Infinitary combinatorics in condensed mathematics and strong homology

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Joint work with Jeffrey Bergfalk.

Introduction

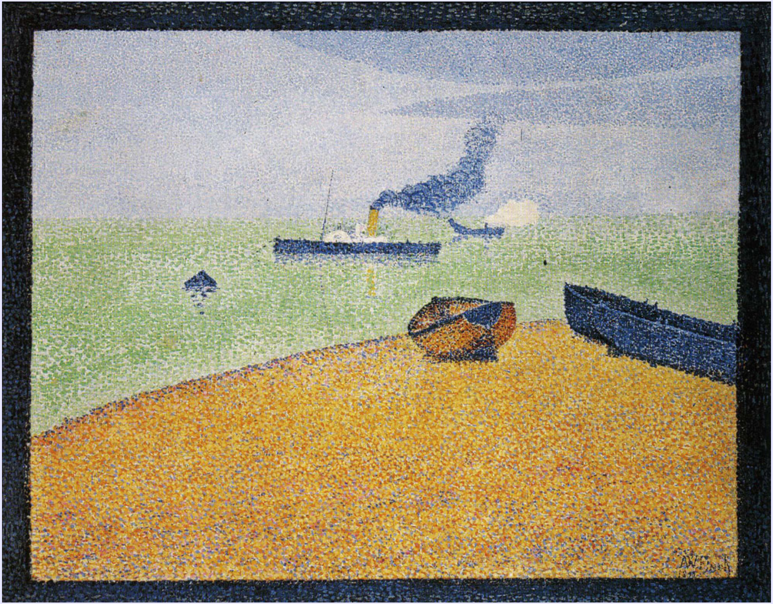
This talk will contain a few recent applications of set-theoretic techniques to the study of strong homology and condensed mathematics. Roughly speaking, they can all be seen as trying to answer questions about how well-behaved the world can consistently be. A quick and overly simplistic summary of the results:

- The world of separable objects is consistently quite well-behaved.
- The world of nonseparable objects is always poorly behaved.

Structure of the talk

1. Introduction to derived limits
2. Consistent positive results in the separable setting:
    - Strong homology
    - Condensed mathematics
3. Negative results in the nonseparable setting

I. Derived limits



Inverse systems

Given a directed partial order Λ, an *inverse system of abelian groups indexed by* Λ is a structure

$$\mathbf{X} = \langle X_u, \pi_{uv} \mid u \leq v \in \Lambda \rangle$$

such that

- each X_u is an abelian group ($X_u \in \mathrm{Ab}$);
- each $\pi_{uv} : X_v \to X_u$ is a group homomorphism;
- for all $u \leq v \leq w$, we have $\pi_{uw} = \pi_{uv} \circ \pi_{vw}$.

Given an inverse system $\mathbf{X}$, we can form its (inverse) limit $\lim \mathbf{X}$. Concretely, this can be represented as

$$\left\{ \mathbf{x} \in \prod_{u \in \Lambda} X_u \;\middle|\; \forall u \leq v \;\; \mathbf{x}(u) = \pi_{uv}(\mathbf{x}(v)) \right\}.$$

The systems $\mathbf{A}[H]$

Given a function $f \in {}^\omega\omega$ and $H \in \mathrm{Ab}$, let

$$I(f) := \{(k, m) \in \omega \times \omega \mid m < f(k)\}$$

and $A_f[H] = \bigoplus_{I(f)} H$. Given $f \le g$ in ${}^\omega\omega$, there is a projection map $\pi_{fg} : A_g[H] \to A_f[H]$. We thus obtain an inverse system

$$\mathbf{A}[H] = \langle A_f[H], \pi_{fg} \mid f \le g \in {}^\omega\omega \rangle.$$

Note that $\lim \mathbf{A}[H] = \bigoplus_\omega \prod_\omega H$. We omit "$H$" from the notation if $H = \mathbb{Z}$.

Short exact sequences

Recall that a pair of group homomorphisms

$$X \xrightarrow{\pi} Y \xrightarrow{\sigma} Z$$

is *exact* at Y if $\ker(\sigma) = \mathrm{im}(\pi)$. A *short exact sequence* is a sequence

$$0 \to X \xrightarrow{\pi} Y \xrightarrow{\sigma} Z \to 0$$

that is exact at X, Y, and Z. This notion extends to any *abelian category*, including the category Ab^{Λ} of all inverse systems of abelian groups indexed by a fixed directed set Λ.

Exactness of $\lim$

The functor $\lim : \mathsf{Ab}^\Lambda \to \mathsf{Ab}$ is *left exact* but not exact, i.e., if

$$0 \to \mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xrightarrow{\mathbf{g}} \mathbf{Z} \to 0$$

is exact in Ab^Λ, then the induced sequence

$$0 \to \lim \mathbf{X} \xrightarrow{\lim \mathbf{f}} \lim \mathbf{Y} \xrightarrow{\lim \mathbf{g}} \lim \mathbf{Z} \to 0,$$

is exact at $\lim \mathbf{X}$ and $\lim \mathbf{Y}$, but might not be exact at $\lim \mathbf{Z}$. Concretely, this failure of exactness comes from the fact that even if a morphism $\mathbf{g} : \mathbf{Y} \to \mathbf{Z}$ consists of surjective maps, the limit map $\lim \mathbf{g} : \lim \mathbf{Y} \to \lim \mathbf{Z}$ need not be surjective.

Derived limits

Derived limits measure the failure of the inverse limit functor to be exact. For each $0 < n < \omega$, there is a derived functor $\lim^n : \mathsf{Ab}^\Lambda \to \mathsf{Ab}$ such that every short exact sequence

$$0 \to \mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to 0$$

induces a *long* exact sequence

$$\begin{aligned} 0 &\longrightarrow \lim \mathbf{X} \longrightarrow \lim \mathbf{Y} \longrightarrow \lim \mathbf{Z} \\ &\longrightarrow \lim^1 \mathbf{X} \longrightarrow \lim^1 \mathbf{Y} \longrightarrow \lim^1 \mathbf{Z} \\ &\longrightarrow \lim^2 \mathbf{X} \longrightarrow \lim^2 \mathbf{Y} \longrightarrow \lim^2 \mathbf{Z} \longrightarrow \dots \end{aligned}$$

$\lim^1 \mathbf{A}$

Given $f \in {}^\omega\omega$, let $B_f = \prod_{I(f)} H$, and let

$$\mathbf{B} = \langle B_f, \pi_{fg} \mid f \leq g \in {}^\omega\omega \rangle.$$

We get a short exact sequence $0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{B}/\mathbf{A} \to 0$. The system $\mathbf{B}$ is very well-behaved; in particular, all of its derived limits are 0. Therefore, the initial segment of the long exact sequence derived from the above short exact sequence is

$$0 \to \lim \mathbf{A} \to \lim \mathbf{B} \to \lim \mathbf{B}/\mathbf{A} \to \lim^1 \mathbf{A} \to 0.$$

In particular, $\lim^1 \mathbf{A} = 0$ if and only if the map $\lim \mathbf{B} \to \lim \mathbf{B}/\mathbf{A}$ is surjective.

$\lim^1 \mathbf{A}$

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Elements of $\lim \mathbf{B}/\mathbf{A}$ are of the form $\langle [\varphi_f] \mid f \in {}^\omega\omega \rangle$ where

1. $\varphi_f : I(f) \to \mathbb{Z}$;
2. $[\varphi_f] = \{\varphi' : I(f) \to \mathbb{Z} \mid \varphi' =^* \varphi_f\}$;
3. for all $f \leq g$, we have $\varphi_g \restriction I(f) =^* \varphi_f$.

A sequence $\langle [\varphi_f] \mid f \in {}^\omega\omega \rangle$ is in the image of the map $\lim \mathbf{B} \to \lim \mathbf{B}/\mathbf{A}$ if and only if there is a single function $\psi : \omega \times \omega \to \mathbb{Z}$ such that $\psi \restriction I(f) =^* \varphi_f$ for all $f \in {}^\omega\omega$.

$\lim^1 \mathbf{A}$

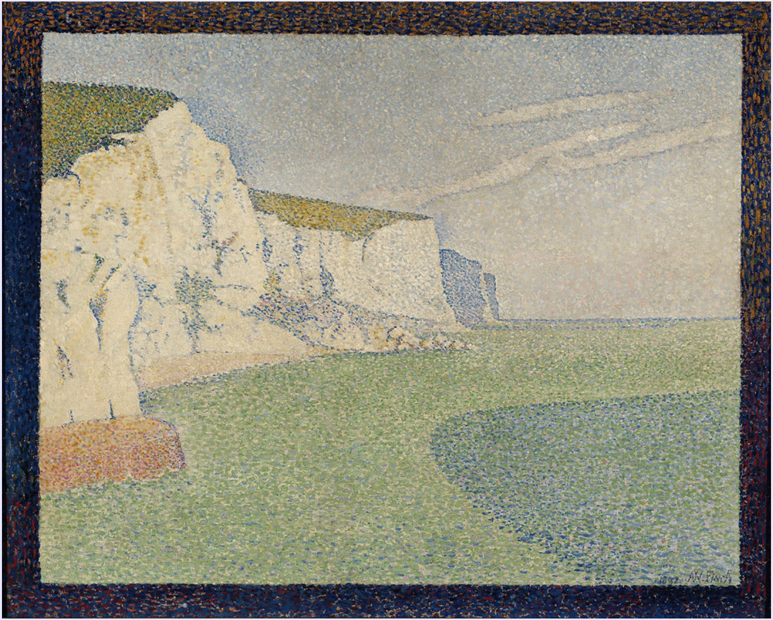
In other words, $\lim^1 \mathbf{A} = 0$ if and only if for every family of the form

$$\Phi = \langle \varphi_f : I(f) \to \mathbb{Z} \mid f \in {}^\omega\omega \rangle,$$

- if Φ is **coherent**, i.e., $\varphi_f =^* \varphi_g$ for all $f, g \in {}^\omega\omega$,
- then Φ is **trivial**, i.e., there exists a function $\psi : \omega \times \omega \to \mathbb{Z}$ such that $\psi \restriction I(f) =^* \varphi_f$ for all $f \in {}^\omega\omega$.

Similar higher-dimensional characterizations exist for the higher derived limits of $\mathbf{A}$.

II. Strong homology



Additivity of strong homology

Strong homology is a homology theory of topological spaces that is strong shape invariant. Strong homology is designed to better deal with certain pathological topological spaces than, say, singular homology.

Definition (Additivity of homology)

A homology theory is *additive* on a class of topological spaces $\mathcal{C}$ if, for every natural number p and every family $\{X_i \mid i \in J\}$ such that each X_i and $\coprod_J X_i$ are in $\mathcal{C}$, we have

$$\bigoplus_J \mathrm{H}_p(X_i) \cong \mathrm{H}_p(\coprod_J X_i).$$

Question (Mardešić–Prasolov)

Is strong homology additive?

Additivity of strong homology

Let X^n denote the n-dimensional infinite earring space, i.e., the one-point compactification of an infinite countable sum of copies of the n-dimensional open unit ball. Let $\bar{\mathrm{H}}_p(X)$ denote the p^{th} strong homology group of X.

Theorem (Mardešić–Prasolov, '88)

Suppose that $0 \leq p < n$ are natural numbers. Then

$$\bigoplus_{\omega} \bar{\mathrm{H}}_p(X^n) = \bar{\mathrm{H}}_p(\coprod_{\omega} X^n)$$

if and only if $\lim^{n-p} \mathbf{A} = 0$.

Consequently, if strong homology is additive on closed subsets of Euclidean space, then $\lim^n \mathbf{A} = 0$ for all $n \geq 1$.

Some history

- (Mardešić–Prasolov, '88) CH $\Rightarrow \lim^1 \mathbf{A} \neq 0$.
- (Dow–Simon–Vaughan, '89) $\mathfrak{d} = \aleph_1 \Rightarrow \lim^1 \mathbf{A} \neq 0$.
- (Dow–Simon–Vaughan, '89) PFA $\Rightarrow \lim^1 \mathbf{A} = 0$.
- (Todorčević, '98) OCA $\Rightarrow \lim^1 \mathbf{A} = 0$.
- (Kamo, '94) After adding $\aleph_2$-many Cohen reals to any model of ZFC, $\lim^1 \mathbf{A} = 0$.

More recent history

- (Bergfalk, ‘17) PFA $\Rightarrow \lim^2 \mathbf{A} \neq 0$.
- (Bergfalk–LH, ‘21) After adding weakly-compact-many Hechler reals to any model of ZFC, we have $\lim^n \mathbf{A} = 0$ for all $0 < n < \omega$.
- (Bergfalk–Hrušák–LH, ‘23) After adding $\beth_\omega$-many Cohen reals to any model of ZFC, we have $\lim^n \mathbf{A} = 0$ for all $0 < n < \omega$.
- (Bannister, ‘24) In either of the above models, we in fact have $\lim^n \mathbf{A}[H] = 0$ for all $0 < n < \omega$ and all $H \in \mathrm{Ab}$.
- (Bannister–Bergfalk–Moore, ‘23, Bannister, ‘24) In either of the above models, strong homology is additive on the class of locally compact separable metric spaces.

III. Condensed mathematics

Condensed mathematics

Condensed mathematics is a framework, introduced recently by Clausen and Scholze, to allow for the application of algebraic tools in contexts in which algebraic objects carry topologies.

Problem: Classical categories of algebraic objects carrying topologies, such as the category TopAb of topological abelian groups, fail to be abelian categories.

Solution: Embed these classical categories into richer, "condensed" categories. E.g., TopAb embeds into the category Cond(Ab) of condensed abelian groups.

Condensed abelian groups

Let ED denote the class of extremally disconnected compact Hausdorff spaces. A *condensed abelian group* is a contravariant functor $T : \mathsf{ED} \to \mathsf{Ab}$ such that

1. $T(\emptyset) = 0$ (i.e., the one-element group);
2. for all $S_0, S_1 \in \mathsf{ED}$, $T(S_0 \sqcup S_1) = T(S_0) \times T(S_1)$.

Given $X \in \mathsf{TopAb}$, define $\underline{X} \in \mathsf{Cond(Ab)}$ by setting $\underline{X}(S) = \mathrm{Cont}(S, X)$ for all $S \in \mathsf{ED}$. This describes an embedding of TopAb into Cond(Ab); it is fully faithful on the class of compactly generated topological abelian groups.

Cond(Ab) is a (very nice) abelian category; e.g., all limits and colimits exist; arbitrary products, direct sums, and filtered colimits are exact; and the category is generated by compact projective objects.

Pro-abelian groups

A *pro-abelian group* is a topological abelian group that can be expressed as the inverse limit of an inverse system of (discrete) abelian groups.

Question (Clausen–Scholze)

Does the category of pro-abelian groups embed fully faithfully into Cond(Ab) *(at the level of derived categories)?*

This reduces to the following question: is it the case that, for all index sets I, J, and K, and all $0 < n < \omega$, we have

$$\mathrm{Ext}^n_{\mathsf{Cond(Ab)}}\left(\prod_I \bigoplus_J \underline{\mathbb{Z}}, \bigoplus_K \underline{\mathbb{Z}}\right) = 0?$$

(Here $\mathrm{Ext}^n(\cdot, \cdot)$ are the derived functors of $\mathrm{Hom}(\cdot, \cdot)$.)

An equivalence

Clausen and Scholze observed that the following conditions are equivalent:

1. For all $0 < n < \omega$ and every cardinal μ, we have

$$\mathrm{Ext}^n_{\mathsf{Cond(Ab)}}\left(\prod_{\omega}\bigoplus_{\omega}\underline{\mathbb{Z}}, \bigoplus_{\mu}\underline{\mathbb{Z}}\right) = 0.$$

2. Whenever $M_0 \twoheadleftarrow M_1 \twoheadleftarrow M_2 \twoheadleftarrow \cdots$ is a sequential system of countable abelian groups with surjective transition maps and N is *any* abelian group, we have, for all $n \geq 0$,

$$\mathrm{Ext}^n_{\mathsf{Cond(Ab)}}(\lim \underline{M_i}, \underline{N}) \cong \mathrm{colim}\ \mathrm{Ext}^n_{\mathsf{Cond(Ab)}}(\underline{M_i}, \underline{N}).$$

3. $\lim^n \mathbf{A}[H] = 0$ for all $n \geq 1$ and all abelian groups H.

A sketch of an argument

Let us sketch an argument that (3) implies (1), assuming for simplicity that $\mu = 1$. We want to compute $\mathrm{Ext}^n(\prod_\omega \bigoplus_\omega \underline{\mathbb{Z}}, \underline{\mathbb{Z}})$ for $0 < n < \omega$ or, equivalently, $\mathrm{RHom}(\prod_\omega \bigoplus_\omega \underline{\mathbb{Z}}, \underline{\mathbb{Z}})$. Observe that

$$\prod_\omega \bigoplus_\omega \underline{\mathbb{Z}} = \operatorname*{colim}_{f \in {}^\omega\omega} \prod_{I(f)} \underline{\mathbb{Z}}.$$

Colimits can be pulled outside of the first coordinates of RHom, therefore we have

$$\mathrm{RHom}\left(\prod_\omega \bigoplus_\omega \underline{\mathbb{Z}}, \underline{\mathbb{Z}}\right) = \mathrm{Rlim}_f \mathrm{RHom}\left(\prod_{I(f)} \underline{\mathbb{Z}}, \underline{\mathbb{Z}}\right).$$

But $\mathrm{RHom}(\prod_{I(f)} \underline{\mathbb{Z}}, \underline{\mathbb{Z}}) = \bigoplus_{I(f)} \underline{\mathbb{Z}} = \underline{A_f}$. Thus, the right hand side becomes $\mathrm{Rlim}\underline{\mathbf{A}}$. It follows that if $\lim^n \mathbf{A}$ vanishes for all $n > 0$, then so does $\mathrm{Ext}^n(\prod_\omega, \bigoplus_\omega \underline{\mathbb{Z}}, \underline{\mathbb{Z}})$.

The continuum

In particular, it follows from the aforementioned results of Bergfalk–Hrušák–LH and Bannister that, after adding $\beth_\omega$-many Cohen reals, the class of *separable* pro-abelian groups embeds fully faithfully into Cond(Ab). Recent joint work with Casarosa indicates that a large continuum is *necessary* for this result:

Theorem (Casarosa–LH)

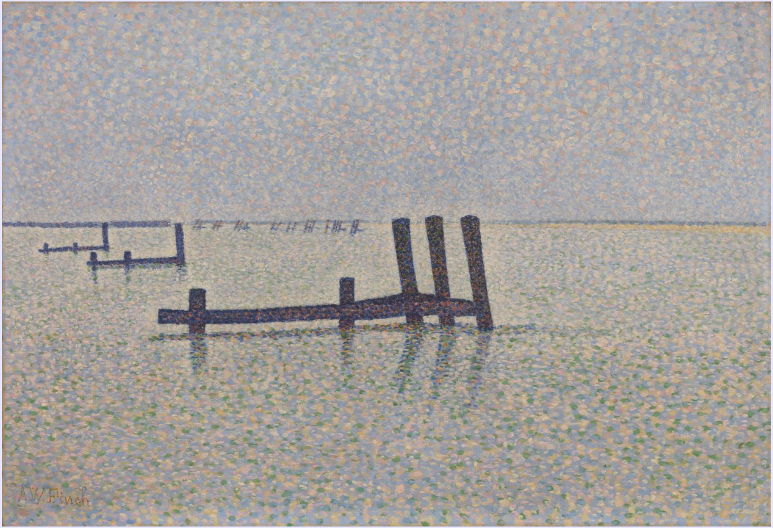
Suppose that $\lim^n \mathbf{A}[H] = 0$ *for all* $0 < n < \omega$ *and all* $H \in \mathsf{Ab}$. *Then* $2^{\aleph_0} > \aleph_\omega$. *More precisely, if* $0 < n < \omega$ *and* $\mathfrak{d} = \aleph_n$, *then*

$$\lim^n \mathbf{A}\left[\bigoplus_{\omega_n} \mathbb{Z}\right] \neq 0.$$

Question

Suppose that $\mathfrak{d} = \aleph_n$. *Must* $\lim^n \mathbf{A} \neq 0$?

IV. The nonseparable world



A generalization

If one runs the above argument to calculate $\mathrm{Ext}^n(\prod_\kappa \bigoplus_\lambda \mathbb{Z}, \mathbb{Z})$ for arbitrary κ and λ, one encounters generalizations of the system $\mathbf{A}$. Given a function $f : \kappa \to [\lambda]^{<\omega}$, let

$$I(f) := \{(i, \alpha) \in \kappa \times \lambda \mid \alpha \in f(i)\}.$$

For two such functions f, g, we say that $f \leq g$ if $f(i) \subseteq g(i)$ for all $i < \kappa$. We can then define groups $A_f := \bigoplus_{I(f)} \mathbb{Z}$ and projection maps $\pi_{fg} : A_g \to A_f$, producing an inverse system

$$\mathbf{A}_{\kappa\lambda} := \langle A_f, \pi_{fg} \mid f \leq g : \kappa \to [\lambda]^{<\omega} \rangle.$$

The above argument can be adapted to show that $\mathrm{Ext}^n(\prod_\kappa \bigoplus_\lambda \mathbb{Z}, \mathbb{Z})$ vanishes for all $n > 0$ if and only if $\lim^n \mathbf{A}_{\kappa\lambda}$ does as well. Note that $\mathbf{A}$ is (equivalent to) $\mathbf{A}_{\omega\omega}$.

Strong homology

A similar story holds for strong homology. Given an infinite cardinal λ, let $X^{n,\lambda}$ denote the one-point compactification of the sum of λ-many copies of the n-dimensional open unit ball.

Theorem (Bergfalk–LH)

Suppose that $0 \leq p < n$ are natural numbers. Then

$$\bigoplus_{\omega} \bar{\mathrm{H}}_p(X^{n,\lambda}) = \bar{\mathrm{H}}_p(\coprod_{\omega} X^{n,\lambda})$$

if and only if $\lim^{n-p} \mathbf{A}_{\omega\lambda} = 0$.

Nonvanishing in ZFC

Proposition (Bergfalk–LH)

$\lim^1 \mathbf{A}_{\omega\omega_1} \neq 0$.

Let us sketch a proof of the proposition. In analogy with the system $\mathbf{A}$, we will construct a family of functions

$$\Phi = \langle \varphi_f : I(f) \to \omega \mid f : \omega \to [\omega_1]^{<\omega} \rangle$$

that is

1. *coherent*, i.e., $\varphi_f =^* \varphi_g$ for all f and g;
2. *nontrivial*, i.e., there is no function $\psi : \omega \times \omega_1 \to \omega$ such that $\psi =^* \varphi_f$ for all f.

Begin by fixing a sequence of functions
$\langle e_\beta : (\beta + 1) \times \omega \to \omega \mid \beta < \omega_1 \rangle$ such that

1. each e_β is finite-to-one; and
2. $e_\alpha =^* e_\beta \restriction (\alpha + 1) \times \omega$ for all $\alpha < \beta < \omega_1$.

Given $f : \omega \to [\omega_1]^{<\omega}$, let

$$\beta_f := \sup \left\{ \bigcup \{ f(i) \mid i < \omega \} \right\},$$

and define $\varphi_f : I(f) \to \omega$ by letting $f(i, \alpha) = e_{\beta_f}(i, \alpha)$ for all $(i, \alpha) \in I(f)$. The coherence of Φ follows from the coherence of $\langle e_\beta \mid \beta < \omega_1 \rangle$. It remains to show that Φ is nontrivial.

Fix an arbitrary $\psi : \omega \times \omega_1 \to \omega$. We will find $f : \omega \to [\omega_1]^{<\omega}$ such that $\psi \neq^* \varphi_f$.

For each $i < \omega$, fix $k_i < \omega$ for which there are infinitely many $\alpha < \omega_1$ such that $\psi(i, \alpha) = k_i$. Find $\beta < \omega_1$ large enough such that, for all $i < \omega$, there are infinitely many $\alpha < \beta$ for which $\psi(i, \alpha) = k_i$.

Recall that $e_\beta : (\beta + 1) \times \omega \to \omega$ is finite-to-one. Therefore, for each $i < \omega$, we can fix $\alpha_i < \beta$ such that $e_\beta(i, \alpha_i) \neq k_i = \psi(i, \alpha_i)$.

Now define $f : \omega \to [\omega_1]^{<\omega}$ by setting $f(i) = \{\alpha_i, \beta\}$ for all $i < \omega$. Then, for each $i < \omega$, we have

$$\varphi_f(i, \alpha_i) = e_\beta(i, \alpha_i) \neq k_i = \psi(i, \alpha_i).$$

But then $\varphi_f \neq^* \psi$, so ψ does not trivialize Φ. $\square$

Limits to good behavior

It follows that:

1. the "uncountable earring space" is a ZFC counterexample to the additivity of strong homology;
2. the pro-abelian group "$\prod_{\omega}$" $\bigoplus_{\omega_1} \mathbb{Z}$ provides a ZFC counterexample to the category of pro-abelian groups embedding fully faithfully into Cond(Ab).

References

All paintings by Alfred William Finch.

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Thank you!