_α}. Let $z_\alpha = (c_\alpha, x_\alpha)$. There are δ rounds and in the end we have $\bar{z} = \langle z_\alpha : \alpha < \delta \rangle$ and $\bar{y} = \langle y_\alpha : \alpha < \delta \rangle$. We say that player **II** *wins* this sequence of rounds, if $p_{\bar{z},\bar{y}} \in \mathrm{Part}(\mathcal{A}_0, \mathcal{B}_1)$. Otherwise player **I** wins this sequence of rounds.

The above definition is useful as an intuitive model of the game. However, it is not mathematically precise because we have not defined what choosing an element means. The idea is that a player is free to choose any element. Also, we are really interested in the existence of a winning strategy for a player by means of which he can win every sequence of rounds. The following exact definition of a winning strategy is our mathematical model for the intuitive concept of a game.

Definition 9.47 A *strategy* of **II** in $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$ is a sequence

$$\tau = \langle \tau_\alpha : \alpha < \delta \rangle$$

of functions such that

$$\mathrm{dom}(\tau_\alpha) = \mathrm{Seq}_{\alpha+1}(A_0, A_1)$$

and

$$\mathrm{rng}(\tau_\alpha) \subseteq A_0 \cup A_1$$

for each $\alpha < \delta$. If $\bar{z} \in \mathrm{Seq}_\delta(A_0, A_1)$ and $\bar{y} = \langle y_\alpha : \alpha < \delta \rangle$, where

$$y_\alpha = \tau_\alpha(\bar{z}\restriction_{\alpha+1})$$

for all $\alpha < \delta$, then we denote $p_{\bar{z},\bar{y}}$ by $p_{\bar{z},\tau}$. The strategy τ of **II** is a *winning strategy* if $p_{\bar{z},\tau} \in \mathrm{Part}(\mathcal{A}_0, \mathcal{A}_1)$ for all $\bar{z} \in \mathrm{Seq}_\delta(A_0, A_1)$. A *strategy* of **I** in $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$ is a sequence

$$\rho = \langle \rho_\alpha : \alpha < \delta \rangle$$

of functions such that

$$\mathrm{rng}(\rho_\alpha) \subseteq \{0, 1\} \times (A_0 \cup A_1)$$

and $\mathrm{dom}(\rho_\alpha)$ is defined inductively as follows: $\mathrm{dom}(\rho_\alpha)$ is the set of sequences $\bar{y} = \langle y_\beta : \beta < \alpha \rangle$ such that for all $\beta < \alpha$,

$$y_\beta \in \begin{cases} A_1, & \text{if } \rho_\beta(\bar{y}\restriction_\beta) = (0, x) \text{ for some } x \\ A_0, & \text{if } \rho_\beta(\bar{y}\restriction_\beta) = (1, x) \text{ for some } x. \end{cases}$$

If $\bar{y} = \langle y_\alpha : \alpha < \delta \rangle \in \mathrm{dom}(\rho)$ (i.e $\langle y_\alpha : \alpha < \beta \rangle \in \mathrm{dom}(\rho_\beta)$ for all $\beta < \delta$) and $\bar{z} = \langle z_\alpha : \alpha < \delta \rangle$ satisfy

$$\rho_\alpha(\bar{y}\restriction_\alpha) = z_\alpha,$$

then $p_{\bar{z},\bar{y}}$ is denoted by $p_{\rho,\bar{y}}$. The strategy ρ is a *winning strategy* of **I** if there is no $\bar{y} \in \mathrm{dom}(\rho)$ such that $p_{\rho,\bar{y}} \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$. We say that a player *wins* $EF_\delta(\mathcal{A}_0, \mathcal{A}_1)$ if he has a winning strategy in it.

Remark If $\tau = \langle \tau_\alpha : \alpha < \delta \rangle$ is a strategy of **II** in $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$ and $\sigma < \delta$, then $\tau\restriction_\sigma = \langle \tau_\alpha : \alpha < \sigma \rangle$ is a strategy of **II** in $\mathrm{EF}_\sigma(\mathcal{A}_0, \mathcal{A}_1)$. If τ is winning, then so is $\tau\restriction_\sigma$. Moreover, if $\bar{z} \in \mathrm{Seq}_\delta(A_0 \cup A_1)$, then $p_{\bar{z}\restriction_\sigma, \tau\restriction_\sigma} \subseteq p_{\bar{z},\tau}$. If δ is a limit ordinal, we have $p_{\bar{z},\tau} = \cup_{\sigma<\delta} p_{\bar{z}\restriction_\sigma, \tau\restriction_\sigma}$ and τ is winning if and only if $\tau\restriction_\sigma$ is winning for all $\sigma < \delta$.

Example 9.48 Let $L = \emptyset$. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be two L-structures of cardinalities κ and λ respectively. Let us first assume $\delta \leq \kappa \leq \lambda$. Then **II** wins $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$. Her winning strategy $\tau = \langle \tau_\alpha : \alpha < \delta \rangle$ is defined as follows. Let $F(X) \in X$ for every non-empty $X \subseteq A_0 \cup A_1$. Suppose τ_α is defined for $\alpha < \sigma$, where $\sigma < \delta$. We define τ_σ. For any $\bar{z} = \langle (c_\alpha, x_\alpha) : \alpha < \sigma \rangle$, let $Y_{\bar{z}} = \{x_\zeta : \zeta < \sigma\} \cup \{\tau_\zeta(\bar{z}\restriction_{\zeta+1}) : \zeta + 1 < \sigma\}$. Let now

$$\tau_\sigma(\bar{z}\restriction_{\sigma+1}) = \begin{cases} \tau_\zeta(\bar{z}\restriction_{\zeta+1}) & \text{if } x_\sigma = x_\zeta, \zeta < \sigma \\ x_\zeta & \text{if } x_\sigma = \tau_\zeta(\bar{z}\restriction_{\zeta+1}), \zeta < \sigma \\ F(A_{1-c_i} \setminus Y_{\bar{z}\restriction_\sigma}) & \text{otherwise.} \end{cases}$$

It is clear that for all $\bar{z}$ the relation $p_{\bar{z},\tau}$ is a partial isomorphism. In fact it suffices that $p_{\bar{z},\tau}$ is a one-one function, since $L = \emptyset$.

Let us then assume $\kappa < \lambda$ and $\kappa < \delta$. Then **I** wins $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{B}_1)$. His winning strategy $\rho = \langle \rho_\alpha : \alpha < \delta \rangle$ is defined as follows. Let $A_0 = \{u_\eta : \eta < \kappa\}$.

$$\rho_\alpha(\langle y_\eta : \eta < \alpha \rangle) = \begin{cases} u_\alpha & \text{if } \alpha < \kappa \\ F(A_1 \setminus \{\rho_\zeta(\langle y_\eta : \eta < \zeta \rangle) : \zeta < \alpha\}) & \text{if } \kappa \leq \alpha < \delta. \end{cases}$$

The intuitive argument behind Example 9.48 based on Definition 9.46 can be described very succinctly: If $\delta \leq \kappa \leq \lambda$, the strategy of **II** is to copy the old moves if **I** plays an old element and choose some new element if **I** plays a new element. The assumption $\delta \leq \kappa < \lambda$ guarantees that there are enough elements to choose from. If $\kappa < \delta$, the strategy of **I** is to first enumerate A_0 during the first κ rounds of the game and then pick an element $x_\kappa \in A_1$, which has not been played yet by **II**. Then **II** has no elements in A_0 left to play and he loses the game.

Lemma 9.49 ***(i)*** *If* **II** *wins the game* $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ *and* $\beta < \alpha$*, then* **II** *wins the game* $\mathrm{EF}_\beta(\mathcal{A}_0, \mathcal{A}_1)$.

(ii) *If* **I** *wins the game* $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ *and* $\alpha < \beta$*, then* **I** *wins the game* $\mathrm{EF}_\beta(\mathcal{A}_0, \mathcal{A}_1)$.

(iii) *There is no* α *such that both* **II** *and* **I** *win* $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$.

Proof (i) If $\langle \tau_\xi : \xi < \alpha \rangle$ is a winning strategy of **II** in $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$, then $\langle \tau_\xi : \xi < \beta \rangle$ is a winning strategy of **II** in $\mathrm{EF}_\beta(\mathcal{A}_0, \mathcal{A}_1)$.

(ii) If $\langle \rho_\xi : \xi < \alpha \rangle$ is a winning strategy of **I** in $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$, then $\langle \rho_\xi : \xi < \beta \rangle$ is a winning strategy of **I** in $\mathrm{EF}_\beta(\mathcal{A}_0, \mathcal{A}_1)$, where

$$\rho_\xi(\langle y_\eta : \eta < \xi \rangle) = \rho_\alpha(\langle y_\eta : \eta < \alpha \rangle)$$

for $\alpha \leq \xi < \beta$.

(iii) Suppose $\langle \tau_\xi : \xi < \alpha \rangle$ is a winning strategy of **II** and $\langle \rho_\xi : \xi < \alpha \rangle$ a winning strategy of **I** in $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$. Define inductively

$$\begin{array}{rcl} x_\xi & = & \rho_\xi(\langle y_\eta : \eta < \xi \rangle) \\ y_\xi & = & \tau_\xi(\langle x_\eta : \eta \leq \xi \rangle). \end{array}$$

If $\bar{z} = \langle x_\xi : \xi < \alpha \rangle$ and $\bar{y} = \langle y_\xi : \xi < \alpha \rangle$, then $p_{\bar{z},\bar{y}}$ is a partial isomorphism because **II** wins, and not a partial isomorphism because **I** wins, a contradiction. □

Lemma 9.50 *(i) If $\mathcal{A}_0 \cong \mathcal{A}_1$, then* **II** *wins* $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ *for all* α.
(ii) If $\mathcal{A}_0 \ncong \mathcal{A}_1$, then **I** *wins* $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ *for all* $\alpha \geq |A_0| + |A_1|$.

Proof (i) Suppose $f : \mathcal{A}_0 \cong \mathcal{A}_1$. Let

$$\tau_\xi(\langle (c_\eta, x_\eta) : \eta \leq \xi \rangle) = \begin{cases} f(x_\xi) & \text{if } c_\xi = 0 \\ f^{-1}(x_\xi) & \text{if } c_\xi = 1. \end{cases}$$

Then $\langle \tau_\xi : \xi < \alpha \rangle$ is a winning strategy of **II** in $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$.

(ii) Let $\{0,1\} \times (A_0 \cup A_1) = \{z_\xi : \xi < \alpha\}$ and $\rho = \langle \rho_\xi : \xi < \alpha \rangle$, where

$$\rho_\xi(\langle y_\eta : \eta < \xi \rangle) = z_\xi$$

for $\xi < \alpha$. For any $\bar{y} = \langle y_\xi : \xi < \alpha \rangle$ the relation $p_{\rho,\bar{y}}$ is a partial isomorphism between $\mathcal{A}_0$ and $\mathcal{A}_1$. Since no isomorphism exists, ρ is a winning strategy of **I**. □

Corollary *(i) If* **I** *wins* $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$*, then* $\mathcal{A}_0 \ncong \mathcal{A}_1$.
(ii) If **II** *wins* $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$*, where* $\alpha \geq |A_0| + |A_1|$*, then* $\mathcal{A}_0 \cong \mathcal{A}_1$.

There is always at least one α for which **II** wins $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$, namely $\alpha = 0$. If $\mathcal{A}_0 \cong \mathcal{A}_1$, then by Lemma 9.49 and 9.50 there cannot be any α for which **I** wins $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$. But if $\mathcal{A}_0 \ncong \mathcal{A}_1$, then **I** wins $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ from some α onwards.

There may be ordinals α for which neither player has a winning strategy (Exercises 9.29 and 9.30 below). Then the game is non-determined. The game of length ω_1 may also be non-determined, see [MSV93]. There may also be a limit ordinal α such that **II** wins $\mathrm{EF}_\beta(\mathcal{A}_0, \mathcal{A}_1)$ for each $\beta < \alpha$ but not $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$. We already know that this can happen if $\alpha = \omega$.

Lemma 9.51 *Let L be a vocabulary and α an ordinal. The relation*

$$\mathcal{A}_0 \sim_\alpha \mathcal{A}_1 \Leftrightarrow \exists \textit{ wins } \mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$$

is an equivalence relation on $\mathrm{Str}(L)$.

Proof Reflexivity of $\sim_\alpha$ follows from Lemma 9.50(i). In fact, **II** wins the game $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_0)$ with the trivial strategy $\tau_\xi(\langle (c_\eta, x_\eta) : \eta \le \xi \rangle) = x_\xi$. Symmetry is also trivial: Suppose **II** wins $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ with $\tau = \langle \tau_\xi : \xi < \alpha \rangle$. The following strategy $\tau' = \langle \tau'_\xi : \xi < \alpha \rangle$ is winning for **II** in $\mathrm{EF}_\alpha(\mathcal{A}_1, \mathcal{A}_0)$:

$$\tau'((c_\eta, x_\eta) : \eta \le \xi) = \tau((1 - c_\eta, x_\eta) : \eta \le \xi).$$

To see this, suppose $\bar{z} = \langle z_\xi : \xi < \alpha \rangle$ is given. Then $p_{\bar{z},\tau}$ is a partial isomorphism between $\mathcal{A}_0$ and $\mathcal{A}_1$, and the relation

$$p'_{\bar{z},\tau} = \{(b, a) : (a, b) \in p_{\bar{z},\tau}\}$$

is a partial isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_0$, witnessing the victory of **II** in $\mathrm{EF}_\alpha(\mathcal{A}_1, \mathcal{A}_0)$. To prove transitivity of $\sim_\alpha$, suppose $\tau = \langle \tau_\alpha : \xi < \alpha \rangle$ is a winning strategy of **II** in $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ and $\tau' = \langle \tau'_\xi : \xi < \alpha \rangle$ is a winning strategy of **II** in $\mathrm{EF}_\alpha(\mathcal{A}_1, \mathcal{A}_2)$. We describe a winning strategy $\tau'' = \langle \tau''_\xi : \xi < \alpha \rangle$ of **II** in $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_2)$. The idea is that **II** plays $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ and $\mathrm{EF}_\alpha(\mathcal{A}_1, \mathcal{A}_2)$ simultaneously. Suppose $\bar{z}'' = \langle (c''_\eta, x''_\eta) : \eta \le \xi \rangle \in \mathrm{Seq}_{\xi+1}(A_0, A_1)$. We define by induction over $\eta \le \xi$ the sequences $\bar{z} = \langle (c_\eta, x_\eta) : \eta \le \xi \rangle$, $\bar{z}' = \langle (c'_\eta, x'_\eta) : \eta \le \xi \rangle$, and $\tau'' = \langle \tau''_\xi : \xi < \alpha \rangle$ as follows:

If	c''_η	=	0	1
Then	(c_η, x_η)	=	$(0, x''_\eta)$	$(1, \tau'_\eta(\bar{z}'\restriction_\eta))$
	(c'_η, x'_η)	=	$(0, \tau_\eta(\bar{z}\restriction_\eta))$	$(1, x''_\eta)$
	$\tau''_\eta(\bar{z}''\restriction_\eta)$	=	$\tau'_\eta(\bar{z}'\restriction_\eta)$	$\tau_\eta(\bar{z}\restriction_\eta)$

Now $\langle \tau''_\xi : \xi < \alpha \rangle$ is a winning strategy of **II** in $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_2)$. □

The relations $\sim_\alpha$ form a sequence of finer and finer partitions of $\mathrm{Str}(L)$, starting from the one-class partition $\sim_0$ and eventually approaching the ultimate refinement $\cong$ of every $\sim_\alpha$.

9.4 A Quasi-order of Partially Ordered Sets

Before we define the dynamic version of the transfinite game EF_α we develop some useful theory of po-sets.

Definition 9.52 Suppose $\mathcal{P}$ and $\mathcal{P}'$ are po-sets. We define

$$\mathcal{P} \le \mathcal{P}'$$

if there is a mapping $f : P \to P'$ such that for all $x, y \in P$:

$$x <_{\mathcal{P}} y \to f(x) <_{\mathcal{P}'} f(y).$$

We write $\mathcal{P} < \mathcal{P}'$, if $\mathcal{P} \leq \mathcal{P}'$ and $\mathcal{P}' \not\leq \mathcal{P}$, and we write $\mathcal{P} \equiv \mathcal{P}'$, if $\mathcal{P} \leq \mathcal{P}'$ and $\mathcal{P}' \leq \mathcal{P}$.

Note that $\leq$ is a transitive relation among po-sets. The $\equiv$-classes of $\leq$ form a quasi-ordered class. This quasi-order is the topic of this section. It is not a total order, for there are incomparable po-sets, for example $(\omega, <)$ and its inverse ordering $(\omega, >)$. For simplicity, we call $\leq$ itself the quasi-order of po-sets, without recourse to the $\equiv$-classes.

Definition 9.53 Suppose $\mathcal{P}$ is a po-set. The tree $\sigma\mathcal{P}$ is defined as follows. Its domain is the set of functions s with $\mathrm{dom}(s) \in \mathrm{On}$ such that for all $\alpha, \beta \in \mathrm{dom}(s)$

$$\alpha < \beta \to s(\alpha) <_{\mathcal{P}} s(\beta).$$

The order is

$$s \leq s' \leftrightarrow s = s' \restriction_{\mathrm{dom}(s)}.$$

$\sigma'\mathcal{P}$ is the suborder of $\sigma\mathcal{P}$ consisting of sequences $s \in \sigma\mathcal{P}$ of successor length.

The σ-operation was introduced by Kurepa [Kur56] and studied further e.g. in [HV90] and [TV99].

Example 9.54 For any ordinal α let B_α be the tree of descending sequences $\beta_0 > \ldots > \beta_n$ of elements of α ordered by end-extension. Show that $\alpha \leq \beta$ (as ordinals) if and only if $B_\alpha \leq B_\beta$ as po-sets. Every well-founded tree is $\equiv$-equivalent to some B_α. (See Exercise 9.36.)

Lemma 9.55 ***(i)*** *$\sigma'\mathcal{P} \leq \mathcal{P}$.*
(ii) *$\sigma\mathcal{P} \not\leq \mathcal{P}$.*
(iii) *$\sigma'\mathcal{P} < \sigma\mathcal{P}$.*
(iv) *If T is a tree, then $T \equiv \sigma' T$.*

Proof (i) If $s \in \sigma'\mathcal{P}$, let $f(s) = s(\mathrm{dom}(s) - 1)$. Then $f : \sigma'\mathcal{P} \to \mathcal{P}$ is order-preserving.

(ii) Suppose $f : \sigma\mathcal{P} \to \mathcal{P}$ were order-preserving. Define inductively $s : \mathrm{On} \to \mathcal{P}$ by $s(\alpha) = f(s\restriction_\alpha)$. Since $\alpha < \beta$ implies $s(\alpha) <_{\mathcal{P}} s(\beta)$, we get the result that $\mathcal{P}$ is a proper class, a contradiction.

(iii) $\sigma'\mathcal{P} \leq \sigma\mathcal{P}$ trivially. On the other hand, if $\sigma\mathcal{P} \leq \sigma'\mathcal{P}$, then $\sigma\mathcal{P} \leq \mathcal{P}$ contrary to (ii),

(iv) We know already $\sigma' T \leq T$. Suppose $t \in T$ and $\langle t_\alpha : \alpha \leq \beta \rangle$ is the set of $t' \in T$ with $t' \leq_T t$ in ascending order. Let $\mathrm{dom}(s) = \beta + 1$ and $s_t(\alpha) = t_\alpha$. then $s_t \in \sigma' T$ and $t \mapsto s_t$ is order-preserving. □

Example 9.56 $Q \not\leq \sigma Q$ since σQ is well-founded while Q is not. In particular $Q \not\leq \sigma' Q$. Hence $\sigma' Q < Q$. Note that $\sigma' Q$ is a special tree while σQ is non-special. (See Exercise 9.40.)

Lemma 9.57 *There is no sequence $\mathcal{P}_0, \mathcal{P}_1, \ldots$ so that $\sigma\mathcal{P}_{n+1} \leq \mathcal{P}_n$ for all $n < \omega$.*

Proof Suppose $f_n : \sigma\mathcal{P}_{n+1} \to \mathcal{P}_n$ is order-preserving. For each fixed α, let $s^n_\alpha \in \mathcal{P}_n$ so that

$$f_n(\langle s^{n+1}_\beta : \beta < \alpha\rangle) = s^n_\alpha.$$

Then each $\mathcal{P}_n$ is a proper class, a contradiction. □

Definition 9.58 Suppose $\mathcal{P}$ and $\mathcal{P}'$ are po-sets. The game $G(\mathcal{P}, \mathcal{P}')$ is defined as follows. Player **I** plays $p_0 \in \mathcal{P}$, then player **II** plays $p'_0 \in \mathcal{P}'$. After this **I** plays $p_1 \in \mathcal{P}$ with $p_0 <_{\mathcal{P}} p_1$, and then player **II** plays $p'_1 \in \mathcal{P}'$ with $p'_0 <_{\mathcal{P}'} p'_1$, and so on. At limits player **I** moves first $p_\nu \in \mathcal{P}$ with $p_\alpha <_{\mathcal{P}} p_\nu$ for all $\alpha < \nu$. Then **II** moves $p'_\nu \in \mathcal{P}'$ with $p'_\alpha <_{\mathcal{P}} p'_\nu$ for all $\alpha < \nu$. If a player cannot move, he loses and the other player wins. Since $\mathcal{P}$ and $\mathcal{P}'$ are sets, one of the players eventually wins.

Lemma 9.59 ***(i)*** *$\sigma'\mathcal{P} \leq \mathcal{P}'$ if and only if* **II** *wins $G(\mathcal{P}, \mathcal{P}')$.*
(ii) *If $\mathcal{P}$ is a tree, then $\mathcal{P} \leq \mathcal{P}'$ if and only if* **II** *wins $G(\mathcal{P}, \mathcal{P}')$.*

Proof (i) Suppose $f : \sigma'\mathcal{P} \to \mathcal{P}'$ is order-preserving. If **I** has played $p_0 < \ldots < p_\alpha$ in $G(\mathcal{P}, \mathcal{P}')$, **II** plays $p'_\alpha = f((p_0, \ldots, p_\alpha))$. In this way she ends up the winner. Conversely, suppose **II** wins $G(\mathcal{P}, \mathcal{P}')$ and $s \in \sigma'\mathcal{P}$ with $\mathrm{dom}(s) = \alpha + 1$. Let us play $G(\mathcal{P}, \mathcal{P}')$ so that **I** plays $p_\beta = s(\beta)$ for $\beta \leq \alpha$ and **II** uses her winning strategy. After **I** plays p_α, **II** plays p'_α. If we define $f(s) = p'_\alpha$, we get an order-preserving mapping $\sigma'\mathcal{P} \to \mathcal{P}$. This ends the proof of (i). (ii) follows from (i) and Lemma 9.55 (iv). □

Lemma 9.60 *$\sigma\mathcal{P}' \leq \mathcal{P}$ if and only if* **I** *wins $G(\mathcal{P}, \mathcal{P}')$.*

Proof Suppose $f : \sigma\mathcal{P}' \to \mathcal{P}$ is order-preserving. If **II** has played

$$p'_0 < \ldots < p'_\beta < \ldots \ (\beta < \alpha) \tag{9.6}$$

in $G(\mathcal{P}, \mathcal{P}')$, **I** plays $p_\alpha = f((p'_0, \ldots, p'_\beta, \ldots))$ in $\mathcal{P}'$. In this way **I** wins $G(\mathcal{P}, \mathcal{P}')$. On the other hand, if **I** wins $G(\mathcal{P}, \mathcal{P}')$ and (9.6) is an ascending chain in $\mathcal{P}'$, we can let **I** play against the moves $p'_0, \ldots, p'_\beta, \ldots$ of **II** in $G(\mathcal{P}, \mathcal{P}')$. Finally **I** plays p_α according to his winning strategy. We let

$$f((p'_0, \ldots, p'_\beta, \ldots)) = p_\alpha.$$

Now $f : \sigma\mathcal{P}' \to \mathcal{P}$ is order-preserving. □

Example 9.61 Suppose $S \subseteq \omega_1$. Let $T(S)$ be the tree of closed ascending sequences of elements of S. Choose disjoint stationary sets S_1 and S_2. Then $T(S_1) \not\leq T(S_2)$ and $T(S_2) \not\leq T(S_1)$ (Exercise 9.35). Thus the game $G(T(S_1), T(S_2))$ is non-determined.

Definition 9.62 We use $\mathcal{T}_{\lambda,\kappa}$ to denote the class of trees of cardinality $\leq \lambda$ without branches of length κ.

The simplest uncountable tree in $\mathcal{T}_{\kappa,\kappa}$ is the κ*-fan* which consists of branches of all lengths $< \kappa$ joined at the root, or in symbols,

$$F_\kappa = \{s_\alpha : 0 < \alpha < \kappa\}, s_\alpha = \langle a^\alpha_\beta : \beta < \alpha \rangle,$$

$$a^\alpha_0 = 0, a^\alpha_\beta = (\alpha, \beta) \text{ for } \beta > 0,$$

ordered by end-extension. Aronszajn trees are in $\mathcal{T}_{\aleph_1,\aleph_1}$. The trees $T(S)$ of Example 9.61 are in $\mathcal{T}_{2^\omega,\aleph_1}$.

Definition 9.63 A tree T is a *persistent* if for all $t \in T$ and all $\alpha < \mathrm{ht}(T)$ there is $t' \in T$ such that $t <_T t'$ and $\mathrm{ht}(t') \geq \alpha$.

Persistency is a kind of non-triviality assumption for a tree. It means that from any node you can go as high as you like. The κ-fan is certainly non-persistent. On the other hand, the tree

$$T^\kappa_p = (F_\kappa)^{<\omega}, (s_{\alpha_0}, ..., s_{\alpha_{n-1}}) \leq (s_{\beta_0}, ..., s_{\beta_{m-1}}) \iff$$

$$n \leq m \text{ and } \alpha_i = \beta_i \text{ for } i < n$$

is persistent and indeed the $\leq$-smallest persistent tree in $\mathcal{T}_{\kappa,\kappa}$ (Exercise 9.41).

Definition 9.64 A po-set $\mathcal{P}$ is a *bottleneck* in a class K of po-sets if $\mathcal{P}' \leq \mathcal{P}$ or $\mathcal{P} \leq \mathcal{P}'$ for all $\mathcal{P}'$ in the class K. A tree T is a *strong bottleneck* for a class K if the game $G(T, \mathcal{P})$ is determined for all $\mathcal{P} \in K$.

Every well-founded tree is a strong bottleneck in the class of all trees. If $S \subseteq \omega_1$ is bistationary, then $T(S)$ is by Example 9.61 not a bottleneck in the class of all trees. The smallest persistent tree T^κ_p is a strong bottleneck in the class $\mathcal{T}_{\kappa,\kappa}$ (Exercise 9.42). It is an interesting problem whether there are bottlenecks in the class $\mathcal{T}_{\kappa,\kappa}$ above T^κ_p. The following partial result is known:

Theorem 9.65 *Suppose κ is a regular cardinal and and $\mathcal{P}$ is the forcing notion for adding κ^+ Cohen subsets to κ, then $\mathcal{P}$ forces that there are no bottlenecks in the class $\mathcal{T}_{\kappa,\kappa}$ above T^κ_p.*

Proof Suppose T is a bottleneck. Let $\alpha < \kappa^+$ such that $T \in V[G_\alpha]$. Let A_α be the Cohen subset of κ added at stage α. Note that A_α is a bistationary subset of κ. We first show that $\Vdash T \not\leq T(A_\alpha)$. Suppose

$$p \Vdash \hat{f} : T(A_\alpha) \to T \text{ is strictly increasing.}$$

When we force with A_α, calling the forcing notion $\mathcal{P}'$, an uncountable branch appears in $T(A_\alpha)$, hence also in T. The product forcing $\mathcal{P}_\alpha \star \mathcal{P}'$ contains a κ-closed dense set (Exercise 9.45). Hence it cannot add a branch of length κ to T. We have shown that $T(A_\alpha) \not\leq T$ in $V[G]$. Since T is a bottleneck, $T \leq T(A_\alpha)$. By repeating the same with $-A_\alpha$ we get $T \leq T(-A_\alpha)$. In sum, $T \leq T(A_\alpha) \otimes T(-A_\alpha)$ (see Exercise 9.44 for the definition of $\otimes$). But $T(A_\alpha) \otimes T(-A_\alpha) \leq T_p^\kappa$ (Exercise 9.46). Hence $T \leq T_p^\kappa$. □

It is also known ([TV99]) that if $V = L$, then there are no bottlenecks in the class $\mathcal{T}_{\aleph_1,\aleph_1}$ above $T_p^{\aleph_1}$.

9.5 The Transfinite Dynamic Ehrenfeucht-Fraïssé Game

In this section we introduce a more general form of the Ehrenfeucht-Fraïssé Game. The new game generalizes both the usual Ehrenfeucht-Fraïssé Game and the dynamic version of it. In this game player **I** makes moves not only in the models in question but also moves up a po-set, move by move. The game goes on as long as **I** can move. This game generalizes at the same time the games $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ and $\mathrm{EFD}_\delta(\mathcal{A}_0, \mathcal{A}_1)$. Therefore we denote it $\mathrm{EF}_\mathcal{P}$ rather than by $\mathrm{EFD}_\mathcal{P}$.

If $\mathcal{P}$ is a po-set, let $\mathrm{b}(\mathcal{P})$ denote the least ordinal δ so that $\mathcal{P}$ does not have an ascending chain of length δ.

Definition 9.66 Suppose $\mathcal{A}_0$ and $\mathcal{A}_1$ are L-structures and $\mathcal{P}$ is a po-set. The *Transfinite Dynamic Ehrenfeucht-Fraïssé Game* $\mathrm{EF}_\mathcal{P}(\mathcal{A}_0, \mathcal{A}_1)$ is like the game $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$ except that on each round **I** chooses an element $c_\alpha \in \{0, 1\}$, an element $x_\alpha \in A_{c_\alpha}$ and an element $p_\alpha \in \mathcal{P}$. It is required that

$$p_0 <_\mathcal{P} \ldots <_\mathcal{P} p_\alpha <_\mathcal{P} \ldots .$$

Finally **I** cannot play a new p_α anymore because $\mathcal{P}$ is a set. Suppose **I** has played $\bar{z} = \langle (c_\beta, x_\beta) : \beta < \alpha \rangle$ and **II** has played $\bar{y} = \langle y_\beta : \beta < \alpha \rangle$. If $p_{\bar{z},\bar{y}}$ is a partial isomorphism between $\mathcal{A}_0$ and $\mathcal{A}_1$, **II** has won the game, otherwise **I** has won.

Thus a winning strategy of **I** in $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ is a sequence $\rho = \langle \rho_\alpha : \alpha < \mathrm{b}(\mathcal{P}) \rangle$ and a strategy of **II** is a sequence $\tau = \langle \tau_\alpha : \alpha < \mathrm{b}(\mathcal{P}) \rangle$. Note that

$$\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1) \text{ is the same game as } \mathrm{EF}_{(\alpha,<)}(\mathcal{A}_0, \mathcal{A}_1),$$

and

$$\mathrm{EFD}_\alpha(\mathcal{A}_0, \mathcal{A}_1) \text{ is the same game as } \mathrm{EF}_{(\alpha,>)}(\mathcal{A}_0, \mathcal{A}_1).$$

Naturally, if α is finite, the games $\mathrm{EF}_{(\alpha,<)}(\mathcal{A}_0, \mathcal{A}_1)$ and $\mathrm{EF}_{(\alpha,>)}(\mathcal{A}_0, \mathcal{A}_1)$ are one and the same game. But if α happens to be infinite, there is a big difference: The first is a transfinite game while the second can only go on for a finite number of moves.

The ordering $\mathcal{P} \leq \mathcal{P}'$ of po-sets has a close connection to the question who wins the game $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$, as the following two results manifest:

Lemma 9.67 *If* **II** *wins the game* $\mathrm{EF}_{\mathcal{P}'}(\mathcal{A}_0, \mathcal{A}_1)$ *and* $\mathcal{P} \leq \mathcal{P}'$*, then* **II** *wins the game* $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$*. If* **I** *wins the game* $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ *and* $\mathcal{P} \leq \mathcal{P}'$*, then* **I** *wins the game* $\mathrm{EF}_{\mathcal{P}'}(\mathcal{A}_0, \mathcal{A}_1)$*.*

Proof Exercise 9.50. □

Proposition 9.68 *Suppose* **II** *wins* $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ *and* **I** *wins* $\mathrm{EF}_{\mathcal{P}'}(\mathcal{A}_0, \mathcal{A}_1)$*. Then* $\sigma\mathcal{P} \leq \mathcal{P}'$*.*

Proof Suppose **II** wins $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ with τ and **I** wins $\mathrm{EF}_{\mathcal{P}'}(\mathcal{A}_0, \mathcal{A}_1)$ with ρ. We describe a winning strategy of **I** in $G(\mathcal{P}', \mathcal{P})$, and then the claim follows from Lemma 9.60. Suppose $\rho_0(\emptyset) = (c_0, x_0, p'_0)$. The element p'_0 is the first move of **I** in $G(\mathcal{P}', \mathcal{P})$. Suppose **II** plays $p_0 \in \mathcal{P}$. Let

$$y_0 = \tau_0(((c_0, x_0, p_0))),$$
$$(c_1, x_1, p'_1) = \rho_1((y_0)).$$

The element p'_1 is the second move of **I** in $G(\mathcal{P}', \mathcal{P})$. More generally the equations

$$y_\beta = \tau_\beta(\langle (c_\gamma, x_\gamma, p_\gamma) : \gamma \leq \beta \rangle)$$
$$(c_\alpha, x_\alpha, p'_\alpha) = \rho_\alpha(\langle y_\beta : \beta < \alpha \rangle)$$

define the move p'_α of **I** in $G(\mathcal{P}', \mathcal{P})$ after **II** has played $\langle p_\beta : \beta < \alpha \rangle$. The game can only end if **II** cannot move p_α at some point, so **I** wins. □

Suppose $\mathcal{A}_0 \not\cong \mathcal{A}_1$. Then there is a least ordinal

$$\delta \leq \mathrm{Card}(A_0) + \mathrm{Card}(A_1)$$

such that **II** does not win $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$. Thus for all $\alpha + 1 < \delta$ there is a winning strategy for **II** in $\mathrm{EF}_{\alpha+1}(\mathcal{A}_0, \mathcal{A}_1)$. Let $K(= K(\mathcal{A}_0, \mathcal{A}_1))$ be the set of all winning strategies of **II** in $\mathrm{EF}_{\alpha+1}(\mathcal{A}_0, \mathcal{A}_1)$ for $\alpha + 1 < \delta$. We can make K a tree by letting

$$\langle \tau_\xi : \xi \leq \alpha \rangle \leq \langle \tau'_\xi : \xi \leq \alpha' \rangle$$

if and only if $\alpha \leq \alpha'$ and $\forall \xi \leq \alpha (\tau_\xi = \tau'_\xi)$.

Definition 9.69 We call K, as defined above, the *canonical Karp tree* of the pair $(\mathcal{A}_0, \mathcal{A}_1)$.

Note that even when δ is a limit ordinal K does not have a branch of length δ, for otherwise **II** would win $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$.

Lemma 9.70 *Suppose $\mathcal{P}$ is a po-set. Then*

$$\exists \text{ wins } \mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1) \iff \sigma'\mathcal{P} \leq K.$$

Proof $\Rightarrow$ Suppose **II** wins $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ with τ. If $s = \langle s_\xi : \xi \leq \alpha \rangle \in \sigma'\mathcal{P}$, we can define a strategy τ' of **II** in $\mathrm{EF}_{\alpha+1}(\mathcal{A}_0, \mathcal{A}_1)$ as follows

$$\tau'_\xi(\langle (c_\eta, x_\eta) : \eta \leq \xi \rangle) = \tau_\xi(\langle (c_\eta, x_\eta, s_\eta) : \eta \leq \xi \rangle).$$

Since K does not have a branch of length δ, $\alpha < \delta$, and hence $\tau' \in K$. The mapping $s \mapsto \tau'$ is an order-preserving mapping $\sigma'\mathcal{P} \to K$.

$\Leftarrow$ Suppose $f : \sigma'\mathcal{P} \to K$ is order-preserving. We can define a winning strategy of **II** in $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ by the equation

$$\tau_\alpha(\langle (c_\xi, x_\xi, s_\xi) : \eta \leq \xi \rangle) = f(\langle s_\xi : \xi \leq \alpha \rangle)(\langle (c_\xi, x_\xi, s_\eta) : \xi \leq \alpha \rangle).$$

□

Proposition 9.71 *Suppose δ is a limit ordinal and* **II** *wins* $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ *for all $\alpha < \delta$. the following are equivalent:*

(i) **II** *wins* $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$.
(ii) **II** *wins* $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ *for every po-set $\mathcal{P}$ with no branches of length δ.*

Proof To prove (ii)→(i), suppose **II** does not win $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$. Let $\mathcal{P} = K(\mathcal{A}_0, \mathcal{A}_1)$. Then $\sigma\mathcal{P}$ does not have branches of length δ, hence by (ii) **II** wins $\mathrm{EF}_{\sigma\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ and we get $\sigma\mathcal{P} \leq \mathcal{P}$ from Lemma 9.70, a contradiction with Lemma 9.55. The other direction (i)→(ii) is trivial. □

Note Suppose $\kappa = \mathrm{Card}(A_0) + \mathrm{Card}(A_1)$. Then we can compute $\mathrm{Card}(K) \leq \sup_{\alpha<\delta}(\kappa^{\kappa^\alpha})^\alpha = \sup_{\alpha<\delta}\kappa^{\kappa^\alpha}$. If GCH and κ is regular, then $\mathrm{Card}(K) \leq \kappa^+$. Furthermore, if we assume GCH, we can assume $\mathrm{Card}(\mathcal{P}) \leq \kappa$ in (ii) above (Hyttinen). For $\delta = \omega$ this does not depend on GCH. **II** wins $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ if and only if **II** wins $\mathrm{EF}_{\sigma'\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$. So from the point of view of the existence of a winning strategy for **II** we could always assume that $\mathcal{P}$ is a tree.

Corollary **II** *never wins* $\mathrm{EF}_{\sigma K}(\mathcal{A}_0, \mathcal{A}_1)$.

Definition 9.72 A po-set $\mathcal{P}$ is a *Karp po-set* of the pair $(\mathcal{A}_0, \mathcal{A}_1)$ if **II** wins $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ but not $\mathrm{EF}_{\sigma\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$. If a Karp po-set is a tree, we call it a *Karp tree*.

By Lemma 9.70 and the above corollary, there are always Karp trees for every pair of non-isomorphic structures.

Suppose **I** wins $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ with the strategy ρ. Let S_ρ be the set of sequences $\bar{y} = \langle y_\xi : \xi \leq \alpha \rangle \in \mathrm{dom}(\rho)$ such that

$$p_{\rho \upharpoonright \alpha+1, y} \in \mathrm{Part}(\mathcal{A}_0, \mathcal{A}_1).$$

Thus S_ρ is the set of sequences of moves of **II** before she loses $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$, when **I** plays ρ. We can make S_ρ a tree by ordering it as follows

$$\langle y_\xi : \xi \leq \alpha \rangle \leq \langle y'_\xi : \xi \leq \alpha' \rangle$$

if and only if $\alpha \leq \alpha'$ and $\forall \xi \leq \alpha (y_\xi = y'_\xi)$.

Lemma 9.73 **I** *wins* $\mathrm{EF}_{\sigma S_\rho}(\mathcal{A}_0, \mathcal{A}_1)$.

Proof The following equation defines a winning strategy ρ' of **I** in the game $\mathrm{EF}_{\sigma S_\rho}(\mathcal{A}_0, \mathcal{A}_1)$:

$$\rho'_\alpha \langle y_\xi : \xi < \alpha \rangle) = \langle c_\alpha, x_\alpha, \langle (y_\xi : \xi \leq \beta \rangle : \beta < \alpha \rangle,$$

where

$$\rho_\alpha(\langle y_\xi : \xi < \alpha \rangle) = (c_\alpha, x_\alpha, p_\alpha).$$

□

Lemma 9.74 $\sigma S_\rho \leq \mathcal{P}$.

Proof Suppose $s = \langle \langle y_\xi : \xi \leq \beta \rangle : \beta < \alpha \rangle \in \sigma S_\rho$, where

$$\beta_0 < \beta_1 < \ldots < \beta_\eta < \ldots (\eta < \alpha).$$

Let $\delta = \sup_{\eta < \alpha} \beta_\eta$ and

$$\rho_\delta(\langle y_\xi : \xi < \delta \rangle) = (c_\delta, x_\delta, p_\delta).$$

We define $f(s) = p_\delta$. Then $f : \sigma S_\rho \to \mathcal{P}$ is order-preserving. □

Note that Lemma 9.74 implies $\mathcal{P} \not\leq S_\rho$. In particular, if **I** wins $\mathrm{EF}_\delta(\mathcal{A}_0, \mathcal{A}_1)$ with ρ, then S_ρ is a tree with no branches of length δ.

Suppose $\mathcal{P}_0$ is such that $\sigma\mathcal{P}_0 \leq \mathcal{P}$ and **I** wins $\mathrm{EF}_{\sigma\mathcal{P}_0}$. So $\mathcal{P}_0$ could be S_ρ. Suppose furthermore that there is no $\mathcal{P}_1$ such that $\sigma\mathcal{P}_1 \leq \mathcal{P}_0$ and **I** wins $\mathrm{EF}_{\sigma\mathcal{P}_1}$. Lemma 9.57 implies that this assumption can always be satisfied.

Lemma 9.75 **I** *does not win* $\mathrm{EF}_{\mathcal{P}_0}(\mathcal{A}_0, \mathcal{A}_1)$.

Proof Suppose **I** wins $\mathrm{EF}_{\mathcal{P}_0}(\mathcal{A}_0, \mathcal{A}_1)$ with ρ'. Then **I** wins $\mathrm{EF}_{\sigma S_{\rho'}}(\mathcal{A}_0, \mathcal{A}_1)$ and $\sigma S_{\rho'} \leq \mathcal{P}_0$, contrary to the choice of $\mathcal{P}_0$. □

Definition 9.76 A po-set $\mathcal{P}$ is a *Scott po-set* of $(\mathcal{A}_0, \mathcal{A}_1)$ if **I** wins the game $\mathrm{EF}_{\sigma\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ but not the game $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$. If a Scott po-set is a tree, we call is a *Scott tree*. If $\mathcal{P}$ is both a Scott and a Karp po-set, it is called a *determined* Scott po-set.

By Lemma 9.73 and Lemma 9.75, S_ρ is always a Scott tree of $(\mathcal{A}_0, \mathcal{A}_1)$, so Scott trees always exist. Note that

$$\mathrm{Card}(S_\rho) \leq \sup_{\alpha < \mathrm{b}(\mathcal{P})} (\mathrm{Card}(A_0) + \mathrm{Card}(A_1))^\alpha.$$

Lemma 9.77 *Suppose* **I** *wins* $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$ *with* ρ *and* $K = K(\mathcal{A}_0, \mathcal{A}_1)$. *Then* $K \leq S_\rho$.

Proof Suppose $\tau \in K$. Let **II** play τ against ρ in $\mathrm{EF}_{\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$. The resulting sequence $\bar{y}$ of moves of **II** is an element of S_ρ. The mapping $\tau \mapsto \bar{y}$ is order-preserving. □

Suppose **II** wins $\mathrm{EF}_{\mathcal{P}_0}(\mathcal{A}_0, \mathcal{A}_1)$ and **I** wins $\mathrm{EF}_{\mathcal{P}_1}(\mathcal{A}_0, \mathcal{A}_1)$ with ρ. Figure 9.7 shows the resulting picture.

In summary, we have proved:

Theorem 9.78 *Suppose* **II** *wins* $\mathrm{EF}_{\mathcal{P}_0}(\mathcal{A}_0, \mathcal{A}_1)$ *and* **I** *wins* $\mathrm{EF}_{\mathcal{P}_1}(\mathcal{A}_0, \mathcal{A}_1)$. *Then there are trees* T_0 *and* T_1 *such that*

(i) $\sigma'\mathcal{P}_0 \leq T_0 \leq T_1 \leq \mathcal{P}_1$.
(ii) **II** *wins* $\mathrm{EF}_{T_0}(\mathcal{A}_0, \mathcal{A}_1)$ *but not* $\mathrm{EF}_{\sigma T_0}(\mathcal{A}_0, \mathcal{A}_1)$.
(iii) **I** *wins* $\mathrm{EF}_{\sigma T_1}(\mathcal{A}_0, \mathcal{A}_1)$ *but not* $\mathrm{EF}_{T_1}(\mathcal{A}_0, \mathcal{A}_1)$.

Example 9.79 Suppose **I** wins $\mathrm{EF}_\omega(\mathcal{A}_0, \mathcal{A}_1)$. By Proposition 7.19 there is a unique $\delta = \delta(\mathcal{A}_0, \mathcal{A}_1)$ such that **II** wins $\mathrm{EF}_{(\delta,>)}(\mathcal{A}_0, \mathcal{A}_1)$ and **I** wins $\mathrm{EF}_{(\delta+1,>)}(\mathcal{A}_0, \mathcal{A}_1)$. Then $(\delta, >)$ is both a Karp and a Scott po-set for $\mathcal{A}_0$ and $\mathcal{A}_1$.

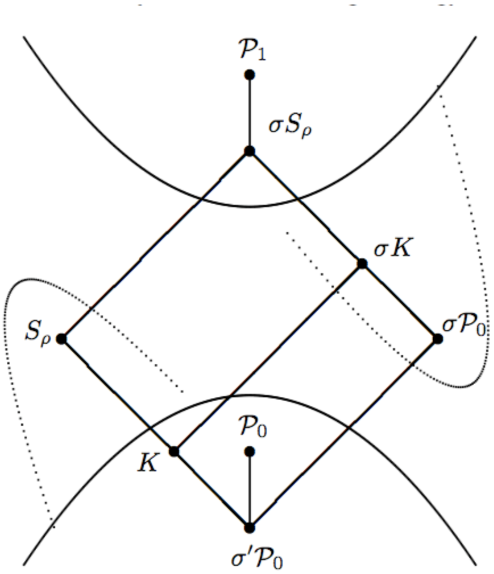

Figure 9.7 The boundary between **II** winning and **I** winning.

Example 9.80 Suppose **II** wins $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$ but not $\mathrm{EF}_{\alpha+1}(\mathcal{A}_0, \mathcal{A}_1)$. Then $(\alpha, <)$ is a Karp tree (in fact a Karp well-order) of $\mathcal{A}_0$ and $\mathcal{A}_1$. This follows from the fact that $\sigma(\alpha, <) \equiv (\alpha + 1, <)$.

Example 9.81 Suppose **I** wins $\mathrm{EF}_{\alpha+1}(\mathcal{A}_0, \mathcal{A}_1)$ but not $\mathrm{EF}_\alpha(\mathcal{A}_0, \mathcal{A}_1)$. Then $(\alpha, <)$ is a Scott tree (in fact a Scott well-order) of $\mathcal{A}_0$ and $\mathcal{A}_1$.

If T is a tree, $T + 1$ is the tree which is obtained from T by adding a new element at the end of every maximal branch of T. Note that $T + 1$ may be uncountable even if T is countable.

Lemma 9.82 *Suppose $S \subseteq \omega_1$ is bistationary, $\mathcal{A}_0 = \Phi(S)$, $\mathcal{A}_1 = \Phi(\emptyset)$ and $\mathcal{P} = T(\omega_1 \setminus S) + 1$. Then* **I** *wins* $\mathrm{EF}_{\sigma\mathcal{P}}(\mathcal{A}_0, \mathcal{A}_1)$.

Proof Suppose **I** has played already $(c_\beta, x_\beta, p_\beta)$ and **II** has played y_β for $\beta < \alpha$. Suppose **I** now has to decide how to play $(c_\alpha, x_\alpha, p_\alpha)$ in $\mathrm{EF}_\mathcal{P}(\mathcal{A}_0, \mathcal{A}_1)$. We assume that **I** has played in such a way that

1. $p_\beta = \langle\langle\delta_\delta : \delta \leq \gamma\rangle : \gamma < \beta\rangle$ $(\in \sigma(T(\omega_1 \setminus S) + 1)$.
2. $x_{\nu+2n} < y_{\nu+2n+1}$ in $\mathcal{A}_0$.
3. $x_{\nu+2n+1} < y_{\nu+2n+2}$ in $\mathcal{A}_1$.

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9.6 Topology of Uncountable Models

Countable models with countable vocabulary can be thought of as points in the Baire space ω^ω. Likewise, models $\mathcal{M}$ of cardinality κ with vocabulary of cardinality κ can be thought of as points $f_{\mathcal{M}}$ in the set κ^κ. We can make κ^κ a topological space by letting the sets

$$N(f, \alpha) = \{g \in \omega^\kappa : f \restriction \alpha = g \restriction \alpha\},$$

where $\alpha < \kappa$, form the basis of the topology. Let us denote this *generalized Baire space* κ^κ by $\mathcal{N}_\kappa$. Now properties of models of size κ correspond to subsets of $\mathcal{N}_\kappa$. In particular, modulo coding, isomorphism of structures of cardinality κ becomes an "analytic" property in this space.

One of the basic questions about models of size κ that we can try to attack with methods of logic is the question which of those models can be identified up to isomorphism by means of a set of invariants. Shelah's Main Gap Theorem gives one answer: If $\mathcal{M}$ is any structure of cardinality $\kappa \geq \omega_1$ in a countable vocabulary, then the first order theory of $\mathcal{M}$ is either of the two types:

Structure Case All uncountable models elementary equivalent to $\mathcal{M}$ can be characterized in terms of dimension-like invariants.

Non-structure Case In every uncountable cardinality there are non-isomorphic models elementary equivalent to $\mathcal{M}$ that are extremely difficult to distinguish from each other by means of invariants.

The game-theoretic methods we have developed in this book help us to analyze further the non-structure case. For this we need to develop some basic topology of $\mathcal{N}_\kappa$. A set $A \subseteq \mathcal{N}_\kappa$ is *dense* if A meets every non-empty open set. The space $\mathcal{N}_\kappa$ has a dense subset of size $\kappa^{<\kappa}$ consisting of all eventually constant functions. If the *Generalized Continuum Hypothesis GCH* is assumed, then $\kappa^{<\kappa} = \kappa$ for all regular κ and $\kappa^{<\kappa} = \kappa^+$ for singular κ.

Theorem 9.87 (Baire Category Theorem) *Suppose A_α, $\alpha < \kappa$, are dense open subsets of $\mathcal{N}_\kappa$. Then $\bigcap_\alpha A_\alpha$ is dense.*

Proof Let $f_0 \in \mathcal{N}_\kappa$ and $\alpha_0 < \kappa$ be arbitrary. If f_ξ and α_ξ for $\xi < \beta$ have been defined so that

$$\alpha_\zeta < \alpha_\xi \text{ and } f_\xi \in N(f_\zeta, \alpha_\zeta)$$

for $\zeta < \xi < \beta$, then we define f_β and α_β as follows: Choose some $g \in \mathcal{N}_\kappa$ such that $g \in N(f_\xi, \alpha_\xi)$ for all $\xi < \beta$ and let $\alpha_\beta = \sup_{\xi<\beta} \alpha_\xi$. Since A_β is dense, there is $f_\beta \in A_\beta \cap N(g, \alpha_\beta)$. When all f_ξ and α_ξ for $\xi < \kappa$ have been defined, we let f be such that $f \in N(f_\xi, \alpha_\xi)$ for all $\xi < \kappa$. Then $f \in \bigcap_\alpha A_\alpha \cap N(f_0, \alpha_0)$. $\square$

Definition 9.88 A subset A of $\mathcal{N}_\kappa$ is said to be $\mathbf{\Sigma}^1_1$ (or *analytic*) if it is a projection of a closed subset of $\mathcal{N}_\kappa \times \mathcal{N}_\kappa$. A set is $\mathbf{\Pi}^1_1$ (or *co-analytic*) if its complement is analytic. Finally, a set is $\mathbf{\Delta}^1_1$ if it is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.

Example 9.89 Examples of analytic sets relevant if κ is a regular cardinal $> \omega$, are

$$\mathrm{CUB}_\kappa = \{f \in \mathcal{N}_\kappa : \{\alpha < \kappa : f(\alpha) = 0\} \text{ contains a club}\}$$

and

$$\mathrm{NS}_\kappa = \{f \in \mathcal{N}_\kappa : \{\alpha < \kappa : f(\alpha) \neq 0\} \text{ contains a club}\}.$$

The set of α-sequences of elements of κ for various $\alpha < \kappa$ form a tree $\mathcal{N}_{<\kappa}$ under the subsequence relation. Any subset T of $\mathcal{N}_{<\kappa}$ which is closed under subsequences is called a *tree* in this section. A κ-branch of such a tree is any linear subtree (branch) of height κ. Let us denote $\langle g(\beta) : \beta < \alpha\rangle$ by $\bar{g}(\alpha)$.

Lemma 9.90 *A set $A \subseteq \mathcal{N}_\kappa$ is analytic iff there is a tree $T \subseteq \mathcal{N}_{<\kappa} \times \mathcal{N}_{<\kappa}$ such that for all f:*

$$f \in A \iff T(f) \textit{ has a } \kappa\textit{-branch}, \tag{9.7}$$

where $T(f) = \{\bar{g}(\alpha) : (\bar{g}(\alpha), \bar{f}(\alpha)) \in T\}$. Such a tree is called a tree representation *of A.*

Proof Suppose first A is analytic and $B \subseteq \kappa^\kappa \times \kappa^\kappa$ is a closed set such that

$$f \in A \iff \exists g((f, g) \in B).$$

Let

$$T = \{(\bar{f}(\alpha), \bar{g}(\alpha)) : (f, g) \in B, \alpha < \kappa\}.$$

Clearly now $f \in A$ if and only if $T(f)$ has a κ-branch. Conversely, suppose such a T exists. Let B be the set of (f, g) such that $(\bar{f}(\alpha), \bar{g}(\alpha)) \in T$ for all $\alpha < \kappa$. The set B is closed and its projection is A. $\square$

Respectively, a set is co-analytic if and only if there is a tree $T \subseteq \mathcal{N}_{<\kappa} \times \mathcal{N}_{<\kappa}$ such that for all f:

$$f \in A \iff T(f) \text{ has no } \kappa\text{-branches}, \tag{9.8}$$

Let $\mathcal{T}_\kappa$ denote the class of all trees without κ-branches. Let $\mathcal{T}_{\lambda,\kappa}$ denote the set of subtrees of $\lambda^{<\kappa}$ of cardinality $\leq \lambda$ without any κ-branches.

Proposition 9.91 *Suppose B is a co-analytic subset of $\mathcal{N}_\kappa$ and T is as in (9.8). For any tree $S \in \mathcal{T}_\kappa$ let*

$$B_S = \{f \in B : T(f) \le S\}.$$

Then

$$B = \bigcup_{S \in \mathcal{T}_{\lambda,\kappa}} B_S,$$

where $\lambda = \kappa^{<\kappa}$.

Proof Clearly $B_S \subseteq B$ if $S \in \mathcal{T}_\kappa$. Conversely, suppose $f \in B$. Then of course $f \in B_{T(f)}$. It remains to observe that $|T(f)| \le \kappa^{<\kappa}$. □

Suppose $A \subseteq B$ is analytic and S is a tree as in (9.7). Let

$$T' = \{(\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha)) : \bar{g}(\alpha) \in T(f), \bar{h}(\alpha) \in S(f)\}. \tag{9.9}$$

Note that $|T'| \le \kappa^{<\kappa}$ and T' has no κ-branches, for such a branch would give rise to a triple (f, g, h) which would satisfy $f \in A \setminus B$. Note also that if $f \in A$, then there is a κ-branch $\{\bar{h}(\alpha) : \alpha < \kappa\}$ in $S(f)$, and hence the mapping

$$\bar{g}(\alpha) \mapsto (\bar{f}(\alpha), \bar{g}(\alpha), \bar{h}(\alpha))$$

witnesses

$$T(f) \le T'.$$

We have proved:

Proposition 9.92 (Covering Theorem for $\mathcal{N}_\kappa$) *Suppose B is a co-analytic subset of $\mathcal{N}_\kappa$ and S is as in (9.8). Suppose $A \subseteq B$ is analytic. Then*

$$A \subseteq B_T$$

for some $T \in \mathcal{T}_{\lambda,\kappa}$, where $\lambda = \kappa^{<\kappa}$.

The idea is that the sets B_T, $T \in \mathcal{T}_{\lambda,\kappa}$ cover the co-analytic set B completely, and moreover any analytic subset of B can be already covered by a single B_T. Especially if B happens to be $\mathbf{\Delta}^1_1$, then there is $T \in \mathcal{T}_{\lambda,\kappa}$ such that $B = B_T$.

Corollary (Souslin-Kleene Theorem for $\mathcal{N}_\kappa$) *Suppose B is a $\mathbf{\Delta}^1_1$ subset of $\mathcal{N}_\kappa$. Then*

$$B = B_T$$

for some $T \in \mathcal{T}_{\lambda,\kappa}$, where $\lambda = \kappa^{<\kappa}$.

Corollary (Luzin Separation Theorem for $\mathcal{N}_\kappa$) *Suppose A and B are disjoint analytic subsets of $\mathcal{N}_\kappa$. Then there is a set of the form C_T for some co-analytic set C and some $T \in \mathcal{T}_{\lambda,\kappa}$, where $\lambda = \kappa^{<\kappa}$, that separates A and B, i.e. $A \subseteq C$ and $C \cap B = \emptyset$.*

In the case of classical descriptive set theory, which corresponds to assuming $\kappa = \omega$, the sets B_T are Borel sets. If we assume CH, then CUB and NS cannot be separated by a Borel set.

Proposition 9.93 *If $\kappa^{<\kappa} = \kappa$, then the sets B_T are analytic. If in addition T is a strong bottleneck, then B_T is $\mathbf{\Delta}^1_1$.*

Let us call a family $\mathcal{B}$ of elements of $\mathcal{T}_{\lambda,\kappa}$ *universal* if for every $T \in \mathcal{T}$ there is some $S \in \mathcal{B}$ such that $T \leq S$. If $\mathcal{T}_{\lambda,\kappa}$ has a universal family of size μ, and $\kappa^{<\kappa} = \kappa$, then by the above results every co-analytic set in $\mathcal{N}_\kappa$ is the union of μ analytic sets. By results in [MV93] it is consistent relative to the consistency of ZFC that $\mathcal{T}_{\kappa^+}$, $2^\kappa = \kappa^+$, has a universal family of size κ^{++} while $2^{\kappa^+} = \kappa^{+++}$.

Definition 9.94 The class of *Borel* subsets of $\mathcal{N}_\kappa$ is the smallest class containing the open sets and the closed sets which is closed under unions and intersections of length κ.

Note that every closed set in $\mathcal{N}_\kappa$ is the union of $\kappa^{<\kappa}$ open sets (Exercise 9.57). So if $\kappa^{<\kappa} = \kappa$, then the definition of Borelness can be simplified.

Theorem 9.95 *Assume $\kappa^{<\kappa} = \kappa > \omega$. Then $\mathcal{N}_\kappa$ has two disjoint analytic sets that cannot be separated by Borel sets.*

Proof Note that κ is a regular cardinal. Every Borel set A has a "Borel code" c such that $A = B_c$. Let us suppose $A = B_c$ separates the disjoint analytic sets $\mathcal{CUB}_\kappa$ and NS_κ defined in Example 9.89. For example, $\mathcal{CUB} \subseteq A$ and $A \cap \text{NS}_\kappa = \emptyset$. Let $\mathcal{P} = (2^{<\kappa}, \leq)$ be the Cohen forcing for adding a generic subset for κ. Let G be $\mathcal{P}$-generic and $g = \bigcup \mathcal{P}$. Now either $g \in A$ or $g \notin A$. Let us assume, w.l.o.g., that $g \in A$. Let $p \Vdash \check{g} \in B_{\check{c}}$. Let $M \prec (H(\mu), \in, <^*)$ for a large μ such that $\kappa, p, \mathcal{P}, TC(c) \in M$, $M^{<\kappa} \subseteq M$, and $<^*$ is a well-order of $H(\mu)$. Since $\kappa^{<\kappa} = \kappa > \omega$, we may also assume $|M| = \kappa$. Since $\mathcal{P}$ is $< \kappa$-closed, it is easy to construct a $\mathcal{P}$-generic G' over M in V such that

$$\{\alpha < \kappa : M \models \text{"}(\check{g})_{G'}(\alpha) \neq 0\text{"}\} \text{ contains a club.} \tag{9.10}$$

It is easy to show that $B_c = (B_{\check{c}})_{G'}$. Since

$$M \models \text{"}p \Vdash \check{g} \in B_{\check{c}}\text{"},$$

whence $(\check{g})_{G'} \in B_c$ and therefore $(\check{g})_{G'} \notin \text{NS}_\kappa$. This contradicts (9.10). □

Example 9.96 Suppose $\mathcal{M}$ is a structure with $M = \kappa$. We call the analytic set

$$\{\mathcal{N} : N = \kappa \text{ and } \mathcal{N} \cong \mathcal{M}\}$$

the *orbit* of $\mathcal{M}$. Let $\mathcal{N} \not\cong \mathcal{M}$. Now player **I** has an obvious winning strategy ρ in $\mathrm{EF}_\kappa(\mathcal{M}, \mathcal{N})$: he simply makes sure that all elements of both models are played. Obviously there are many ways to play all the elements but any of them will do. Let us consider the co-anaytic set $B = \{f_\mathcal{N} : N = \kappa \text{ and } \mathcal{N} \not\cong \mathcal{M}\}$. Let $S(\mathcal{N})$ be the Scott tree S_ρ of the pair $(\mathcal{M}, \mathcal{N})$. Let us choose a tree representation T of B in such a way that for all $\mathcal{N}$ with $N = \kappa$, $T(f_\mathcal{N}) = S(\mathcal{N})$. If now $f_\mathcal{N} \in B_{T'}$, then player **I** wins $\mathrm{EF}_{T'}(\mathcal{M}, \mathcal{N})$.

Recall that if $\mathcal{M}$ is a countable structure and α is the Scott height of $\mathcal{M}$, then **I** wins $\mathrm{EFD}_{\alpha+\omega}(\mathcal{M}, \mathcal{N})$ whenever $\mathcal{M} \not\cong \mathcal{N}$ and N is countable. Equivalently, using the notation of Example 9.54, player **I** wins $\mathrm{EF}_{B_{\alpha+\omega}}(\mathcal{M}, \mathcal{N})$ whenever $\mathcal{M} \not\cong \mathcal{N}$ and N is countable. We now generalize this property of $B_{\alpha+\omega}$ to uncountable structures.

Definition 9.97 Suppose κ is an infinite cardinal and $\mathcal{M}$ is a structure of cardinality κ. A tree T is a *universal Scott tree* of a structure $\mathcal{M}$ if T has no branches of length κ and player **I** wins $\mathrm{EF}_{\sigma T}(\mathcal{M}, \mathcal{N})$ whenever $\mathcal{M} \not\cong \mathcal{N}$ and $|N| = |M|$.

The idea of the universal Scott tree is that the tree T alone suffices as a clock for player **I** to win all the 2^κ different games $\mathrm{EF}_T(\mathcal{M}, \mathcal{N})$ where $\mathcal{M} \not\cong \mathcal{N}$ and $|N| = |M|$. Universal Scott trees exist: there is always a universal Scott tree of cardinality $\leq 2^\kappa$ as we can put the various Scott trees of the pairs $(\mathcal{M}, \mathcal{N})$, $\mathcal{M} \not\cong \mathcal{N}$, $|M| = |N|$, each of them of the size $\leq \kappa^{<\kappa}$, together into one tree. So the question is: How small universal Scott trees does a given structure have?

If $\kappa^{<\kappa} = \lambda$ and $\mathcal{T}_{\lambda,\kappa}$ has a universal family of size μ, then every structure of size κ has a universal Scott tree of size μ.

If we allowed T to have a branch of length κ, any such tree would be a universal Scott tree of any structure of cardinality κ.

We ask whether **I** wins $\mathrm{EF}_{\sigma T}(\mathcal{M}, \mathcal{N})$ rather than in $\mathrm{EF}_T(\mathcal{M}, \mathcal{N})$ in order to preserve the analogy with the concept of a Scott tree. A universal Scott tree T in our sense would give rise to a universal Scott tree σT in the latter sense. Note that $|\sigma T| = |T|^{<\kappa}$, so this is the order of magnitude of a difference in the size of universal Scott trees in the two possible definitions.

Proposition 9.98 *Suppose $\kappa^{<\kappa} = \kappa$ and $\mathcal{M}$ is a structure with $M = \kappa$. the following are equivalent:*

(1) *The orbit of $\mathcal{M}$ is $\mathbf{\Delta}^1_1$.*

(2) $\mathcal{M}$ has a universal Scott tree of cardinality κ.

Proof Suppose first (2) is true. Then

$$\mathcal{M} \not\cong \mathcal{N} \iff \text{player } \mathbf{I} \text{ wins } \mathrm{EF}_{\sigma T}(\mathcal{M}, \mathcal{N}).$$

The existence of a winning strategy of **I** can be written in $\mathbf{\Pi}^1_1$ form since we assume $\kappa^{<\kappa} = \kappa$. Assume then (1). Let ρ be a strategy of player **I** in $\mathrm{EF}_\kappa(\mathcal{M}, \mathcal{N})$ in which he simply enumerates the universes. Note that this is independent of $\mathcal{N}$. Let $S(\mathcal{N})$ be the Scott tree S_ρ of the pair $(\mathcal{M}, \mathcal{N})$. Let us consider the co-anaytic set $B = \{f_\mathcal{N} : N = \kappa \text{ and } \mathcal{N} \not\cong \mathcal{M}\}$. Let us choose a tree representation T of B as in Example 9.96. If now $f_\mathcal{N} \in B_{T'}$, then player **I** wins $\mathrm{EF}_{T'}(\mathcal{M}, \mathcal{N})$. By the above Souslin-Kleene theorem, (1) implies the existence of a tree T' such that $B = B'_T$. Thus for any $\mathcal{N}$ with $N = \kappa$, $\mathcal{M} \not\cong \mathcal{N}$ implies that player **I** wins $\mathrm{EF}_{T'}(\mathcal{M}, \mathcal{N})$. Thus T' is a universal Scott tree of $\mathcal{M}$. Moreover, $|T'| = \kappa^{<\kappa} = \kappa$. □

The question whether the orbit of $\mathcal{M}$ is $\mathbf{\Delta}^1_1$ is actually highly connected to stability-theoretic properties of the first order theory of $\mathcal{M}$, see [HT91] for more on this.

9.7 Historical Remarks and References

Excellent sources for stronger infinitary languages are the textbook [Dic75], the handbook chapter [Dic85] and the book chapter [Kue75]. The Ehrenfucht-Fraïssé game for the logics $L_{\infty\lambda}$ appeared in [Ben69] and [Cal72]. Proposition 9.32, Proposition 9.45 and the corollary of Proposition 9.45 are due to Chang [Cha68]. The concept of Definition 9.40 and its basic properties were isolated independently by Dickmann [Dic75] and Kueker [Kue75]. Theorem 9.31 is from [She78].

Looking at the origins of the transfinite Ehrenfeucht-Fraïssé Game, one can observe that the game plays a role in [She78], and is then systematically studied, first in the framework of back-and-forth sets in [Kar84], and then explicitly as a game in [Hyt87], [Hyt90], [HV90] and [Oik90].

The importance of trees in the study of the transfinite Ehrenfeucht-Fraïssé Game was first recognized in [Kar84] and [Hyt87]. The crucial property of trees, or more generally partial orders, is Lemma 9.55 part (ii), which goes back to Kurepa [Kur56]. A more systematic study of the quasi-order $\mathcal{P} \leq \mathcal{P}'$ of partial orders, with applications to games in mind, was started in [HV90], where Lemma 9.57, Definition 9.58, Lemma 9.59 and Lemma 9.60 originate. The important role of the concept of persistency (Definition 9.63) gradually emerged and was explicitly isolated and exploited in [Huu95]. Once it became clear that trees may be incomparable by $\leq$, the concept of bottleneck arose quite naturally. Definition 9.64 is from [TV99]. The relative consistency of the non-existence of non-trivial bottlenecks (Theorem 9.65) was proved in [MV93]. For more on the structure of trees see [TV99] and [DV04].

The point of studying trees in connection with the transfinite Ehrenfeucht-Fraïssé Game is that there are two very natural tree structures behind the game. The first tree that arises from the game is the tree of sequences of moves, as in Lemma 9.73. This tree originates in [Kar84]. The second, and in a sense more powerful tree is the tree of strategies of a player, as in Definition 9.69 and the subsequent Proposition 9.71. This idea originates from [Hyt87].

The "transfinite" analogues of Scott ranks are the Scott and Karp trees, introduced in [HV90]. Because of problems of incomparability of some trees, the picture of the "Scott watershed" is much more complicated than in the case of games of length ω, as one can see by comparing Figure 7.4 and Figure 9.7. Proposition 9.85 and Theorem 9.86 are from [Tuu90].

There is a form of infinitary logic the elementary equivalence of which corresponds exactly to the existence of winning strategy for $\mathbf{II}$ in EF_α, in the spirit of the Strategic Balance of Logic. These infinitary logics are called *infinitely deep languages*. Their formulas are like formulas of $L_{\kappa\lambda}$ but there are infinite descending chains of subformulas. Thus, if we think of the syntax of a formula as a tree, the tree may have transfinite rank. These languages were introduced in [HR76] and studied in [Kar79], [Ran81], [Kar84], [Hyt90] and [Tuu92]. See [Vää95] for a survey on the topic.

There is also a transfinite version of the Model Existence Game, the other leg of the Strategic Balance of Logic, with applications to undefinability of (generalized) well-order and Separation Theorems, see [Tuu92] and [Oik97].

It was recognized already in [She78] that the roots of the problem of extending the Scott Isomorphism Theorem to uncountable cardinalities lie in stability theoretic properties of the models in question. This was made explicit in the context of transfinite Ehrenfeucht-Fraïssé Games in [HT91]. It turns out that there is indeed a close connection between the structure of Scott and Karp trees of elementary equivalent uncountable models and the stability theoretic properties such as superstability, DOP and OTOP, of the (common) first order theory. For more on this, see [Hyt92], [HST93], and [HS99].

A good testing field for the power of long Ehrenfeucht-Fraïssé games turned out to be the area of almost free groups, where it seemed that the applicability of the infinitary languages $L_{\kappa\lambda}$ had been exhausted. For results in this direction, see [MO93], [EFS95], [SV02] and [Väi03].

An alternative to considering transfinite Ehrenfeucht-Fraïssé Games is to study isomorphism in a forcing extension. Isomorphism in a forcing extension is called potential isomorphism. The basic reference is [NS78]. See also [HHR04].

Early on it was recognized that the trees $T(S)$ (see Example 9.61) are very useful and in some sense fundamental in the area of transfinite Ehrenfeucht-Fraïssé Games. The question arose, whether there is a largest such tree for $S \subseteq \omega_1$ bistationary. Quite unexpectedly the existence of a largest such tree turned out to be consistent relative to the consistency of ZF. The name "Canary trees" was coined for them, because such a tree would indicate whether some stationary set was killed. See [MS93] and [HR01] for results on the Canary tree.

While the Ehrenfeucht-Fraïssé game of length ω is almost trivially determined, the Ehrenfeucht-Fraïssé game of length ω_1 (and also of length $\omega + 1$) can be non-determined, see [Hyt92], [MSV93] and [HSV02]. This has devastating consequences for attempts to use transfinite Ehrenfeucht-Fraïssé games to classify uncountable models. It is a phenomenon closely related to the incomparability of non-well-founded trees by the relation $\leq$. This non-determinism is ultimately also the reason why the simple picture Figure 7.4 becomes Figure 9.7.

Some of the complexities of uncountable models can be located already on the topological level, as is revealed by the study of the spaces $\mathcal{N}_\kappa$. These spaces were studied under the name of κ-metric spaces in [Sik50], [JW78] and [Tod81b]. Their role as spaces of models, in the spirit of [Vau73], was emphasized in [MV93]. For more on the topology of uncountable models, see [Vää91], [Vää95] and [SV00]. See [Vää08] for an informal exposition of some basic ideas. Theorem 9.95 is from [SV00].

Exercise 9.22 is from [NS78]. Exercises 9.29 and 9.30 are from [Hyt87]. Exercise 9.35 is from [HV90]. Exercise 9.40 is from [Kur56]. Exercise 9.41 is from [Huu95]. Exercise 9.47 is from [Tod81a]. Exercise 9.56 is due to Lauri Hella.

Exercises

9.1 Show that player **II** wins $EF_\omega^{\aleph_0}(\mathcal{M}, \mathcal{M}')$ if and only if she has a winning strategy in $EF_\omega(\mathcal{M}, \mathcal{M}')$.

9.2 Show that **I** wins $\mathrm{EFD}_2^{\omega_1}(\mathcal{M}, \mathcal{N})$ if $\mathcal{M} = (\mathbb{Q}, <)$ and $\mathcal{N} = (\mathbb{R}, <)$.

9.3 Show that in Example 9.2 player **I** has a winning strategy already in $\mathrm{EFD}_2^{\omega_1}(\mathcal{M}, \mathcal{M}')$.

9.4 Show that $\mathcal{M} \simeq_p \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are as in Example 9.4.

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10
Generalized Quantifiers

10.1 Introduction

First order logic is not able to express "there exists infinitely many x such that ..." nor "there exists uncountably many x such that ...". Also, if we restrict ourselves to finite models, first order logic is not able to express "there exists an even number of x such that ...". These are examples of new logical operations called *generalized quantifiers*. There are many others, such as the Magidor-Malitz quantifiers, cofinality quantifiers, stationary logic, and so on. We can extend first order logic by adding such new quantifiers. In the case of "there exists infinitely many x such that ..." the resulting logic is not axiomatizable, but in the case of "there exists uncountably many x such that ..." the new logic is indeed axiomatizable. The proof of the completeness theorem for this quantifier is non-trivial going well beyond the Completeness Theorem of first order logic.

10.2 Generalized Quantifiers

Generalized quantifiers occur everywhere in our language. Here are some examples:

Two thirds voted for John
Exactly half remains.
Most wanted to leave.
Some but not all liked it.
Between 10% and 20% were students.
Hardly anybody touched the cake.
The number of white balls **is even**.
There are infinitely many primes.
There are uncountably many reals.

These are instances of generalized quantifiers in natural language[1]. The mathematical study of quantifiers provides an exact framework in which such quantifiers can be investigated. An overall goal is to find *invariants* for such objects, that is, to classify them and find the characteristic properties of quantifiers in each class. Typical questions that we study are: which quantifier is "definable" in terms of another given quantifier, which quantifiers can be axiomatized, which satisfy the Compactness Theorem, etc. We start with a very general concept of a quantifier and then later we impose restrictions. Usually in the literature the generalized quantifiers are assumed to be what we call bijection closed (see Definition 10.16).

Definition 10.1 A *weak (generalized) quantifier* is a mapping Q which maps every non-empty set A to a subset of $\mathcal{P}(A)$. A *weak (generalized) quantifier on* a domain A is any subset of $\mathcal{P}(A)$.

Virtually all quantifiers we consider are quantifiers in the first sense, i.e. mappings $A \longmapsto Q(A)$. However, most actual results and examples are about a fixed given domain A, whence the concept of a quantifier *on* a domain. The domain is assumed to be a set.

The set-theoretic nature of a quantifier (as a mapping) is somewhat problematic. We cannot call a quantifier a function in the set-theoretical sense since its domain consists of all possible non-empty sets. However, this problem does not arise in practice. Our quantifiers are in general definable so we can treat them as classes. If we have to talk about all quantifiers, definable or not, we have to restrict ourselves to considering domains A contained in one sufficiently

[1] Quantifiers occurring in natural language are usually of a slightly more complex form, such as "Two thirds of the people voted for John", "Exactly half of the cake remains", "Most students wanted to leave", "Some but not all viewers liked it".

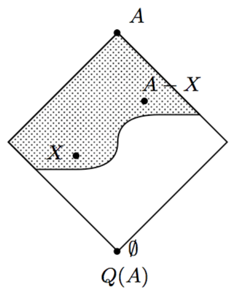

Figure 10.1 Generalized quantifier

big "monster domain". There are no considerations here that would make this necessary.

Example 10.2 1. The *existential quantifier* $\exists$ is the mapping

$$\exists(A) = \{X \subseteq A : X \neq \emptyset\}.$$

2. The *universal quantifier* $\forall$ is the mapping

$$\forall(A) = \{X \subseteq A : X = A\} = \{A\}.$$

3. The *counting quantifier* $\exists^{\geq n}$ is the mapping

$$\exists^{\geq n}(A) = \{X \subseteq A : |X| \geq n\},$$

where we assume n is a natural number.

4. The *infinity quantifier* $\exists^{\geq \omega}$ is the mapping

$$\exists^{\geq \omega}(A) = \{X \subseteq A : X \text{ is infinite}\}.$$

5. The *finiteness quantifier* $\exists^{<\omega}$ is the mapping

$$\exists^{<\omega}(A) = \{X \subseteq A : X \text{ is finite}\}.$$

6. The following subsets of $\mathcal{P}(\mathbb{N})$ are weak quantifiers on $\mathbb{N}$:

$[\{5\}] = \{X \subseteq \mathbb{N} : 5 \in X\}$
$[X_0] = \{X \subseteq \mathbb{N} : X_0 \subseteq X\}$, where $X_0 \subseteq \mathbb{N}$ is fixed
$[X_0]^* = \{X \subseteq \mathbb{N} : X_0 \cap X \neq \emptyset\}$, where $X_0 \subseteq \mathbb{N}$ is fixed.

We can draw pictures of quantifiers on a domain A by thinking of $\mathcal{P}(A)$ as a Boolean algebra under $\subseteq$, as in Figure 10.1.

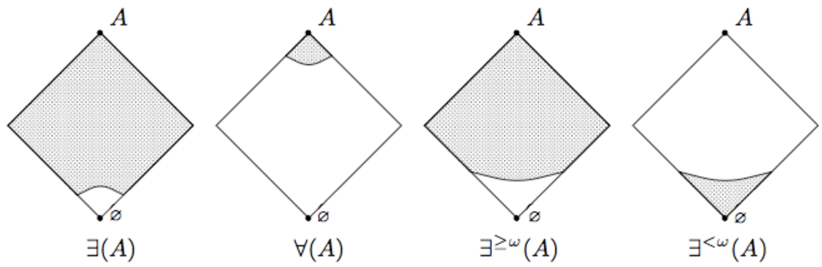

Figure 10.2 Some generalized quantifiers

The reason we call $\{X \subseteq A : X \neq \emptyset\}$ the existential quantifier on A is the following:

$$\begin{aligned} \mathcal{A} \models \exists x \varphi(x) &\iff \{a \in A : \mathcal{A} \models \varphi(a)\} \neq \emptyset \\ &\iff \{a \in A : \mathcal{A} \models \varphi(a)\} \in \exists(A). \end{aligned}$$

Respectively

$$\begin{aligned} \mathcal{A} \models \forall x \varphi(x) &\iff \{a \in A : \mathcal{A} \models \varphi(a)\} = A \\ &\iff \{a \in A : \mathcal{A} \models \varphi(a)\} \in \forall(A). \end{aligned}$$

Later we will associate with every quantifier Q an extension of first order logic based on the above idea.

Some quantifiers make only sense in a *finite context*. By this we mean that only finite domains A are considered. If we allow countable domains too we work in a *countable context*.

Example 10.3 (Finite context) 1. The *even-cardinality quantifier* Q^{even} is the mapping (see Figure 10.3)

$$Q^{\text{even}}(A) = \{X \subseteq A : |X| \text{ is even}\}.$$

Similarly

$$Q^{D}(A) = \{X \subseteq A : |X| \in D\} \text{ for any } D \subseteq \mathbb{N}.$$

2. The *at-least-one-half quantifier* $\exists^{\geq\frac{1}{2}}$ is the mapping (see Figure 10.4)

$$\exists^{\geq\frac{1}{2}}(A) = \{X \subseteq A : |X| \geq |A|/2\}.$$

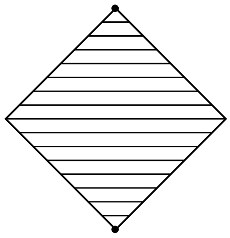

Figure 10.3 Even cardinality quantifier

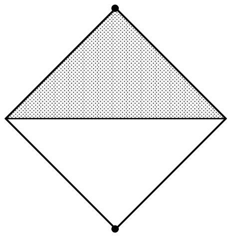

Figure 10.4 At-least-one-half quantifier

The quantifier $\exists^{\geq r}$ is defined similarly

$$\exists^{\geq r} = \{X \subseteq A : |X| \geq r \cdot |A|\} \text{ for any real } r \in [0, 1].$$

Thus at-least-two-thirds would be the quantifier $\exists^{\geq \frac{2}{3}}$. It is obvious how to define the quantifier less-than-two-third, in symbols $\exists^{< \frac{2}{3}}$, and more generally $\exists^{<r}$, $\exists^{\leq r}$ and $\exists^{>r}$.

3. The *most quantifier* $\exists^{\mathrm{most}}$ is the mapping

$$\exists^{\mathrm{most}}(A) = \{X \subseteq A : |X| > |A - X|\}.$$

We can define Boolean operations for weak quantifiers in a natural way:

$$(Q \cap Q')(A) = Q(A) \cap Q'(A)$$
$$(Q \cup Q')(A) = Q(A) \cup Q'(A)$$
$$(-Q)(A) = \{X \subseteq A : X \notin Q(A)\}.$$

These operations obey familiar laws of Boolean algebras, such as idempotency, commutativity, associativity, distributivity and the de Morgan laws:

$$-(Q \cap Q') = -Q \cup -Q'$$
$$-(Q \cup Q') = -Q \cap -Q'.$$

The quantifier $-Q$ is called the *complement* of Q. There is also another kind of complement of a quantifier, the quantifier

$$(Q-)(A) = \{A \setminus X : X \in Q(A)\}$$

called the *postcomplement* of Q.

Example 10.4 The complement of "everybody" is "not everybody", while the postcomplement of "everybody" is "nobody". The complement of $\exists^{\geq\frac{2}{3}}$ is $\exists^{<\frac{2}{3}}$, while the postcomplement of $\exists^{\geq\frac{2}{3}}$ is $\exists^{<\frac{1}{3}}$.

The postcomplement satisfies $(Q-)- = Q$, but does not obey the de Morgan laws. Rather:

$$(Q \cap Q')- = (Q'-) \cap (Q-)$$
$$(Q \cup Q')- = (Q'-) \cup (Q-)$$

Note that complement and postcomplement obey the following associativity law:

$$(-Q)- = -(Q-).$$

Thus we may leave out parentheses and write simply $-Q-$. The existential and the universal quantifier have a special relationship called *duality*, exemplified by the equation

$$\exists = -\forall- \text{ and } \forall = -\exists- .$$

Duality is an important phenomenon among quantifiers and gives rise to the following definition:

Definition 10.5 The *dual* of a weak quantifier Q is the quantifier

$$\breve{Q} = -Q-,$$

that is, the mapping

$$\breve{Q}(A) = \{X \subseteq A : A - X \notin Q(A)\}.$$

The dual of a weak quantifier on a domain is defined in the same way. (See Figure 10.5)

Example 10.6 1. The dual of $\exists$ is $\forall$ and vice versa: the dual of $\forall$ is $\exists$.
2. The dual of $\exists^{\geq\omega}$ is the quantifier *all-but-finite*

$$\forall^{<\omega}(A) = \{X : |A - X| \text{ is finite}\} = (\exists^{<\omega})-$$

and vice versa: the dual of $\forall^{<\omega}$ is the quantifier $\exists^{\geq\omega}$.

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Note that $Q^{\mathrm{cf}}_{\omega}(M)$ is by no means monotone, so it is a quite different object from what we are used to.

A *weak cofinality model* is a pair $(\mathcal{M}, Q)$, where $\mathcal{M}$ is an ordinary model and $Q \subseteq \mathcal{P}(M \times M)$. Likewise, we can add a new quantifier symbol $\mathcal{Q}$ to $L_{\omega\omega}$ and define

$$(\mathcal{M}, Q) \models_s \mathcal{Q}xy\varphi(x,y) \iff \{(a,b) : (\mathcal{M}, Q) \models_{s[a/x,b/y]} \varphi\} \in Q.$$

What kind of axioms should $\varphi \in L_{\omega\omega}(\mathcal{Q})$ be consistent with in order to have a model of the form $(\mathcal{M}, Q^{\mathrm{cf}}_{\omega})$? We have some obvious candidates such as

(LO) $\mathcal{Q}xy\varphi(x,y) \to \mathcal{Q}^{\mathrm{LO}}xy\varphi(x,y)$, where

$$\begin{array}{rcl} \mathcal{Q}^{\mathrm{LO}}xy\varphi(x,y) & = & \forall x \neg\varphi(x,x) \wedge \\ & & \forall x \forall y \forall z((\varphi(x,y) \wedge \varphi(y,z)) \to \varphi(x,z)) \wedge \\ & & \forall x \forall y(\varphi(x,y) \vee \varphi(y,x) \vee \approx xy) \end{array}$$

and

(NLE) $\mathcal{Q}xy\varphi(x,y) \to \forall x \exists y \varphi(x,y)$.

Let us define

$$\mathcal{Q}^{*}xy\varphi(x,y) = \mathcal{Q}^{\mathrm{LO}}xy\varphi(x,y) \wedge \forall x \exists y \varphi(x,y) \wedge \neg \mathcal{Q}xy\varphi(x,y).$$

Thus $\mathcal{Q}^{*}xy\varphi(x,y)$ "says" that φ is a linear order without last element but the cofinality is not ω. So it is a formalization of

$$Q^{\mathrm{cf}}_{>\omega}(M) = \{R \subseteq M \times M : \ R \text{ is a linear order of } M \text{ with cofinality} > \omega\}.$$

Let us make some observations about the case

$$R \in Q^{\mathrm{cf}}_{\omega}(M) \quad \& \quad S \in Q^{\mathrm{cf}}_{>\omega}(M). \tag{10.15}$$

First of all we may observe that there is no order-preserving mapping

$$f : (M, <_S) \to (M, <_R)$$

$$x <_S y \to f(x) <_R f(y)$$

whose range is cofinal in $<_S$. Why? Suppose f is one such. Let $a_0 <_R a_1 <_R \cdots$ be cofinal in $<_R$. We define $b_0 <_S b_1 <_S \cdots$ as follows. If $n = 0$ or b_{n-1} is defined let $a_n <_R z = f(n)$. Let $b_{n-1} <_S b_n$ be such that also $n <_S b_n$. Now b_n is defined. Let b be such that $b_n <_S b$ for all n (remember that $S \in Q^{\mathrm{cf}}_{>\omega}(M)$). Let n be such that $f(b) <_R a_n$. Then

$$a_n <_R f(b_n) <_R f(b) <_R a_n,$$

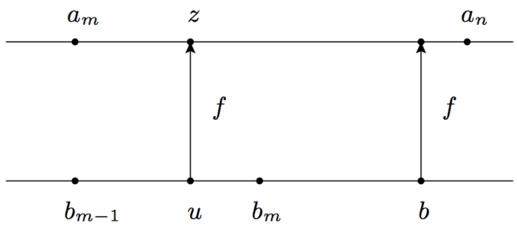

Figure 10.23

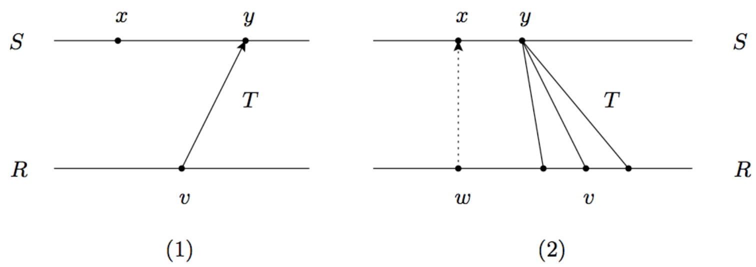

Figure 10.24

a contradiction.

We use now a similar inference but with a relation instead of a function:

Lemma 10.81 *If (10.15) holds, then there is* no *relation* $T \subseteq M \times M$ *such that*

(1) $\forall x \exists y >_S x \exists v (vTy)$
(2) $\forall w \exists x \forall y >_S x \forall v (vTy \rightarrow w <_R v)$.

Proof Let $a_0 <_R a_1 <_R \cdots$ be cofinal in $<_R$. We define $b_0 <_S b_1 <_S \cdots$ as follows. If $n = 0$ b_0 is arbitrary. If b_{n-1} is defined choose (by (1)) some $y_n >_S b_n$ and v_n such that $v_n T y_n$. Use (2) to find x such that for all $y >_S x$ and all v, vTy implies $\max(a_n, v_n) <_R v$. Let $b_{n+1} >_S x$ and $v_{n+1} T b_{n+1}$. By (10.15) there is b such that $b_n <_S b$ for all $n \in \mathbb{N}$. By (1) there is $y >_S b$ and vTy. For some n $a_n <_R v$. This is a contradiction. □

Shelah's Axiom is

I	**II**	Explanation
$\mathcal{Q}xy\varphi(x,y)$		a played formula
	$\mathcal{Q}^{\mathrm{LO}}xy\varphi(x,y)\wedge$ $\forall x\exists y\varphi(x,y)$	
$\mathcal{Q}^{*}xy\varphi(x,y)$		a played formula
	$\mathcal{Q}^{\mathrm{LO}}xy\varphi(x,y)\wedge$ $\forall x\exists y\varphi(x,y)$	
$\mathcal{Q}xy\varphi(x,y)$ $\mathcal{Q}^{*}xy\psi(x,y)$		played formulas
	$\varphi(c,d)$ $\neg\psi(c,d)$ $\neg\varphi(c,d)$ $\psi(c,d)$	or
$\varphi\vee\neg\varphi$		φ any sentence
	φ $\neg\varphi$	or

Theorem 10.83 (Model Existence Theorem for Cofinality Logic) *Suppose L is a vocabulary of cardinality $\leq\kappa$ and T is a set of L-sentences of $L_{\omega\omega}(\mathcal{Q})$. TFAE*

(1) *T has a model $(\mathcal{M},Q)$ satisfying (LO)+(NLE).*
(2) *Player* **II** *has a winning strategy in* $\mathrm{MEG}_{\kappa}^{\mathcal{Q},\mathrm{cf}}(T,L)$.

Proof If $(\mathcal{M},Q)\models$ (T) + (LO) +(NLE), then clearly (2) holds. Conversely, suppose (2) holds. We let Player **I** play the obvious enumeration strategy. Let H be the set of responses of **II**, using her winning strategy. By construction, H gives rise to a model of (T) + (LO) +(NLE). Now the details: Let H be the set of responses of **II**, using her winning strategy, to a maximal play of **I**. Let $\mathcal{M}$ be defined from H as before. We define a weak cofinality quantifier Q on M as follows:

$$Q=\{\{([c],[d]):\varphi(c,d)\in H\}:\mathcal{Q}xy\varphi(x,y)\in H\}.$$

Now we show $(\mathcal{M},Q)\models T$ by proving the following claim. By our previous work we have

1. $\approx tt \in H$
2. If $\varphi(c) \in H$ and $\approx ct \in H$ then $\varphi(t) \in H$
3. If $\varphi \wedge \psi \in H$, then $\varphi \in H$ and $\psi \in H$
4. If $\varphi \vee \psi \in H$, then $\varphi \in H$ and $\psi \in H$
5. If $\forall x\varphi(x) \in H$, then $\varphi(c) \in H$ for all $c \in C$
6. If $\exists x\varphi(x) \in H$, then $\varphi(c) \in H$ for some $c \in C$.

Now we can note further

7. If $\varphi \notin H$, then $\neg\varphi \in H$ ($\neg\varphi$ has to be written in NNF).

The reason for 7 is simply that **I** can play $\varphi \vee \neg\varphi$ whenever he wants.

Claim

$$\varphi \in H \iff M \models \varphi$$

Proof Note that:

- If $\varphi \in H$ and $\psi \in H$, then $\varphi \wedge \psi \in H$ for otherwise $\neg(\varphi \wedge \psi) \in H$ whence $\neg\varphi \in H$ or $\neg\psi \in H$. This is not possible as then $M \models \varphi \wedge \neg\varphi$ or $M \models \psi \wedge \neg\psi$.
- If $\varphi \in H$ or $\psi \in H$, then $\varphi \vee \psi \in H$ for otherwise $\neg\varphi \in H$ and $\neg\psi \in H$.
- If $\varphi(c) \in H$ for all $c \in C$, then $\forall x\varphi(x) \in H$ for otherwise $\neg\forall x\varphi(x)$, which in NNF is $\exists x\neg\varphi(x)$ is in H, leading to the conclusion that $M \models \varphi(c) \wedge \neg\varphi(c)$ for some c.
- If $\varphi(c) \in H$ for some $c \in C$, then $\exists x\varphi(x) \in H$ for otherwise $\neg\exists x\varphi(x) \in H$, leading to a contradiction.
- If $\mathcal{Q}xy\varphi(x,y) \in H$, then $M \models \mathcal{Q}xy\varphi(x,y)$, for let $R = \{([c],[d]) : M \models \varphi(c,d)\}$. By the induction hypothesis

  $$R = \{([c],[d]) : \varphi(c,d) \in H\}.$$

  By construction, $R \in Q(M)$.
- If $\mathcal{Q}^*xy\varphi(x,y) \in H$, then $M \models \mathcal{Q}^*xy\varphi(x,y)$, for let $R = \{([c],[d]) : M \models \varphi(c,d)\}$. As above, $R = \{([c],[d]) : \varphi(c,d) \in H\}$. By construction, R is a linear order without last element. If $M \models \mathcal{Q}xy\varphi(x,y)$, then $R = \{([c],[d]) : \psi(c,d) \in H\}$ for some ψ such that $\mathcal{Q}xy\psi(x,y) \in H$. By the rules of the game, there are c and d such that $\varphi(c,d) \in H \nleftrightarrow \psi(c,d) \in H$, contrary to the choice of ψ.
- If $M \models \mathcal{Q}xy\varphi(x,y)$ then $\mathcal{Q}xy\varphi(x,y) \in H$, for otherwise $\neg\mathcal{Q}xy\varphi(x,y) \in H$. By induction hypothesis, $M \models \mathcal{Q}xy\varphi(x,y)$ implies

  $$\begin{aligned} R &= \{([c],[d]) : \varphi(c,d) \in H\} \\ &= \{([c],[d]) : M \models \varphi(c,d)\} \end{aligned}$$

is a linear order without last element and $\mathcal{Q}^{\mathrm{LO}}xy\varphi(x,y) \wedge \forall x \exists y \varphi(x,y) \in H$. If $\mathcal{Q}^*xy\varphi(x,y) \in H$, 9 leads to a contradiction. Hence $\neg\mathcal{Q}^*xy\varphi(x,y) \in H$, whence $\mathcal{Q}xy\varphi(x,y) \in H$.

- If $M \models \mathcal{Q}^*xy\varphi(x,y)$, then $\mathcal{Q}^*xy\varphi(x,y) \in H$, for otherwise $\neg\mathcal{Q}^*xy\varphi(x,y) \in H$. Since $\mathcal{Q}^{\mathrm{LO}}xy\varphi(x,y) \wedge \forall x \exists y \varphi(x,y) \in H$, we have $\mathcal{Q}xy\varphi(x,y) \in H$. By 8, $M \models \mathcal{Q}xy\varphi(x,y)$, a contradiction.

□

Theorem 10.84 (Weak Compactness of Cofinality Logic) *If T is a set of sentences of $L_{\omega\omega}(\mathcal{Q})$ and every finite subset has a weak cofinality model satisfying (LO) + (NLE), then so does the whole T.*

Proof As in Theorem 10.63. □

Theorem 10.85 (Weak Omitting Types Theorem of Cofinality Logic) *Assume κ is an infinite cardinal. Let L be a vocabulary of cardinality $\leq \kappa$, T an $L_{\omega\omega}(\mathcal{Q})$-theory and for each $\xi < \kappa$, Γ_ξ is a set $\{\varphi^\xi_\alpha(x) : \alpha < \kappa\}$ of $L_{\omega\omega}(\mathcal{Q})$-formulas in the vocabulary L. Assume that*

1. *If $\alpha \leq \beta < \kappa$, then $T \vdash \varphi^\xi_\beta(x) \rightarrow \varphi^\xi_\alpha(x)$.*
2. *For every $L_{\omega\omega}(\mathcal{Q})$-formula $\psi(x)$, for which $T \cup \{\psi(x)\}$ is consistent, and for every $\xi < \kappa$, there is an $\alpha < \kappa$ such that $T \cup \{\psi(x)\} \cup \{\neg\varphi^\xi_\alpha(x)\}$ is consistent.*

Then T has a weak cofinality model which omits Γ.

Proof As in Theorem 6.62. □

Definition 10.86 The *union* of an elementary chain $(\mathcal{M}_\alpha, Q_\alpha)$ of weak cofinality models of (LO) + (NLE) is $(\mathcal{M}, Q)$, where $\mathcal{M} = \bigcup_\alpha \mathcal{M}_\alpha$ and

$$Q = \{R \subseteq M \times M : R \text{ is a linear order without last element and there is } \alpha < \kappa \text{ such that } R \cap (M_\beta \times M_\beta) \in Q_\beta \text{ for all } \beta \geq \alpha\}.$$

Lemma 10.87 (Union Lemma) *The union of an elementary chain is an elementary extension of each member of the chain.*

Proof We will do only the case of $\mathcal{Q}xy\varphi(x,y)$. Suppose first $(\mathcal{M}, Q) \models_s \mathcal{Q}xy\psi(x,y)$, where s is an assignment into M_α. Then $R \in Q$ where for $a, b \in M$

$$aRb \longleftrightarrow (\mathcal{M}, Q) \models_{s[a/x,b/y]} \psi(x,y).$$

By definition there is $\beta \geq \alpha$ such that $R \cap (M_\gamma \times M_\gamma) \in Q_\gamma$ for $\gamma \geq \beta$. By the induction hypothesis, for $a, b \in M_\gamma$

$$aRb \longleftrightarrow (\mathcal{M}_\gamma, Q_\gamma) \models_{s[a/x,b/y]} \psi(x,y).$$

i.e.

$$(\mathcal{M}_\gamma, Q_\gamma) \models_s \mathcal{Q}xy\psi(x,y).$$

By assumption $(\mathcal{M}_\alpha, Q_\alpha) \models_s \mathcal{Q}xy\psi(x,y)$. Conversely, suppose $(\mathcal{M}, Q) \not\models_s \mathcal{Q}xy\psi(x,y)$. Then for all $\beta \geq \alpha$: $R \cap (M_\beta \times M_\beta) \notin Q_\beta$ where for $a, b \in M$

$$aRb \iff (\mathcal{M}, Q) \models_{s[a/x,b/y]} \psi(x,y).$$

By the induction hypothesis for $a, b \in M_\beta$

$$aRb \iff (\mathcal{M}_\beta, Q_\beta) \models_{s[a/x,b/y]} \psi(x,y)$$

i.e.

$$(\mathcal{M}_\beta, Q_\beta) \not\models_s \mathcal{Q}xy\varphi(x,y)$$

and hence $(\mathcal{M}_\alpha, Q_\alpha) \not\models_s \mathcal{Q}xy\varphi(x,y)$. □

Lemma 10.88 *For every infinite weak cofinality model $(\mathcal{M}, Q)$ and every $\kappa \geq |M|$ there is $(\mathcal{M}', Q')$ such that $(\mathcal{M}, Q) \prec (\mathcal{M}', Q')$ and every linear order on $\mathcal{M}$, which has no last element and which is $L_{\omega\omega}(\mathcal{Q})$-definable on $\mathcal{M}$ with parameters, has cofinality κ.*

Proof See Exercise 10.109. □

Lemma 10.89 (Main Lemma) *Suppose $(\mathcal{M}, Q)$ is an infinite weak cofinality model of (SA) and $\varphi(x,y)$ is a formula of $L_{\omega\omega}(\mathcal{Q})$ such that $(\mathcal{M}^*, Q) \models \mathcal{Q}xy\varphi(x,y)$. Then there is a weak cofinality model $(\mathcal{M}', Q')$ such that*

(1) $(\mathcal{M}, Q) \prec (\mathcal{M}', Q')$
(2) *For some $b \in M' \setminus M$ we have $(\mathcal{M}', Q') \models \varphi(a,b)$ for all $a \in M$*
(3) *For every $\psi(x,y)$ such that $(\mathcal{M}, Q) \models \mathcal{Q}^*xy\psi(x,y)$ and every $d \in M'$ we have $(\mathcal{M}', Q') \models \psi(d,a)$ for some $a \in M$.*

Proof In the light of Lemma 10.88 may assume, without loss of generality, that the model $\mathcal{M}$ and the vocabulary L have an infinite cardinality κ, and every linear order on $\mathcal{M}$ which has no last element and which is $L_{\omega\omega}(\mathcal{Q})$-definable on $\mathcal{M}$ with parameters, has cofinality κ. Let c be a new constant symbol and T the theory

$$\{\theta : (\mathcal{M}^*, Q) \models \theta\} \cup \{\varphi(a,c) : a \in M\}.$$

The useful criterion, familiar from the proof of Lemma 10.78, is in this case very simple:

$$T \cup \{\theta(c)\} \text{ is consistent iff } (\mathcal{M}^*, Q) \models \forall x \exists y(\varphi(x,y) \wedge \theta(y)). \quad (10.16)$$

Proof of (10.16). Suppose $(\mathcal{M}^*, Q) \models \forall x \exists y(\varphi(x,y) \wedge \theta(y))$. Let $T_0 \subseteq T$

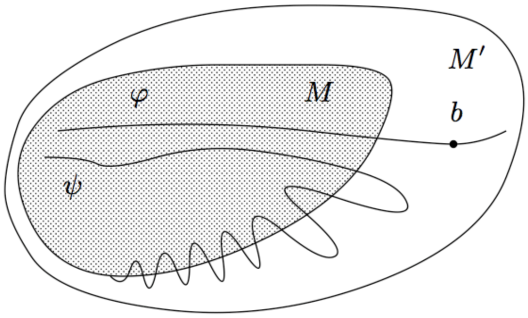

Figure 10.26

be finite. Let $a_0, \ldots, a_n$ be the constants occurring in T_0. Let b be φ-above every a_i in $\mathcal{M}$. By assumption there is d such that $(\mathcal{M}^*, Q) \models \varphi(b, d) \wedge \theta(d)$. Thus $T_0 \cup \{\theta(c)\}$ is consistent. Hence $T \cup \{\theta(c)\}$ is consistent. Conversely, suppose $(\mathcal{N}^*, Q') \models T \cup \{\theta(c)\}$. If $a \in M$, then $(\mathcal{N}^*, Q') \models \varphi(a, c) \wedge \theta(c)$, so $(\mathcal{M}^*, Q) \models \exists y(\varphi(a, y) \wedge \theta(y))$. □

Since $(\mathcal{M}^*, Q) \models \forall x \exists y(\varphi(x, y) \wedge \not\approx yy)$, we conclude that T itself is consistent. Let $\psi_\xi(x, y)$, $\xi < \kappa$, be a complete list of all formulas such that

$$(\mathcal{M}^*, Q) \models \mathcal{Q}^* xy\psi_\xi(x, y).$$

Let w^ξ_α, $\alpha < \kappa$, be a cofinal strictly ψ_ξ-increasing sequence in $\mathcal{M}$. Let for each ξ Γ_ξ be the type

$$\Gamma_\xi = \{\psi_n(w^\xi_\alpha, x) : \alpha < \kappa\}.$$

Thus Γ_n "says" that x is $<_{\psi_\xi}$-above every element of M. This is the situation we want to avoid, so we want to omit each type Γ_ξ. To prove using Theorem 10.85 that all the sets Γ_ξ can be simultaneously omitted suppose $\exists x\theta(x, c)$ is consistent with T. By (10.16)

$$(\mathcal{M}^*, Q) \models \forall x \exists y \exists v(\theta(v, y) \wedge \varphi(x, y)).$$

If there is no $\alpha < \kappa$ such that

$$\exists y(\theta(y, c) \wedge \neg\psi_\xi(w^\xi_\alpha, y))$$

is consistent with T, then for all $\alpha < \kappa$ (by (10.16))

$$(\mathcal{M}^*, Q) \models \exists x \forall y \forall v((\varphi(x, y) \wedge \theta(v, y)) \to \psi_\xi(w^\xi_\alpha, v))$$

i.e.

$$(\mathcal{M}^*, Q) \models \forall w \exists x \forall y \forall v((\varphi(x,y) \wedge \theta(v,y)) \rightarrow \psi_\xi(w,v))$$

contrary to $(\mathcal{M}^*, Q) \models$ (SA).

By the Omitting Types Theorem there is a countable weak cofinality model $(\mathcal{M}', Q')$ of T which omits each Γ_ξ. This is clearly as required. □

Lemma 10.90 (Precise Extension Lemma) *Suppose $(\mathcal{M}, Q)$ is an infinite weak cofinality model satisfying (SA). There is an elementary extension $(\mathcal{N}, R)$ of $(\mathcal{M}, Q)$ such that for all formulas $\varphi(x,y)$ of $L_{\omega\omega}(\mathcal{Q})$ of the vocabulary of $\mathcal{M}^*$ the following are equivalent:*

(1) $(\mathcal{M}^*, Q) \models \mathcal{Q}xy\varphi(x,y)$

(2) $(\mathcal{M}^*, Q) \models \mathcal{Q}^{\mathrm{LO}}xy\varphi(x,y) \wedge \forall x \exists y \varphi(x,y)$ *and there is $b \in N \setminus M$ such that $(\mathcal{N}^*, R) \models \varphi(a,b)$ for all $a \in M$.*

Such $(\mathcal{N}, R)$ is called a precise extension *of $(\mathcal{M}, Q)$.*

Proof Let $\varphi_0(x,y), \varphi_1(x,y)$ list all $\varphi(x,y)$ with $(\mathcal{M}^*, Q) \models \mathcal{Q}xy\varphi(x,y)$. By the Main Lemma there is an elementary chain

$$(\mathcal{M}_0, Q_0) \prec (\mathcal{M}_1, Q_1) \prec \cdots$$

such that

(3) $(\mathcal{M}_0, Q_0) = (\mathcal{M}, Q)$

(4) There is $b_n \in M_{n+1} \setminus M_n$ such that $(\mathcal{M}^*_{n+1}, Q_{n+1}) \models \varphi_n(a, b_n)$ for all $a \in M_n$

(5) If $(\mathcal{M}^*_n, Q_n) \models \mathcal{Q}^*xy\varphi(x,y)$, then for all $b \in M_{n+1}$ there is $a \in M_n$ such that $(\mathcal{M}^*_{n+1}, Q_{n+1}) \models \psi(b,a)$.

Let $(\mathcal{N}, R)$ be the union of this chain. Then by the Union Lemma $(\mathcal{M}, Q) \prec (\mathcal{N}, R)$. Conditions (1) and (2) clearly hold. □

Theorem 10.91 (Completeness Theorem for Cofinality Logic) *Suppose T is a theory in $L_{\omega\omega}(\mathcal{Q})$. Then the following conditions are equivalent:*

(1) *T has a model $(\mathcal{M}, Q^{\mathrm{cf}}_\omega)$*

(2) *T has a weak cofinality model satisfying (SA)*

(3) *$T \cup \{$(LO)$\} \cup \{$(NLE)$\} \cup \{$(SA)$\}$ is consistent.*

Proof To prove (3) → (1) we start with an $\aleph_1$-saturated model $(\mathcal{M}, Q)$ of $T \cup \{$(LO)$\} \cup \{$(NLE)$\} \cup \{$(SA)$\}$. Thus in $(\mathcal{M}, Q)$ every definable linear order

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