CHAPTER 4

A Remark on Nondeterminacy in IF Logic

JOUKO VÄÄNÄNEN

ABSTRACT. A sentence of IF-logic is nondetermined in a model if neither player has a winning strategy in the two-player semantic game on that model. It is well-known that there are sentences[1] of IF-logic that are nondetermined in every model with at least two elements. In fact, only the first-order definable sentences of IF-logic are determined in all models.[2] Thus every non first-order IF-sentence is nondetermined in some models. In this paper we take a closer look at some familiar examples of sentences of IF-logic and models in which they are determined. It turns out that if we want to use the familiar examples of IF-sentences to characterize well-known mathematical structures, we observe that the relevant IF-sentences are nondetermined in exactly the standard models. This raises the question of what the game-theoretical interpretation of these sentences tells us, if neither player has a winning strategy in the standard, that is, intended models. We consider also classes of structures with some standardness property and make a similar observation.

1. Introduction

By IF-logic we mean the extension of first-order logic,[3] introduced in Hintikka & Sandu (1989), in which quantifiers of the form

$$\exists x/y_1, \ldots, y_n\, \phi$$

$$\forall x/y_1, \ldots, y_n\, \phi$$

are allowed. For player II to win the semantic game associated with these formulas she has to have a winning strategy in the ordinary sense and in addition, if the game is played again and players have played similarly apart from the elements $y_1, \ldots, y_n$, then II has played the same element x. A similar restriction concerns winning strategies of player I. We write $\mathcal{M} \models \phi$, and say that ϕ is true in $\mathcal{M}$ (or $\mathcal{M}$ is a model of ϕ) and that ϕ has a model, if player II has a winning strategy in the associated semantical game. An IF-sentence ϕ is said to be valid if ϕ is true in every model. For the complexity of testing the validity of an IF-sentence, see Väänänen (2001).

A crucial property of IF-logic is its close relationship with Σ^1_1-sentences of second-order logic. A Σ^1_1-sentence has the form

$$\exists R_1 \ldots \exists R_n \phi,$$

where $R_1, \ldots, R_n$ are relation variables and ϕ is first-order. The relationship we use is

FACT 1 (The Σ^1_1-reduction). If ϕ is a sentence of IF-logic, then there is a Σ^1_1-sentence Φ such that for all models $\mathcal{M}$ we have

$$\mathcal{M} \models \phi \iff \mathcal{M} \models \Phi.$$

The converse is also true by Enderton (1970) and Walkoe (1970).

An immediate consequence of Fact 1 and the Compactness Theorem of first-order logic, is the following fact, which we use in an essential way:

FACT 2 (The Compactness Theorem). Suppose L is an arbitrarily vocabulary and Σ any set of sentences of IF-logic in the vocabulary of L. If every finite subset of Σ has a model, then Σ itself has a model.

Another immediate consequence of Fact 1 is:

FACT 3 (The Löwenheim-Skolem Theorem). Suppose L is an arbitrarily vocabulary and ϕ any sentence of IF-logic in the vocabulary of L. If ϕ has an infinite model or arbitrary large finite models, then ϕ has models of all infinite cardinalities.

We say that an IF-sentence ϕ is *determined in the model* $\mathcal{M}$ if one of the players has a winning strategy in the semantic game on ϕ and $\mathcal{M}$. The *dual* $\widetilde{\phi}$ of an IF-sentence ϕ is defined as follows:

$$\begin{array}{rcl}
\widetilde{\phi} & = & \neg\phi \text{ if } \phi \text{ atomic} \\
\widetilde{\neg\phi} & = & \phi \text{ if } \phi \text{ atomic} \\
\widetilde{(\phi \vee \psi)} & = & \widetilde{\phi} \wedge \widetilde{\psi} \\
\widetilde{(\phi \wedge \psi)} & = & \widetilde{\phi} \vee \widetilde{\psi} \\
\widetilde{\exists x/\vec{y}\, \phi} & = & \forall x/\vec{y}\, \widetilde{\phi} \\
\widetilde{\forall x/\vec{y}\, \phi} & = & \exists x/\vec{y}\, \widetilde{\phi}
\end{array}$$

It follows from the definitions that

II has a winning strategy in the semantic game on ϕ and $\mathcal{M}$

if and only if

I has a winning strategy in the semantic game on $\widetilde{\phi}$ and $\mathcal{M}$.

Thus the determinacy of an IF-sentence ϕ in a model $\mathcal{M}$ means the same as the truth of $\phi \vee \widetilde{\phi}$ in $\mathcal{M}$, in other words that either ϕ or its dual is true in

$\mathcal{M}$. John Burgess (2003) has pointed out that the dual operation is not a semantical operation in the sense that there are IF-sentences ϕ and ψ which have the same models but their duals do not have the same models.

2. IF-characterizable structures

The truth of a first-order sentence in a model $\mathcal{M}$ can be characterized in many important cases by means of validity in IF-logic. For example, suppose $\mathcal{M}$ is any of the following models:

$(\mathbb{N}, +, \cdot, 0, 1, <)$ The standard model of arithmetic.

$(\mathcal{P}(\mathbb{N}), \mathbb{N}, \in, +, \cdot, 0, 1, <)$ The standard model of second-order arithmetic.

$(\mathbb{R}, +, \times, 0, 1, <)$ The ordered field of real numbers.

$(\mathbb{Q}, +, \times, 0, 1)$ The field of rational numbers (we use Julia Robinson's definition of the integers in this field, see Robinson 1949).

$(\mathbb{R}^n, +, 0; \mathbb{R}, +, \times, 0, 1, <; \cdot)$ The Euclidean vector space $\mathbb{R}^n$ as a two-sorted structure (the last operation $\cdot$ is the operation of multiplying a vector by a scalar).

$(\omega, <)$ The order-type of (standard) natural numbers.

$(V_\omega, \in)$ The smallest level of cumulative hierarchy of sets that is a model of Zermelo's set theory (The cumulative hierarchy is defined as follows: $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, $V_\nu = \bigcup_{\alpha<\nu} V_\alpha$ for limit ν.)

$(V_\kappa, \in)$ The smallest level of cumulative hierarchy of sets, where κ is strongly inaccessible. A very natural standard model of Zermelo-Fraenkel's axioms for set theory.

Then there is an IF-sentence ϕ such that for all $\mathcal{N}$ we have[4]

$$\mathcal{N} \models \phi \iff \mathcal{N} \ncong \mathcal{M}.$$

Naturally, there are many different ϕ with this property. Let us call all such ϕ *IF-characterizations* of $\mathcal{M}$. Note that, for every first-order (or IF) ψ we have

$$\mathcal{M} \models \psi \iff \models \phi \vee \psi.$$

Thus to decide whether $\mathcal{M} \models \psi$ all we have to do is study the validity of $\phi \vee \psi$ in IF-logic for our favorite characterization ϕ of $\mathcal{M}$. This gives a reduction of truth in a structure to validity in IF-logic, emphasized strongly in Hintikka (1996). The usefulness of this reduction is diminished by the high complexity of validity in IF-logic (It is Π_2-complete, see Väänänen (2001) for details.)

THEOREM 1. Suppose $\mathcal{M}$ is an infinite model with an IF-characterization ϕ. Then ϕ is nondetermined in exactly the models isomorphic to $\mathcal{M}$.

PROOF. Suppose $\mathcal{N} \ncong \mathcal{M}$. Then $\mathcal{N} \models \phi$, so ϕ is determined in $\mathcal{N}$. On the other hand, suppose $\mathcal{N} \cong \mathcal{M}$. Then II does not have a winning strategy in the semantic game on ϕ and $\mathcal{N}$. Let us suppose I has. That means

$\mathcal{N} \models \widetilde{\phi}$. Since $\mathcal{N} \cong \mathcal{M}$, $\mathcal{N}$ is an infinite model. By Fact 2, there is $\mathcal{N}' \models \widetilde{\phi}$ of cardinality $> |\mathcal{M}|$. Thus $\mathcal{N}' \not\cong \mathcal{M}$, whence $\mathcal{N}' \models \phi$. But this is impossible, since $\mathcal{N}' \models \widetilde{\phi}$. Thus neither player has a winning strategy in the semantic game on ϕ and $\mathcal{N}$. The theorem is proved. $\square$

We have shown that in the case of the standard infinite structures any IF-characterization of the structure is bound to be nondetermined, and moreover exactly in the very structure that we have characterized. This is not affected by the choice of the characterizing IF-sentence.

Not all familiar mathematical structures are IF-characterizable. We give below three examples:

LEMMA 1. The following structures do not have an IF-characterization:

$(\mathbb{C}, +, \times, 0, 1)$	The field of complex numbers.
$(\mathbb{Q}, <)$	The order-type of rational numbers.
$(\mathbb{Z}, +)$	The abelian group of integers.

PROOF. $\mathcal{M} = (\mathbb{Q}, <)$: Since there are infinite models $\mathcal{N} \equiv \mathcal{M}$ (i.e., $\mathcal{N}$ is elementarily equivalent to $\mathcal{M}$) with $\mathcal{N} \not\cong \mathcal{M}$, there are dense order-types $\mathcal{N} \models \phi$ without endpoints. By Fact 3 there is one of cardinality $\aleph_0$. Thus $\mathcal{N} \cong (\mathbb{Q}, <)$. This contradicts $\mathcal{N} \models \phi$.

$\mathcal{M} = (\mathbb{C}, +, \times, 0, 1)$: Since there are infinite $\mathcal{N} \equiv \mathcal{M}$ with $\mathcal{N} \not\cong \mathcal{M}$, there are algebraically closed fields $\mathcal{N} \models \phi$ of characteristic zero. By Fact 3 there is one of cardinality 2^ω. By 2^ω-categoricity of the theory of algebraically closed fields of characteristic zero, $\mathcal{N} \cong (\mathbb{C}, +, \times, 0, 1)$. This contradicts $\mathcal{N} \models \phi$. $\square$

By the Π_2-completeness of the set of Gödel-numbers of valid IF-sentences, (Väänänen 2001, Theorem 1), truth in almost any structure can be reduced by some recursive function to validity in IF-logic. In particular this concerns any mathematical structure first-order definable in $(V_\omega, \in)$. For any such $\mathcal{M}$ there is a recursive function f mapping a first-order sentence to IF-sentences such that for all ψ:

$$\mathcal{M} \models \psi \iff \models f(\psi).$$

3. IF-characterizable classes of structures

We now extend the discussion on nondeterminacy and characterizations by IF sentences from single structures to classes of structures. By Fact 1, IF-definable model classes are exactly the Σ^1_1-definable model classes. In line with the previous section we investigate, not definability of model classes but characterizability of truth in a model class. Truth of ϕ in a class K, $K \models \phi$, means truth of ϕ in every member of K. The truth of a first-order sentence in all members of a model class K can be characterized in some important cases by means of validity in IF-logic. For example, suppose $\mathcal{M}$ is any of the following model classes:

$K^{<\aleph_0}$	The class of finite structures of a fixed vocabulary L
$K^{<\aleph_0}_{\mathrm{abel}}$	The class of finite abelian groups
$K^c_<$	The class of complete infinite dense order-types $(M,<)$
K_{wf}	The class of well-orderings
K_{st}	The class of transitive models $(M,\in)$ of ZFC^n (i.e., of a large finite part of the ZFC axioms)
$K^{\mathrm{cof}}_{>\aleph_0}$	The class of well-ordered order-types of cofinality $>\aleph_0$
$K^{\mathrm{cd}}_{>\aleph_0}$	The class of well-ordered order-types of cardinality $>\aleph_0$
$K_\in$	The class of levels of the cumulative hierarchy that are models of a fixed large finite part of ZFC
K_{cd}	The class of order-types of cardinal numbers
K_{IC}	The class of order-types of inaccessible cardinal numbers.

Then there is an IF-sentence ϕ such that for all $\mathcal{N}$ we have

$$\mathcal{N} \models \phi \iff \mathcal{N} \notin K,$$

in other words, K is Π^1_1-definable. Naturally there are many different ϕ with this property. Let us call all such ϕ *IF-characterizations* of K. Note that, for every first-order (or even IF) ψ we have

$$K \models \psi \iff \models \phi \vee \psi.$$

Thus to decide whether $K \models \psi$ all we have to do is study the validity of $\phi \vee \psi$ in IF-logic for our favorite characterization ϕ of K.

Let us call a model class K Σ^1_1-*thin* if it does not contain any Σ^1_1-class that has an infinite element or for each n an element with universe of size at least n. All the above classes are Σ^1_1-thin.

THEOREM 2. Suppose K is a Σ^1_1-thin model class consisting of infinite or arbitrarily large finite models, with an IF-characterization ϕ. Then there is a finite number N such that for models of size $\geq N$, ϕ is nondetermined in exactly the models in K.

PROOF. Suppose $\mathcal{N} \notin K$. Then $\mathcal{N} \models \phi$, so ϕ is determined in $\mathcal{N}$. On the other hand, suppose $\mathcal{N} \in K$ is infinite. Then II does not have a winning strategy in the semantic game on ϕ and $\mathcal{N}$. Let us suppose I has. That means $\mathcal{N} \models \widetilde{\phi}$. Let K' be the class of models of $\widetilde{\phi}$. Now K' is a Σ^1_1-class with an infinite element. By the Σ^1_1-thinness of K, there is $\mathcal{N}' \in K' \setminus K$. But this is impossible, since then $\mathcal{N}' \models \widetilde{\phi} \wedge \phi$. Thus neither player has a winning strategy in the semantic game on ϕ and $\mathcal{N}$. Let us then consider the finite $\mathcal{N} \in K$. Still, II does not have a winning strategy in the semantic game on ϕ and $\mathcal{N}$. Let us suppose I has a winning strategy in arbitrarily large such $\mathcal{N}$. Let K' be the class of models of $\widetilde{\phi}$. Now K' is a Σ^1_1-class with arbitrarily large finite models. By the Σ^1_1-thinness of K, there is $\mathcal{N}' \in K' \setminus K$. But this is impossible, since then $\mathcal{N}' \models \widetilde{\phi} \wedge \phi$. Thus there is a finite number N such that for models of size $\geq N$ the sentence ϕ is nondetermined in exactly the models in K. $\square$

We have shown that in the case of the standard classes of mathematical structures any IF-characterization of the class is bound to be nondetermined, apart from some finite number of finite structures, and moreover exactly in the very class of structures we have characterized.

LEMMA 2. The following classes do not have an IF-characterization:

$K_{\aleph_0}$	The class of countable models of a fixed vocabulary L
$K_{<\kappa}$	The class of models of size $< \kappa$ of vocabulary L
K_{nwf}	The class of non-well-founded models (M, E).

PROOF. $K = K_{\aleph_0}$: Since there are infinite $\mathcal{N} \notin K$, there are $\mathcal{N} \models \phi$ of cardinality $\aleph_0$. This contradicts the very definition of ϕ.

$K = K_{nwf}$: Since there are infinite $\mathcal{N} \notin K$, there are non-well-founded $\mathcal{N} \models \phi$. This contradicts the very definition of ϕ. $\square$

University of Helsinki

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Notes

1. E.g. $\forall x \exists y/x\, (x = y)$.
2. Skolemization yields two disjoint Σ^1_1-classes (see Fact 1) from model classes definable by a determined sentence ϕ of IF-logic and its dual (see later for a definition of dual). Craig Interpolation Theorem gives a first-order sentence ϕ' separating these two model classes, and ϕ' gives a first-order definition of ϕ.
3. We assume first-order logic formulated so that negation is allowed only in front of atomic formulas, and the logical operations are $\vee$, $\wedge$, $\exists$, $\forall$.
4. We may ask why we did not define an IF-sentence ϕ to be an IF-characterization of $\mathcal{M}$ if we have for all $\mathcal{N}$: $\mathcal{N} \models \phi \iff \mathcal{N} \cong \mathcal{M}$. But then there would be no infinite IF-characterizable models, an immediate consequence of Fact 2.

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