An ultraproduct approach to spectral theorems

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Motivation

- originally trying to justify methods used by physisists in quantum mechanical calculations
  - using eigenvectors where such don't really exist
  - using finite-dimensional approximation without specifying how they approximate the space being studied
- I talked about a first approach last time I was here
- our next approach was *Rigged Hilbert spaces* and distributions, which turned into a study of ultraproduct approaches to spectral theorems

Background on operators

Recall:

Definition

For a bounded operator A on a Hilbert space, there is a unique operator A^*, the *adjoint of A* that for all u and v satisfies

$$\langle Au, v\rangle = \langle u, A^* v\rangle$$

If it happens that $A = A^*$, A is called *self-adjoint*.

Definition

The *spectrum* of an operator A, denoted $\sigma(A)$, is the set of $\lambda \in \mathbb{C}$ such that

$$A - \lambda I$$

does not have a bounded inverse.

Spectral theorem for bounded self-adjoint operators

Theorem (Spectral theorem)

For A a bounded self-adjoint operator on a Hilbert space H, there exists a measure μ on $\sigma(A)$ and an isomorphism $U : H \to L_2(\sigma(A), \mu)$ such that

$$UAU^{-1}f = M_g f$$

where M_g is the multiplication operator $f(x) \mapsto g(x)f(x)$.

Moreover, if A has a cyclic vector, the multiplication operator is just $f(x) \mapsto xf(x)$.

Cyclic vectors

Definition

For H a Hilbert space and A a bounded self-adjoint operator, a vector φ is called *cyclic* for A if the vectors $A^n\varphi$, $n < \omega$, span a dense subset of H.

Fact

A Hilert space can be decomposed into a direct sum of invariant subspaces, each with a cyclic vector,

So from the spectral theorem point of view, one can work with cyclic vectors and just get a sum of L_2-spaces.

Self-adjoint via unitary

Theorem (Stone)

There is a one-one correspondence between unitary operators U and self-adjoint operators A with spectrum $\subseteq [0,1]$ and not having 0 in the point spectrum, given by $U = e^{2\pi i A}$.

Fact

We can modify the above, to consider spectra $\subset [-\frac{\pi}{2}, \frac{\pi}{2}]$, and a correspondence $U = e^{iA}$.

So, we start with a bounded self-adjoint operator A, with a cyclic vector φ (of norm 1).

We assume $\sigma(A) \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$, and consider $U = e^{iA}$.

Now φ is cyclic also for U, in the sense that the vectors $U^k\varphi$, $k \in \mathbb{Z}$, span a dense set of H.

Finding finite dimensional approximations of (H, U)

Consider the spanning vectors

$$\cdots \quad U^{-n}\varphi \quad U^{-n+1}\varphi \quad \cdots \quad U^{-1}\varphi \quad \varphi \quad U\varphi \quad \cdots \quad U^{n-1}\varphi \quad U^{n}\varphi \quad \cdots$$

Define finite dimensional spaces

$$H_N = \overline{\mathrm{span}\{U^k\varphi : -N \leq k \leq N\}}$$

$$U^{-N}\varphi \quad \cdots \quad U^{-1}\varphi \quad \varphi \quad U\varphi \quad \cdots \quad U^{N}\varphi$$

Approximations of U

Let

$$H_N^- = \overline{\mathrm{span}\{U^k\varphi : -N \leq k < N\}}$$

and

$$H_N^+ = \overline{\mathrm{span}\{U^k\varphi : -N < k \leq N\}}$$

and let W^+ and W^- be their corresponding orthogonal complements in H_N

$$H_N = H_N^- \oplus W^+ = W^- \oplus H_N^+$$

Let U_N be built from

- U on H_N^-
- a unitary operator mapping W^+ to W^-

Eigenvectors

In each H_N

- U_N is unitary and has an eigenvector basis $(u_N(k))_{k<2N+1}$ with corresponding eigenvalues $\lambda_N(k)$,
- the cyclic vector can be written as

$$\varphi = \sum_{k=0}^{2N} \xi_N(k) u_N(k)$$

where each $\xi_N(k)$ is a non-negative real, and $\sum_{k=0}^{2N} \xi_N(k)^2 = 1$.

Note: The spaces H_N extend each other, but the bases do not.

The ultraproduct model

Let $\mathcal{U}$ be a non-principal ultrafilter on ω.

We will work with the metric ultraproduct of the spaces (H_N, U_N).

As φ is cyclic, there is a natural embedding of H into H^m:

Definition

For $P(X, Y) \in \mathbb{C}[X, Y]$, let $P(U, U^{-1})$ be natural interpretation as an operator on H, e.g.,

$$X^2Y(U, U^{-1}) = U \circ U \circ U^{-1} = U.$$

Then $P(U, U^{-1})(\varphi)$ makes sense in almost all H_N, and thus we can define $G^m : H \to H^m$ by

$$G^m(P(U, U^{-1})(\varphi)) = (P(U, U^{-1})(\varphi))_{N<\omega}/\mathcal{U}.$$

Spectral measure in H_N

Remember: in each H_N, $\varphi = \sum_{k=0}^{2N} \xi_N(k) u_N(k)$

Definition

For each $N < \omega$, define a measure μ_N for subsets $X \subset \mathbb{C}$:

$$\mu_N(X) = \sum_{k<2N, \lambda_N(k) \in X} \xi_N(k)^2$$

Note that for all $X \subset \mathbb{C}$, $\mu_N(X) \leq 1$, as $\|\varphi\| = 1$.

Spectral measure from ultraproduct

We construct a spectral measure for U in steps:

1. For each $X \subseteq \mathbb{C}$, let $\mu^n(X)$ be the ultralimit $\lim_{\mathcal{U}} \mu_N(X)$
2. consider a set of *nice* vertical (I_r) and horizontal (J_r) lines for which for all $\delta > 0$ there is $\varepsilon > 0$ such that their "ε-thickenings" I_r^ε and J_R^ε have small "measure"
$$\mu^n(I_r^\varepsilon) < \delta, \quad \mu^n(J_r^\varepsilon) < \delta$$
3. define an outer measure based on the μ^n-value of boxes bounded by nice lines
$$\mu^*(Y) = \inf \left\{ \sum_{k=0}^{\infty} \mu^n(X_k) \mid X_k \text{ a } \textit{nice box}, Y \subseteq \bigcup_{k<\omega} X_k \right\}$$
4. by Caratheodory's construction, find a σ-algebra of sets for which μ^* is a measure

Spectral representation for U

Consider the space $L_2(S, \mu^*)$, where S is a suitable compact subset of $\mathbb{C}$, the complement of which has zero μ^*-measure.

Definition

Let $D(S)$ be the subspace of $C(S)$ that consists of functions

$$f_P(\lambda) = P(\lambda, \bar{\lambda})$$

where $P \in \mathbb{C}[X, Y]$ and $\bar{\lambda}$ is the complex conjugate of λ.
Define U_D and U_D^* by

$$U_D(f_P) = f_{XP} \quad \text{and} \quad U_D^*(f_P) = f_{YP}.$$

Note that $U_D(f_P)(\lambda) = \lambda f_P(\lambda)$, and $U_D^*(f_P)(\lambda) = \bar{\lambda} f_P(\lambda)$.

Theorem

1. *The measure μ^* is zero outside the spectrum of U.*
2. *There is an isometry mapping $L_2(S, \mu^*)$ to H, and (the extension of) U_D to U. (We find it going via H^m.)*
3. *We can transfer the measure μ^* from the unit circle to the real line to get (from the isometry above) an isomorphism between $L_2(\sigma(A), \mu)$ and (H, A).*

Next steps

- currently working on extending the result to unbounded self-adjoint operators
  - also these translate to unitary operators, that can be used for a decomposition
  - we get bounded self-adjoint operators on finite dimensional spaces (but without a uniform bound), and need to take a *guarded* ultraproduct of them – this gives a partially defined operator
- the following step is to look at bounded normal operators via their decomposition into self-adjoint operators

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Thank you!