A Variant of Baumgartner's Axiom with Applications

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• Discuss the relation of this variant to other axioms.

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• A family $\mathcal{P} \subseteq [\omega]^\omega$ has the strong finite intersection property if for every finite $\mathcal{A} \subseteq \mathcal{P}$ the intersection $\bigcap \mathcal{A}$ is infinite. The cardinal $\mathfrak{p}$ is the least size of a family $\mathcal{P}$ with the strong finite intersection property but no pseudointersection i.e. no $A \in [\omega]^\omega$ which is mod finite contained in every member of $\mathcal{P}$.

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• The additivity of the null ideal, $\mathrm{add}(\mathcal{N})$ is the least size of a family of null sets whose union is non null.

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Fact (Bell's Equality)

For all κ *we have* $\mathfrak{p} > \kappa$ *if and only if* $\mathsf{MA}_\kappa(\sigma\text{-centered})$ *holds.*

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Theorem (Brouwer)

For every finite $n < \omega$ the Euclidean space $\mathbb{R}^n$ is CDH.

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Denote the statement above by BA for Baumgartner's Axiom.

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Let κ be a cardinal and X a topological space. We denote by $\mathsf{BA}_\kappa(X)$ the statement that for every pair $A, B \subseteq X$ which are κ-dense there is an autohomeomorphism $h : X \to X$ so that $h``A = B$. If $\kappa = \aleph_1$ we drop the subscript.

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Thus $\mathsf{BA} = \mathsf{BA}_{\aleph_1}(\mathbb{R})$. We are interested in general in the question of what consequences and implications between axioms like these can we expect for various κ and X?

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- For any finite $n > 1$, if X is either $\mathbb{R}^n$ or an n-dimensional compact manifold then $\mathsf{BA}_\kappa(X)$ holds for every $\kappa < \mathfrak{p}$. (Steprāns-Watson, 1989)

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Consequently $\mathsf{BA}_{\aleph_1}(\mathbb{R}^n)$ does not imply BA for any finite $n > 1$.

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Question

Does BA *imply* $\mathfrak{p} > \aleph_1$*?*

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It's worth noting that $\mathfrak{b}$ has (roughly) the same relation to eventual domination that $\mathfrak{t}$ (which we now know is the same as $\mathfrak{p}$) has to eventual inclusion...

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1. (Medini) If there is a perfect analytic subset $A \subseteq X$ all of whose κ dense subsets are homeomorphic then $\kappa \neq \mathfrak{b}$.

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2. (S.) If all κ-dense subsets of X are homeomorphic then $2^{\aleph_0} = 2^\kappa$.

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For every family of size $\aleph_1$ with the strong finite intersection property $\mathcal{P}$ there are $\aleph_1$-dense $A, B \subseteq \mathbb{R}$ so that the standard forcing with finite conditions to add an isomorphism from A to B adds a pseudointersection to $\mathcal{P}$. In particular $\mathrm{MA}_{\aleph_1}(\sigma\text{-centered})$ holds in Baumgartner's original model of BA.

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What we mean more precisely by "the standard forcing" would take us too far afield but feel free to ask me later. In any case, of course this is not a ZFC proof of the conjecture.

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• If $A, B \subseteq \omega^\omega$, then an isometry from A to B is simply a map $f : A \to B$ so that for all $x, y \in A$ we have for all $k < \omega$, $x \restriction k = y \restriction k$ if and only if $f(x) \restriction k = f(y) \restriction k$

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- Similarly a map $f : A \to B$ is Lipschitz if for all $x, y \in A$ we have for all $k < \omega$, $x \upharpoonright k = y \upharpoonright k$ implies $f(x) \upharpoonright k = f(y) \upharpoonright k$

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- Similarly a map $f : A \to B$ is Lipschitz if for all $x, y \in A$ we have for all $k < \omega$, $x \restriction k = y \restriction k$ implies $f(x) \restriction k = f(y) \restriction k$
- It's easy to check that if $f : A \to B$ is Lipschitz it lifts uniquely to a map $\hat{f} : \bar{A} \to \bar{B}$ and and in the case it's an isometry it lifts to a homeomorphism $\hat{f} : \bar{A} \to \bar{B}$. In particular Lipschitz maps defined on a dense subset of ω^ω or 2^ω lift to maps defined on the whole space.

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This suggests the following axioms.

Definition

Let X be either ω^ω or 2^ω. The axiom $\mathrm{BA}_{\mathrm{Isom}}(X)$ states that for every pair $A, B \subseteq X$ which are $\aleph_1$-dense there is an isometry $f : X \to X$ so that $f``A = B$.

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Proof.

We just do ω^ω. For each $s \in \omega^{<\omega}$ let O_s be an $\aleph_1$-sized subset of $[s]$ which is zero on the odd coordinates above $|s|$ and E_s the same with "odd" replaced by "even". The $O = \bigcup O_s$ and E_s are $\aleph_1$-dense. If $f : E \to O$ is a map then by pigeon hole there are s and t and two $x, y \in O_s$ so that $f(x), f(y) \in E_t$ but then f isn't an isometry. $\square$

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• The axiom $\overline{\mathsf{BA}}_{\mathrm{Lip}}(2^\omega)$ states that for every $A, B \subseteq 2^\omega$ which are $\aleph_1$-dense there are countably many injective Lipschitz functions $f_n : 2^\omega \to 2^\omega$ so that $B = \bigcup_{n<\omega} f_n``A$.

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Theorem

All fours axioms $\mathsf{BA}_{\mathrm{Lip}}(\omega^\omega)$, $\mathsf{BA}_{\mathrm{Lip}}(2^\omega)$, $\overline{\mathsf{BA}}_{\mathrm{Lip}}(\omega^\omega)$, *and* $\overline{\mathsf{BA}}_{\mathrm{Lip}}(2^\omega)$ *are consistent and can be forced by ccc forcing over a model of* CH.

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Both of these results are similar to the proofs of the corresponding theorems for BA by Baumgartner.

Applications Between Spaces

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Again, it is not known whether $\mathsf{BA}(\omega^\omega)$ implies $\mathsf{BA}(2^\omega)$ (or vice versa).

Applications To Cardinal Arithmetic

Recall that for any perfect Polish space X we have $\mathsf{BA}(X)$ implies $2^{\aleph_0} = 2^{\aleph_1}$. An similar proof shows the same holds for the stronger Lipschitz versions.

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The proof uses actually the existence of an almost disjoint family of subsets of ω_1 of size $2^{\aleph_1}$ which follows from $2^{\aleph_0} < \aleph_{\omega_1}$ (amongst other hypotheses) but is not a theorem of ZFC (Baumgartner).

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By a result of Bartoszyński-Shelah there is a set $A \subseteq 2^\omega$ of size $\mathfrak{b}$ which cannot be continouously surjected onto any unbounded $B \subseteq \omega^\omega$. Suppose $\mathfrak{b} = \aleph_1$. Let $A' \supseteq A$ be $\aleph_1$-dense. By treating ω^ω as 2^ω minus a countable dense set we can find an unbounded, $\aleph_1$-dense $B \subseteq 2^\omega$ so that $B = \{x_\alpha \mid \alpha \in \omega_1\}$ is homeomorphic to a *scale*: if $\alpha < \beta$ then x_α is eventually dominated by x_β.

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Applications To Cardinal Characteristics II

More surprising is the following.

Theorem (S.)

$\overline{\mathsf{BA}}_{\mathrm{Lip}}(2^\omega)$ *(and hence* $\overline{\mathsf{BA}}_{\mathrm{Lip}}(\omega^\omega)$*) implies that* $\mathrm{add}(\mathcal{N}) > \aleph_1$.

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Note $\mathsf{MA}_{\aleph_1}(\sigma\text{-centered})$ is consistent with $\mathrm{add}(\mathcal{N}) = \aleph_1$ so an immediate consequence of this theorem is that $\overline{\mathsf{BA}}_{\mathrm{Lip}}(2^\omega)$ (and $\overline{\mathsf{BA}}_{\mathrm{Lip}}(\omega^\omega)$) have concrete consequences the classical $\mathsf{BA}(\omega^\omega)$ and $\mathsf{BA}(2^\omega)$ do not.

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I want to sketch a proof of this theorem. Recall that if $h : \omega \to \omega$ is strictly increasing then an *h-slalom* is a function $\varphi : \omega \to [\omega]^{<\omega}$ so that for all n we have $|\varphi(n)| \leq h(n)$.

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• Similarly let us write $f \in \varphi$ if for every $n < \omega$ we have $f(n) \in \varphi(n)$.

Finally for a set $A \subseteq \omega^\omega$ we say an h-slalom φ, captures A if it eventually captures every element.

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Note the point is that the cardinal doesn't depend on which h we choose - however it must be uniform for all A of size $<\kappa$.

Cardinal Characteristics II

Using this we can show that $\overline{\mathsf{BA}}_{Lip}(\omega^\omega)$ implies $\mathrm{add}(\mathcal{N}) > \aleph_1$. The case of 2^ω, which we stated, is similar but requires more definitions so we do the case of ω^ω in the interest of time and simplicity.

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Assume $\mathsf{BA}_{\mathrm{Lip}}(\omega^\omega)$. We will show that every set of size $\aleph_1$ is caught in an h-slalom for $h(n) = n2^{n+1}$. Let A be an arbitrary set of set $\aleph_1$. By possibly making it bigger we can assume that A is $\aleph_1$-dense.

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- By assumption there is an $f : \omega^\omega \to \omega^\omega$ so that $f``B = A$ and f is Lipschitz. Fix such an f. $\square$

Proof.

Fix $s \in \omega^{<\omega}$ and let $\varphi_s : \omega \to [\omega]^{<\omega}$ be defined by $\varphi_s(n) = \{m \mid \exists x \in B_s\, f(x)(n) = m\}$. One can show that this is a 2^{n+1}-slalom.

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• Now observe that if $x \in B_s$ then for every $n < \omega$ we have $f(x)(n) \in \varphi_s(n)$ by construction. In other words, for each $s \in \omega^{<\omega}$ the forward image $f``B_s$ is caught (totally, not eventually) by φ_s. In particular there are countably many 2^{n+1}-slaloms $\{\varphi_s \mid s \in \omega^{<\omega}\}$ so that every element of A is totally caught by (at least) one of them.

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• Now enumerate $\omega^{<\omega}$ as $\{s_n \mid n < \omega\}$ and let $\varphi(n) = \bigcup_{i<n} \varphi_{s_i}(n)$. This is a $n2^{n+1}$-slalom which eventually captures every element of A, completing the proof. □

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I want to finish by sketching this proof. The idea is that any uncountable disjoint pair of Cohen reals are so far from being Lipschitz embeddable one into the other that they cannot be made so while forcing this fragment of MA.

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• If κ is uncountable and $\{c_\alpha \mid \alpha \in \kappa\}$ are mutually Cohen generic then for every uncountable $A, B \subseteq \kappa$ which are disjoint we have that $\{c_\alpha \mid \alpha \in A\}$ Lipschitz loathes $\{c_\alpha \mid \alpha \in B\}$. It's also easy to cook up an example under CH.

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Theorem

Let δ be an ordinal, $A, B \subseteq \omega^\omega$ and assume A Lipschitz loathes B. If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \in \delta\rangle$ is a finite support iteration of ccc forcing notions so that for each $\alpha < \delta$ we have that $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ preserves that $\check{A}$ Lipschitz loathes $\check{B}$", then $\Vdash_\delta$ "$\check{A}$ Lipschitz loathes $\check{B}$".

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The theorem now follows from the following two lemmas combined with the iteration theorem from the previous slide. In what follows fix $A, B \subseteq 2^\omega$ so that A Lipschitz loathes B.

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Lemma

If T is a tree of height ω_1 with no cofinal branch and $\mathbb{P}$ is the standard ccc forcing to specialize T with finite conditions then $\mathbb{P}$ forces that A Lipschitz loathes B.

Therefore the standard iteration to force $\mathrm{MA}_{\aleph_1}(\mathrm{Knaster})+$ "Every Aronszajn tree is special" will preserve that A Lipschitz loathes B which proves the theorem. We remark, somewhat oddly, that in this model A and B are homeomorphic, even though no uncountable subset of A can be Lipschitz injected into B.

Steprāns-Watson, Again

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Question

Does $\overline{\mathsf{BA}}_{\mathrm{Lip}}(\omega^\omega)$ *imply* $\mathfrak{p} > \aleph_1$*?*

Thank You!