Set Theoretic Aspects of Hausdorff Dimension

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Hausdorff Dimension

review

Define a family of outer measures, parameterized by $s > 0$. For $A \subseteq \mathbb{R}^n$,

$$\mathcal{H}^s(A) = \lim_{r \to 0} \inf \left\{ \sum_i d_i^s : \begin{array}{l} \text{there is a cover of } A \text{ by balls } B_i \\ \text{with diameters } d_i < r \end{array} \right\}.$$

Definition (Hausdorff 1918)

The *Hausdorff dimension* of A is as follows.

$$\begin{aligned} \dim_{\mathrm{H}}(A) &= \inf\{s > 0 : \mathcal{H}^s(A) = 0\} \\ &= \sup\left(\{s > 0 : \mathcal{H}^s(A) = \infty\} \cup \{0\}\right) \end{aligned}$$

Remark

The numerical Hausdorff dimension of a set A characterizes the cut of functions x^s which assign A infinite outer-measure.

Examples

Example

- A line segment within $\mathbb{R}^n$ has Hausdorff dimension 1.
- The Sierpinski triangle has Hausdorff dimension $\log 3/\log 2$.
- Almost surely, the path of a Brownian motion in $\mathbb{R}^3$ has Hausdorff dimension 2.

Gauge Functions and General Hausdorff Dimension

Definition

A *gauge function* is a function $f : (0, \infty) \to (0, \infty)$ which has the following properties:

- continuous
- increasing
- $\lim_{x \to 0^+} f(x) = 0$

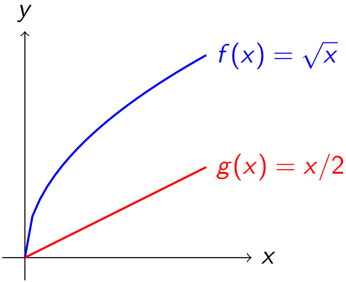
Example

For $s > 0$, x^s is a gauge function.

As above, we can associate a Hausdorff outer measure H^f with any gauge function f.

Gauge Functions and General Hausdorff Dimension

- Write $f \prec g$ to indicate that $\lim_{x \to 0^+} \frac{g(x)}{f(x)} = 0$.
- Say that g has *higher order* than f.



$H^g(A) > 0$ indicates a higher dimension than $H^f(A) > 0$ does.

Gauge Functions and General Hausdorff Dimension

- If $f \prec g$ and $H^f(A) < \infty$ then $H^g(A) = 0$.
- If $f \prec g$ and $H^g(A) > 0$ then $H^f(A) = \infty$.

A set determines a *filter* of gauge functions for which it is null and an *ideal* of gauge functions for which it has positive outer measure.

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A set determines a *filter* of gauge functions for which it is null and an *ideal* of gauge functions for which it has positive outer measure.

Here is a principal example.

Example

Almost surely, the path of a Brownian motion in $\mathbb{R}^3$ has positive *finite* H^f measure for $f(x) = x^2 \log\log(1/x) \prec x^2$. See Falconer (2003).

Capacitability

Quantitative versions of the perfect set property

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Theorem

- *(Davies 1952) If A is analytic and $H^f(A) > 0$ then A has a compact subset C such that $H^f(C) > 0$.*
- *(Davies 1956 for x^s, Sion and Sjerve 1962) If A is analytic and not σ-finite for H^f then A has a compact subset C such that C is not σ-finite for H^f.*

Capacitability

The previous results of Davies are the most that can be proven within *ZFC*.

Theorem

If $V = L$ then the maximal thin Π^1_1-set, $\{x : x \in L_{\omega_1^x}\}$, is a co-analytic set of dimension 1 with no perfect subset.

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Stronger set theoretic hypotheses yield capacitability for larger classes of sets.

Theorem (Crone, Fishman and Jackson (2020); Yinhe Peng, Liuzhen Wu and Liang Yu (2023))

Under the Axiom of Determinacy, for every set A, if A has Hausdoff dimension s then for every $d < s$ there is closed subset C of A such that C has Hausdorff dimension at least d.

Sets of Strong Dimension f

Definition

A set A has *strong dimension f* iff

$$\forall h\,[h \prec f \Rightarrow H^h(A) = \infty]$$

$$\forall g\,[f \prec g \Rightarrow H^g(A) = 0]$$

As a limiting case, A has *strong dimension zero* iff for all g, $H^g(A) = 0$.

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Example

A line, or even a line segment, within $\mathbb{R}^n$ has strong linear dimension.

Size Matters

Repeat. A has strong dimension f iff

- $H^h(A) = \infty$ whenever $h \prec f$ and
- $H^g(A) = 0$ whenever $f \prec g$.

Besicovitch raised the question, "What about $H^f(A)$, the size of A, when A has strong dimension f?"

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Besicovitch raised the question, "What about $H^f(A)$, the size of A, when A has strong dimension f?"

There are three possibilites:

1. $H^f(A) = 0$
2. $H^f(A) > 0$ and A is σ-finite for H^f
3. A is not σ-finite for H^f

Case 1. $H^f(A) = 0$

Theorem (Besicovitch 1956)

Suppose that f is a gauge function such that $H^f(A) = 0$. Then there is an h such that $f \succ h$ and $H^h(A) = 0$.

Consequently, if $H^f(A) = 0$ then A does not have strong dimension f.

Aside: Nonprincipal Dimension

Question (C. A. Rogers 1962)

Suppose that $A \subseteq \mathbb{R}$, *that* $(f_i : i \in \omega)$ *is a sequence of gauge functions such that for all* i, $f_i \succ f_{i+1}$, *and that for all* i, $H^{f_i}(A) = 0$. *Does there exist a gauge function* h *such that for all* i, $f_i \succ h$ *and* $H^h(A) = 0$?

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Although positive for closed sets, Rogers's question is settled negatively by the following.

Theorem (Olsen and Renfro 2006)

Let $\mathbb{L}$ denote the set of Liouville numbers, those with infinite exponent of irrationality. For a gauge function h,

- *if there is an $s > 0$ such that $x^s \prec h$, then $H^h(\mathbb{L}) = 0$*
- *if for every $s > 0$, $h \prec x^s$, then $H^h(\mathbb{L}) = \infty$*

For Rogers's question, use the functions $f_i = x^{\frac{1}{i+1}}$.

Aside: Nonprincipal Dimension

dimension for comeager sets

Definition

If f is a gauge function, define the gauge function $\Gamma_f : \{1/2^n : n \in \omega\} \to \mathbb{R}^+$ by

$$\Gamma_f(x) = \inf_{0<s\leq x} f(s)/s$$

In their analysis of Liouville numbers, Olsen and Renfro showed that $H^f = H^{\Gamma_f}$.

Theorem (Yiping Miao, work in progress)

Let G be the set of arithmetically generic reals. For a gauge function f, $H^f(G) > 0$ iff for all arithmetic gauge functions g, $\Gamma_f \not\succ g$.

Aside: Nonprincipal Dimension

More generally:

Theorem

Suppose that $(f_i : i \in \omega)$ is a sequence of gauge functions such that for all i, $f_i \succ f_{i+1}$. There is a $\mathbf{\Pi}^0_3$ set A such that the following conditions hold.

- *For all i, $H^{f_i}(A) = 0$.*
- *For all gauge functions h, if for all i, $f_i \succ h$, then $H^h(A) = \infty$.*

Case 2. $H^f(A) > 0$ and A is σ-finite for H^f

Remark

- *If $H^f(A)$ is finite and $f \prec g$, then $H^g(A) = 0$.*
- *H^g is countably additive.*

Consequently, if $H^f(A) > 0$ and A is σ-finite for H^f, then A has strong dimension f.

Case 3. A is not σ-finite for H^f

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Theorem (Besicovitch 1956, generalized Rogers 1962)

If A is compact and is not σ-finite for H^f, then there is a g such that $f \prec g$ and A is not σ-finite for H^g.

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Thus, if A is compact and not σ-finite for H^f, then A does not have strong dimension f.

By the Davies/Sion-Sjerve Theorem, the non-σ-finiteness of an analytic set is supported by that of its compact subsets, so we have the same conclusion for analytic sets.

Consistency Results for Case 3

Theorem (Besicovitch 1963)

If CH, or the failure of the Borel Conjecture, then there is a set $A \subset \mathbb{R}^2$ such that A has strong linear dimension and is not σ-finite for linear measure.

Borel Conjecture

Definition

A set $A \subseteq \mathbb{R}$ has *strong measure zero* iff for any sequence of positive real numbers $\{\epsilon_i\}$ there is a sequence of open intervals $\{O_i\}$ such that for each i, O_i has length ϵ_i, and $A \subseteq \cup_{i=1}^{\infty} O_i$.

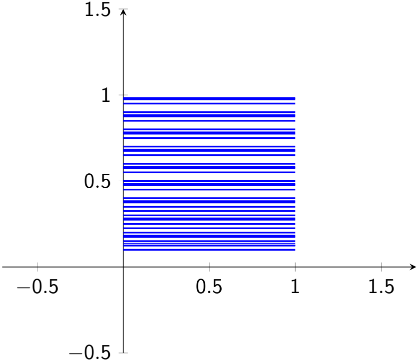
Borel (1919) conjectured that strong measure zero implies countable (BC).

Theorem

- *(Sierpiński 1928) CH implies* $\neg BC$.
- *(Laver 1976)* $Con(ZFC)$ *implies* $Con(ZFC + BC)$.

Besicovitch's Proof Sketch

Let $S \subset [0,1]$ be a counterexample to the Borel Conjecture: uncountable with strong measure zero. Show that $[0,1] \times S$ has strong linear dimension and is not σ-finite for linear measure.



Extending the Borel Conjecture to Dimension

Recall that a set A has strong dimension zero iff for all gauge functions f, $H^f(A) = 0$.

Theorem (Besicovitch 1955)

A set A has strong measure zero iff it has strong dimension zero.

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This suggests the following.

Extended Borel Conjecture (BC^*)

For all $A \subset \mathbb{R}^n$ and for all gauge functions f, A has strong dimension f iff $H^f(A) > 0$ and A is σ-finite for H^f.

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In the context of our discussion, BC^* implies that A does not have strong dimension f when Case 3 applies.

The Consistency of BC^*

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In the proof, show that Laver's model for BC is also one for BC^*.

Laver's Model

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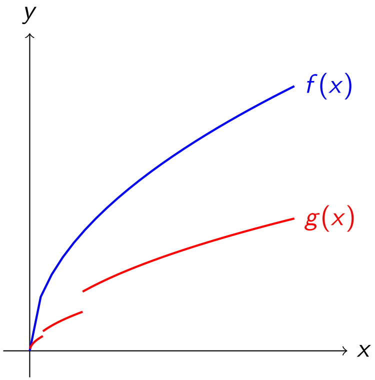
- Each step of the iteration adds a function $G : \omega \to \omega$ which is fast-growing compared to those which precede it.
- For f a gauge function and $G : \omega \to \omega$ Laver-generic, define g so that for $x \in [1/G(k+1), 1/G(k))$, $g(x) = f(x)/(k+1)$.



Heuristic

Proposal. Given a gauge function f and a set $A \subset \mathbb{R}^n$ in $\mathcal{M}[G_{\omega_2}]$ so that

$$\mathcal{M}[G_{\omega_2}] \models \text{ ``}A \text{ is not } \sigma\text{-finite for } H^f.\text{''}$$

Show that $H^g(A) > 0$, where $g \succ f$ is the gauge function induced from f by a Laver generic.

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The last step yields a contradiction to the first two steps.

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The previous argument showed that if $V \models ZFC$ then there is a generic extension $V[G]$ in which BC^* holds.

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- When restricted to refer only to closed sets, the assertion of BC^* is equivalent to a Π^1_3-statement.
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- Since Π^1_3-statements are downward absolute for inner models of ZFC, BC^* is true for closed sets.

This provides an alternate proof of Besicovitch's theorem that closed sets of non-σ-finite measure for H^f do not have strong dimension f.

A Final Challenge

A set $A \subset \mathbb{R}$ determines an ideal $I(A)$ in the gauge functions:

$$I(A) = \{f : H^f(A) > 0\}.$$

Question

- *What is a necessary and sufficient condition on an an ideal I to ensure that there is a compact set A such that $I = I(A)$?*
- *What is a necessary and sufficient condition on an an ideal I to ensure that there is a Borel set A such that $I = I(A)$?*
- *What is a necessary and sufficient condition on an an ideal I to ensure that there is an unrestricted set A such that $I = I(A)$?*

The End