Borel equivalence relations and forcing

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joint work with F. Calderoni

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(The Union Problem): Does hyperhyperfinite imply hyperfinite?

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More general examples:

- Def. Let G be a countable group acting on a space X in a Borel way. The induced orbit equivalence relation E_G is a cber.
- And actually every cber is obtained in such a way (Feldman Moore)

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Theorem (Smythe)

1. $\mathrm{Gen}(\mathbb{P}, M)$ is a G_δ set.
2. $E^M_{\mathbb{P}}$ is a countable Borel equivalence relation.
3. $E^M_{\mathbb{P}}$ is induced by the action of the group of automorphisms of $\mathbb{P}$ that are in M.

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A key point in the proof is that $\mathbb{P}$ is weakly homogeneous iff the action generating $\mathsf{E}^{\mathsf{M}}_{\mathbb{P}}$ is generically ergodic. This is combined with having meager orbits, which follows from $\mathbb{P}$ being atomless.

We show a characterization of smoothness for equivalence relations of the form $\mathsf{E}^{\mathsf{M}}_{\mathbb{P}}$.

A Characterization of Smoothness

Definition

$(\dagger)_p$: for all $p' \leq p$, there are incompatible $q, r \leq p'$, such that there exists distinct generic filters G, H, such that $q \in G, r \in H$ and $V[H] = V[G]$.

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The idea: a translation of the topological characterization of smoothness via condensation, which is weaker than generic ergodicity.

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idea of the proof: take lottery sums of nonisomorphic homogeneous forcings in a tree like fashion.

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For $p = \langle s, A\rangle \in \mathbb{P}$, set $\mathrm{lh}(p) = |s|$.

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Theorem. $E^M_{\mathbb{P}}$ is hyperfinite.

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Then, for comeagerly many x, there are comegearly many such y's.

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Can identify finite partial function from ω to ω with $\mathbb{P}$, and so elements in the Baire space with elements in X.

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