The inner model $C(\mathrm{aa}^+)$

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Extended constructibility

Research program started by Kennedy, Magidor, Väänänen ([2], [3]).

Definition

Suppose $\mathcal{L}^*$ is a logic. The hierarchy (J'_α), α a limit ordinal, of *sets construbtible using* $\mathcal{L}^*$ and the class Tr are defined by transfinite double induction, as follows:

$$\mathrm{Tr} = \{(\alpha, \varphi(\vec{a})) : (J'_\alpha, \in, \mathrm{Tr}\restriction\alpha) \models \varphi(\vec{a}), \varphi(\vec{x}) \in \mathcal{L}^*, \vec{a} \in J'_\alpha, \alpha \in \mathrm{Lim}\},$$

where

$$\mathrm{Tr}\restriction\alpha = \{(\beta, \vec{a}) \in \mathrm{Tr} : \beta < \alpha\},$$

and

$$\begin{aligned} J'_0 &= \emptyset, \\ J'_{\alpha+\omega} &= \mathrm{rud}_{\mathrm{Tr}}(J'_\alpha \cup \{J'_\alpha\}), \\ J'_{\omega\delta} &= \bigcup_{\alpha<\delta} J'_{\omega\alpha} \text{ for limit } \delta. \end{aligned}$$

The class $\bigcup_{\alpha\in\mathrm{Ord}} J'_\alpha$ is denoted by $C(\mathcal{L}^*)$.

Inner model $C(\mathrm{aa})$

- $C(\mathrm{aa})$ obtained from stationary logic
- Stationary logic $\mathcal{L}(\mathrm{aa})$ adds to first-order logic the aa-quantifier, defined as follows:
$$M \models \mathrm{aa}\, s\, \varphi(s, \vec{a})$$
if there is a closed unbounded set $C \subset \mathcal{P}_{\omega_1}(M)$ such that for any $s \in C$,
$$(M, s) \models \varphi(s, \vec{a}).$$

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- The aa-quantifier can express, e.g., that a set is countable, that an ordinal has countable cofinality, or that a linear order is $\aleph_1$-like.

The aa^+-quantifier

- Variant of the aa-quantifier
- Defined intuitively by

$$M \models \mathrm{aa}^+ s\varphi(s, \vec{a})$$

is there is a club $C \subset \mathcal{P}_{\omega_1}(M)$ such that for each $s \in C$

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- For nested aa^+-quantifiers, the clubs are always taken in M not in the next admissibles
- Advantage over the aa-quantifier: the aa^+-quantifier can talk about transitive collapses of well-founded sets in M
- $\mathcal{L}(\mathrm{aa}^+)$ logic which adds the aa^+-quantifier to first order logic
- $C(\mathrm{aa}^+)$ the inner model obtained from $\mathcal{L}(\mathrm{aa}^+)$

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- For transitive A, the next admissible set $A^+ = L_\alpha(A)$, where α is the least such that $(L_\alpha(A), \in) \models KP$.

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- For transitive A, the next admissible set $A^+ = L_\alpha(A)$, where α is the least such that $(L_\alpha(A), \in) \models KP$.
- For a structure $\mathcal{M} = (M, R_i)_{i \in I}$ such that M is not a transitive set or $\in$ is not in the vocabulary of $\mathcal{M}$, a more natural notion of next admissible is based on structures which have the elements of the domain M as urelements
- For $\mathcal{M}$ as above, a set $(\mathcal{M}; A, \in)$ is admissible above $\mathcal{M}$ if $(\mathcal{M}; A, \in) \models KPU$ (KP with urelements) and $M \in A$
- For $\mathcal{M}$ as above, the next admissible of $\mathcal{M}$ is $(\mathcal{M}; A, \in)$ where $A = \bigcap\{B : (\mathcal{M}; B, \in) \text{ admissible above } \mathcal{M}\}$.

Club determinacy

Idea: no definable stationary co-stationary sets.

Definition

The inner model $C(\mathrm{aa})$ is said to be Club Determined if for all α and for all $\varphi(\vec{x}, \vec{t}, s) \in \mathcal{L}(\mathrm{aa})$, and for all finite sequences $\vec{t}$ of countable subsets of J'_α:

$$(J'_\alpha, \in, \mathrm{Tr}\restriction\alpha) \models \forall \vec{x}\, [\, \mathrm{aa}\, s \varphi(\vec{x}, \vec{t}, s) \vee \mathrm{aa}\, s \neg\varphi(\vec{x}, \vec{t}, s)\,].$$

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Club Determinacy

Theorems (Kennedy, Magidor, Väänänen)

- *If there is a proper class of Woodin cardinals, then Club Determinacy holds in* $C(\mathrm{aa})$.
- *Suppose* $C(\mathrm{aa})$ *satisfies Club Determinacy. Then every regular* $\kappa \geq \omega_1^V$ *is measurable in* $C(\mathrm{aa})$.
- *Suppose* $C(\mathrm{aa})$ *satisfies Club Determinacy. Then the first-order theory of* $C(\mathrm{aa})$ *is set forcing absolute.*

Lemma

All the above results hold for $C(\mathrm{aa}^+)$ *with the same proof.*

aa-mice by Kennedy-Magidor-Väänänen

- An aa-mouse is a structure of the form $(J_\alpha^T, \in, T, T^*, (P)_\xi)$.
- For limit $\beta < \alpha$, $T_\beta = \{\varphi(\vec{a}) : (\beta, \vec{a}) \in T\}$ is a complete consistent $\mathcal{L}(\mathrm{aa})$-theory with parameters from J_β^T that extends the first-order theory of $(J_\beta^T, \in, T \restriction \beta)$.
- T^* is a complete consistent $\mathcal{L}(\mathrm{aa})$-theory that extends the first-order theory of $(J_\alpha^T, \in, T)$.

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- For limit $\beta < \alpha$, $T_\beta = \{\varphi(\vec{a}) : (\beta, \vec{a}) \in T\}$ is a complete consistent $\mathcal{L}$(aa)-theory with parameters from J_β^T that extends the first-order theory of $(J_\beta^T, \in, T \restriction \beta)$.
- T^* is a complete consistent $\mathcal{L}$(aa)-theory that extends the first-order theory of $(J_\alpha^T, \in, T)$.
- By starting from a countable aa mouse $(M_o, \in, T, T^*)$ and iterating the aa-ultrapower construction ω_1-many times, the images of the previous iterates $\{j_{\alpha\omega_1}[M_\alpha] : \alpha < \omega_1\}$ form a club in M_{ω_1}.
- This allows one to prove that the predicate $T^*_{\omega_1}$ of the ω_1-iterate is correct about $\mathcal{L}$(aa)-truth.
- Consequently, the ω_1-iterate is a level of the C(aa)-hierarchy.

Theorem (Kennedy, Magidor, Väänänen)

If Club Determinacy holds in $C(\mathrm{aa})$, then $C(\mathrm{aa})$ satisfies the Continuum Hypothesis.

Theorem (Goldberg, Steel [1])

Is Club Determinacy holds in $C(\mathrm{aa})$, then $C(\mathrm{aa})$ satisfies the Ultrapower Axiom.

Theorem (Goldberg, Steel [1])

Is Club Determinacy holds in $C(\mathrm{aa})$, then $C(\mathrm{aa})$ satisfies GCH.

Good formulas

- In stationary logic, $\varphi \to \mathrm{aa}\, s\varphi$, where s does not occur in φ, is valid.
- In $\mathcal{L}(\mathrm{aa}^+)$, $\varphi \to \mathrm{aa}^+ s\varphi$ is not valid in general.
- Consider, e.g., a model M such that axiom φ of KP fails in M. Then $M \models \neg\varphi$, but necessarily $M \models \mathrm{aa}^+ s\varphi$.

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- We want to be able to use $\varphi \to \mathsf{aa}^+ s\varphi$ in the definition of aa^+-mice

Good formulas

- Let $x \in J^T_{\sup T}$ be a shorthand for a first-order formula that says in any model containing J^T_α, where $T \subset \mathrm{Lim} \cap \alpha \times J^T_\alpha$, as a subset that $x \in J^T_\alpha$.

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- For ordinals $\eta_1, \dots, \eta_n$, we let $\Psi(y, \gamma, J^T_{\sup T}, \eta_1, \dots, \eta_n, s_1, \dots, s_k)$ be a shorthand for the formula

$$\begin{aligned} &\exists z\, (\forall w\, (w \in z \leftrightarrow w \in J^T_{\sup T})) \\ &\wedge y \in J_\gamma(w, \in, T, \mathbf{P}_{\eta_1}, \dots, \mathbf{P}_{\eta_n}, s_1, \dots, s_k) \\ &\wedge \neg \exists \gamma' \leq \gamma\, [\, J_{\gamma'}(w, \in, T, \mathbf{P}_{\eta_1}, \dots, \mathbf{P}_{\eta_n}, s_1, \dots, s_k) \models KP\,]. \end{aligned}$$

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- I.e., $\Psi(y, \gamma, J^T_{\sup T}, \eta_1, \ldots, \eta_n, s_1, \ldots, s_k)$ says in any model containing the next admissible $(J^T_\alpha, \in, T, P_{\eta_1}, \ldots, P_{\eta_n}, s_1, \ldots, s_k)^+$ that y is in the next admissible of $(J^T_\alpha, \in, T, P_{\eta_1}, \ldots, P_{\eta_n}, s_1, \ldots, s_k)^+$.

Good formulas

Definition (Good formula)

Suppose φ is an $\mathcal{L}(\mathsf{aa}^+)$-formula in vocabulary τ_ξ^-. We say that a set $\{\theta' : \theta \text{ a subformula of } \varphi\}$ is an existential specification of φ in vocabulary τ_ξ^- if it satisfies:

1. If θ is atomic, then $\theta' = \theta$.
2. If $\theta = \psi \wedge \gamma$, then $\theta' = \psi' \wedge \gamma'$. If $\theta = \neg\psi$, then $\theta' = \neg\psi'$. If $\theta = \mathsf{aa}^+ s\, \psi$, then $\theta' = \mathsf{aa}^+ s\, \psi'$.

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3. If θ is $\exists x\, \psi(x)$, then θ' is $\exists x\, (\psi^* \wedge \psi'(x))$ where ψ^* is one of the following:
   - 3.1 $x \in J^T_{\sup T}$.
   - 3.2 There are $\{\eta_1, \ldots, \eta_n\} \subset \xi$ and $\{k_1, \ldots, k_l\} \subset \omega$, at least one of them nonempty, such that
     - 3.2.1 if $\{k_1, \ldots, k_l\} \neq \emptyset$, then for some m, θ is in the scope of $\mathsf{aa}^+ s_1, \ldots \mathsf{aa}^+ s_m$ in φ, $\{k_1, \ldots, k_l\} \subset m$,
     - 3.2.2 ψ^* is $\exists \beta \in \mathrm{Ord}\,(\, \Psi(x, \beta, J^T_{\sup T}, \eta_1, \ldots, \eta_n, s_{k_1}, \ldots, s_{k_l}))$.

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We say that φ is *good* (in τ_ξ^-) if there is a formula ψ (in τ_ξ^-) and an existential specification $\{\theta' : \theta \text{ a subformula of } \psi\}$ of ψ such that $\varphi = \psi'$.

aa^+-premouse

An aa^+-premouse in vocabulary $\tau_\xi = \{\in, R_T, R_{T^*}\} \cup \{\mathbf{P}_\alpha : \alpha < \xi\}$ is a structure of the form

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where

- $T \subset \{\beta < \alpha : \mathrm{Lim}(\beta)\} \times \mathcal{L}(\text{aa}^+)$, and for all limit ordinals $\beta < \alpha$, $T_\beta = \{\varphi(\vec{a}) : (\beta, \varphi(\vec{a})) \in T\}$ is an $\mathcal{L}(\text{aa}^+)$-theory in vocabulary τ_0^- with parameters from J_β^T.

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- $T \subset \{\beta < \alpha : \mathrm{Lim}(\beta)\} \times \mathcal{L}(aa^+)$, and for all limit ordinals $\beta < \alpha$, $T_\beta = \{\varphi(\vec{a}) : (\beta, \varphi(\vec{a})) \in T\}$ is an $\mathcal{L}(aa^+)$-theory in vocabulary τ_0^- with parameters from J_β^T.
- For each $\varphi \in \mathcal{L}(aa^+)$ in vocabulary τ_0^- and each $\vec{a} \in J_\beta^T$, T_β contains either $\varphi(\vec{a})$ or $\neg\varphi(\vec{a})$.
- T_β is closed under the rules for first-order logic and the axioms $(A0^+)$ - $(A5^+)$, and weakly consistent for good formulas relative to J_β^T
- T_β attempts to describe what $\mathcal{L}(aa^+)$-truth in the relevant next admissibles of $(J_\beta, \in, T\restriction\beta)$, where $T\restriction\beta =_{\mathrm{def}} T\restriction J_\beta^T$, but may be wrong about the aa^+-quantifiers.

aa^+-premouse

- If φ is a first-order formula in vocabulary τ_0^- and $\vec{a} \in J^T_\beta$, then $\varphi(\vec{a}) \in T_\beta$ if and only if $(J^T_\beta, \in, T\restriction\beta) \models \varphi(\vec{a})$.
- Conditions which describe in which way the formulas T_β are about the next admissibles. For example, the following sentence, which intuitively says that J^T_α is a member of the relative next admissible, is always in T_β:

$$\mathrm{aa}^+ s_1 \ldots \mathrm{aa}^+ s_m \,\exists y$$
$$[\,\exists\beta \in \mathrm{Ord}\,(\,\Psi(y, \beta, J^T_{\sup T}, s_1, \ldots, s_m)\wedge$$
$$\forall x\,(\,\exists\gamma \in \mathrm{Ord}\,\Psi(x, \gamma, J^T_{\sup T}, s_1, \ldots, s_m) \to (x \in J^T_{\sup T} \leftrightarrow x \in y)\,)]$$

- ...

aa^{+}-premouse

- $T^* \subset \mathcal{L}(\mathrm{aa}^+) \times J^T_\alpha$ is an $\mathcal{L}(\mathrm{aa}^+)$-theory in the vocabulary τ^-_ξ with parameters from J^T_α.
- T^* is complete for good formulas in vocabulary τ^-_ξ with parameters in J^T_α, closed under first-order axioms and the axioms (A0$^+$) - (A5$^+$), and weakly consistent for good formulas in vocabulary τ^-_ξ.
- T^* is right about first order truth in the next admissible sets built with the predicates P_η but may be wrong about the aa^+-quantifier.
- If a first-order formula φ is good, the $\mathbf{P}$-predicates appearing in φ are among $\mathbf{P}_{\eta_1}, \ldots, \mathbf{P}_{\eta_n}$, and $\vec{a} \in J^T_\alpha$, then

$$\varphi(\vec{a}) \in T^* \Leftrightarrow (J^T_\alpha, \in, T, P_{\eta_1}, \ldots, P_{\eta_n})^+ \models \varphi(\vec{a}).$$

aa$^+$-premouse

Some other conditions

- For any good $\mathcal{L}(\mathrm{aa}^+)$-formula φ and any $\vec{a} \in J^T_\alpha$, $\varphi(\vec{a})$ is in T^* if and only if $\mathrm{aa}^+ \vec{s} \varphi(\vec{a})$ is in T^*, where none of the subset variables in $\vec{s}$ occur in φ.
- For any good $\mathcal{L}(\mathrm{aa}^+)$-formula φ, $\exists x\,(x \in J^T_{\sup T} \wedge \varphi(x, \vec{a}))$ is in T^* if and only if there is some $b \in J^T_\alpha$ such that $\varphi(b, \vec{a})$ is in T^*.
- Coherence between T and T^* for formulas not containing any **P**-predicates: For all limit $\beta < \alpha$, all $\vec{a} \in J^T_\beta$, and all very good φ in vocabulary τ_0^-,

$$\varphi(\vec{a}) \in T_\beta \text{ if and only if } (\varphi(\vec{a})^{(J^T_\beta)})' \in T^*,$$

  where $\varphi(\vec{a})^{(J^T_\beta)}$ is obtained by replacing each occurrence of $x \in J^T_{\sup T}$ by $x \in J^T_\beta$, and if $\mathrm{aa}^+ s$ appears in φ, each $x \in s$ is replaced by $x \in s \wedge x \in J^T_\beta$.

aa^+-embeddings

Definition

Suppose $\mathbf{M} = (M, \in, T, T^*, (P)_\xi)$ is τ_ξ-structure and $\mathbf{N} = (N, \in, \bar{T}, \bar{T}^*, (\bar{P})_\nu)$ is a τ_ν-structure with $\xi \leq \nu$. A function $\pi : M \to N$ is called an aa^+-embedding if it satisfies the following conditions:

1. π is a first-order elementary embedding from $(M, \in, T, (P)_\xi)$ to $(N, \in, \bar{T}, (\bar{P})_\xi)$.
2. For all good $\varphi \in \mathcal{L}(\mathrm{aa}^+)$ in vocabulary τ_ξ^-, and $\vec{a} \in J_\alpha^T$, $\varphi(\vec{a}) \in T^*$ if and only if $\varphi(\pi(\vec{a})) \in \bar{T}^*$.

aa^+-ultrapower

- The aa^+-ultrapower of an aa^+-mouse $(J_\alpha^T, \in, T, T^*)$ will be a structure of the form $(M, E, S, S^*, (P')_{\xi+1})$
- One new predicate P'_ξ which is the image $j[J_\alpha^T]$ of the domain of the mouse under the aa^+-ultrapower embedding j

aa$^+$-ultrapower

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- One new predicate P'_ξ which is the image $j[J^T_\alpha]$ of the domain of the mouse under the aa$^+$-ultrapower embedding j
- For each finite $d = \{\eta_1, \dots, \eta_n\} \subset \xi + 1$, we form an auxiliary model $\mathbf{M}_d = (M_d, E_d, S_d, S^*_d, P^d_{\eta_1}, \dots, P^d_{\eta_n})$.
- The idea is that if the ultrapower $(M, E, S, S^*, (P')_{\xi+1})$ and the auxiliary model $\mathbf{M}_d$ are well-founded, then the transitive collapse of the domain M_d is the next admissible of the collapse of $(M, E, S, S^*, P'_{\eta_1}, \dots, P'_{\eta_n})$.
- This allows us to show that if the aa$^+$-ultrapower and all the auxiliary models $\mathbf{M}_d$ are well-founded, then the ultrapower collapses to an aa$^+$-premouse

aa^+-ultrapower

Definition

Suppose $\mathbf{J}_\alpha^T = (J_\alpha^T, \in, T, T^*, (P)_\xi)$ is an aa^+-premouse.

1. $M' = M'_\emptyset$ is the set of all $\varphi(s, x, \vec{a})$ in vocabulary τ_ξ^-, where $\vec{a} \in J_\alpha^T$, such that $\mathrm{aa}^+ s\, \varphi(s, x, \vec{y})$ is good, and $\mathrm{aa}^+ s\, \exists x\, (x \in J_{\sup T}^T \wedge \varphi(s, x, \vec{a})) \in T^*$.

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Definition

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1. $M' = M'_\emptyset$ is the set of all $\varphi(s, x, \vec{a})$ in vocabulary τ^-_ξ, where $\vec{a} \in J^T_\alpha$, such that $\mathrm{aa}^+ s\, \varphi(s, x, \vec{y})$ is good, and $\mathrm{aa}^+ s\, \exists x\, (x \in J^T_{\sup T} \wedge \varphi(s, x, \vec{a})) \in T^*$.
2. For nonempty $d \in [\xi + 1]^{<\omega}$, there are two cases. If $\xi \notin d$ and $d = \{\eta_1, \ldots, \eta_n\}$, M'_d is the set of all $\varphi(s, x, \vec{a})$ in vocabulary τ^-_ξ, where $\vec{a} \in J^T_\alpha$, such that $\mathrm{aa}^+ s\, \varphi(s, x, \vec{y})$ is good, and

$$\mathrm{aa}^+ s\, \exists x\, \exists \beta \in \mathrm{Ord}\, (\Gamma(x, \beta, J^T_{\sup T}, \eta_1, \ldots, \eta_n) \wedge \varphi(s, x, \vec{a})) \in T^*.$$

If $\xi \in d$ and $d = \{\eta_1, \ldots, \eta_n, \xi\}$, then M'_d is the set of all $\varphi(s, x, \vec{a})$ in vocabulary τ^-_ξ, where $\vec{a} \in J^T_\alpha$, such that $\mathrm{aa}^+ s\, \varphi(s, x, \vec{y})$ is good, and

$$\mathrm{aa}^+ s\, \exists x\, \exists \beta \in \mathrm{Ord}\, (\Gamma(x, \beta, J^T_{\sup T}, \eta_1, \ldots, \eta_n, s) \wedge \varphi(s, x, \vec{a})) \in T^*.$$

aa^+-ultrapower

Definition

The equivalence relations for members of M' and M'_d are defined as follows: For $\varphi(s,x,\vec{a}), \varphi'(s,x,\vec{a}') \in M'$,

$$\varphi(s,x,\vec{a}) \sim \varphi'(s,x,\vec{a}') \text{ if } \mathrm{aa}^+ s\,(f_{\varphi(s,x,\vec{a})}(s) = f_{\varphi'(s,x,\vec{a}')}(s)) \in T^*.$$

For $\varphi(s,x,\vec{a}), \varphi'(s,x,\vec{a}') \in M'_d$, suppose $e, e' \subset d$ are minimal such that $\varphi(s,x,\vec{a}) \in M'_e$ and $\varphi'(s,x,\vec{a}') \in M_{e'}$. We define

$$\begin{aligned}&\varphi(s,x,\vec{a}) \sim_d \varphi'(s,x,\vec{a}') \text{ if}\\ &e = e' \text{ and } \mathrm{aa}^+ s\,(f^{e,\xi}_{\varphi(s,x,\vec{a})}(s) = f^{e',\xi}_{\varphi'(s,x,\vec{a}')}(s)) \in T^*.\end{aligned}$$

aa^+-ultrapower

Definition

Suppose $\mathbf{J}_\alpha^T = (J_\alpha^T, \in, T, T^*, (P)_\xi)$ is an aa^+-premouse. Its aa^+-ultrapower is the $\tau_{\xi+1}$-structure $\mathbf{M} = (M, E, S, S^*, (P')_{\xi+1})$ defined as follows:

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1. M is the set of equivalence classes $[\varphi(s, x, \vec{a})]$ of $\sim$ on M'.
2. $[\varphi(s, x, \vec{a})]E[\varphi'(s, x, \vec{a}')]$ iff $\mathrm{aa}^+ s\, R_\in(f_{\varphi(s,x,\vec{a})}(s), f_{\varphi'(s,x,\vec{a}')}(s)) \in T^*$.

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3. $([\varphi(s, x, \vec{a})], [\varphi'(s, x, \vec{a}')]) \in S$ iff $aa^+ s\, R_T(f_{\varphi(s,x,\vec{a})}(s), f_{\varphi'(s,x,\vec{a}')}(s)) \in T^*$.
4. S^* consists of $\varphi(\mathbf{P}_\xi, [\theta_1(s, x, \vec{a}_1)], \dots, [\theta_n(s, x, \vec{a}_n)])$, where $\varphi(s, x_1, \dots, x_n) \in \mathcal{L}(aa^+)$ is in vocabulary τ^-_ξ and $aa^+ s\, \varphi(s, f_{\theta_1(s,x,\vec{a}_1)}(s), \dots, f_{\theta_n(s,x,\vec{a}_n)}(s)) \in T^*$.
5. $[\varphi(s, x, \vec{a})] \in P'_\gamma$ iff $aa^+ s\, P_\gamma(f_{\varphi(s,x,\vec{a})}(s)) \in T^*$ for $\gamma < \xi$.
6. $P'_\xi = \{j(b) : b \in J^T_\alpha\}$, where $j : J^T_\alpha \to M$ is the canonical embedding defined by $j(b) = [x = b]$.

Auxiliary models $\mathbf{M}_d$

Definition

Suppose $\mathbf{J}_\alpha^T = \langle J_\alpha^T, \in, T, T^*, (P)_\xi \rangle$ is an aa$^+$-premouse. For all $d = \{\eta_1, \ldots, \eta_n\} \in [\xi + 1]^{<\omega}$, the model $\mathbf{M}_d = (M_d, E_d, S_d, S_d^*, P_{\eta_1}^d, \ldots, P_{\eta_n}^d)$ is defined as follows:

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- M_d is the set of equivalence classes $[\varphi(s, x, \vec{a})]_d$ of $\sim_d$ on M_d'.

Auxiliary models $\mathbf{M}_d$

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- M_d is the set of equivalence classes $[\varphi(s, x, \vec{a})]_d$ of $\sim_d$ on M'_d.
- $[\varphi_1(s, x, \vec{a}_1)]_d E_d [\varphi_2(s, x, \vec{a}_2)]_d$ iff aa$^+ s\, R_\in(f^{d_1}_{\varphi_1(s,x,\vec{a}_1)}(s), f^{d_2}_{\varphi_2(s,x,\vec{a}_2)}(s)) \in T^*$, where $d_1, d_2 \subset d$ are minimal such that $\varphi_1(s, x, \vec{a}_1) \in M_{d_1}$ and $\varphi_2(s, x, \vec{a}_2) \in M_{d_2}$.
- $([\varphi_1(s, x, \vec{a}_1)]_d, [\varphi_2(s, x, \vec{a}_2)]_d) \in S_d$ iff aa$^+ s\, R_T(f^{d_1}_{\varphi_1(s,x,\vec{a}_1)}(s), f^{d_2}_{\varphi_2(s,x,\vec{a}_2)}(s)) \in T^*$, where $d_1, d_2 \subset d$ are minimal such that $\varphi_1(s, x, \vec{a}_1) \in M_{d_1}$ and $\varphi_2(s, x, \vec{a}_2) \in M_{d_2}$.

Auxiliary models $\mathbf{M}_d$

- If $\xi \in d$, S_d^* consists of $\varphi(\mathbf{P}_\xi, [\theta_1(s,x,\vec{a}_1)]_d, \ldots, [\theta_n(s,x,\vec{a}_n)]_d)$ in vocabulary τ_d^- such that

$$\mathrm{aa}^+ s\, \varphi(s, f^{d_1,\xi}_{\theta_1(s,x,\vec{a}_1)}(s), \ldots, f^{d_n,\xi}_{\theta_n(s,x,\vec{a}_n)}(s)) \in T^*,$$

where each $d_i \subset d$ is minimal such that $\theta_i(s,x,\vec{a}_i) \in M_{d_i}$.
If $\xi \notin d$, S_d^* consists of $\varphi([\theta_1(s,x,\vec{a}_1)]_d, \ldots, [\theta_n(s,x,\vec{a}_n)]_d)$ in vocabulary τ_d^- such that

$$\mathrm{aa}^+ s\, \varphi(f^{d_1,\xi}_{\theta_1(s,x,\vec{a}_1)}(s), \ldots, f^{d_n,\xi}_{\theta_n(s,x,\vec{a}_n)}(s)) \in T^*,$$

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Auxiliary models $\mathbf{M}_d$

- If $\xi \in d$, S^*_d consists of $\varphi(\mathbf{P}_\xi, [\theta_1(s,x,\vec{a}_1)]_d, \ldots, [\theta_n(s,x,\vec{a}_n)]_d)$ in vocabulary τ^-_d such that

$$\mathrm{aa}^+ s\, \varphi(s, f^{d_1,\xi}_{\theta_1(s,x,\vec{a}_1)}(s), \ldots, f^{d_n,\xi}_{\theta_n(s,x,\vec{a}_n)}(s)) \in T^*,$$

  where each $d_i \subset d$ is minimal such that $\theta_i(s,x,\vec{a}_i) \in M_{d_i}$.
  If $\xi \notin d$, S^*_d consists of $\varphi([\theta_1(s,x,\vec{a}_1)]_d, \ldots, [\theta_n(s,x,\vec{a}_n)]_d)$ in vocabulary τ^-_d such that

$$\mathrm{aa}^+ s\, \varphi(f^{d_1,\xi}_{\theta_1(s,x,\vec{a}_1)}(s), \ldots, f^{d_n,\xi}_{\theta_n(s,x,\vec{a}_n)}(s)) \in T^*,$$

  where each $d_i \subset d$ is minimal such that $\theta_i(s,x,\vec{a}_i) \in M_{d_i}$.
- For all $\eta_i \in d \setminus \{\xi\}$, $[\varphi(s,x,\vec{a})]_d \in P^d_{\eta_i}$ iff $\mathrm{aa}^+ s\, \mathbf{P}_{\eta_i}(f^{d',\xi}_{\varphi(s,x,\vec{a})}(s)) \in T^*$, where $d' \subset d$ is minimal such that $\varphi(s,x,\vec{a}) \in M_{d'}$.
- $P^d_\xi = \{[x = b]_d : b \in J^T_\alpha\}$ if $\xi \in d$.

Key lemma

Lemma

Suppose φ is a first-order formula.

- *If $d = \{\eta_1, \dots, \eta_n\} \in [\xi+1]^{<\omega}$ is nonempty, $\xi \notin d$, and φ is in vocabulary τ_d^-, then the following are equivalent:*
  - $\mathbf{M}_d^- \models \varphi([\theta_1(s,x,\vec{a}_1)]_d, \dots, [\theta_m(s,x,\vec{a}_m)]_d)$,
  - $\mathrm{aa}^+ s\, \varphi_d^*(f^{d_1}_{\theta_1(s,x,\vec{a}_1)}(s), \dots, f^{d_m}_{\theta_m(s,x,\vec{a}_m)}(s)) \in T^*$,

  where each d_i is the minimal subset of d such that $\theta_i(s, x_i, \vec{a}_i) \in d_i$.
- *If $\xi \in d = \{\eta_1, \dots, \eta_n, \xi\} \in [\xi+1]^{<\omega+1}$, and φ is in vocabulary τ_d^-, then the following are equivalent:*
  - $\mathbf{M}_d^- \models \varphi(P^d_\xi, [\theta_1(s,x,\vec{a}_1)]_d, \dots, [\theta_m(s,x,\vec{a}_m)]_d)$,
  - $\mathrm{aa}^+ s\, \varphi_d^*(s, f^{d}_{\theta_1(s,x,\vec{a}_1)}(s), \dots, f^{d}_{\theta_m(s,x,\vec{a}_m)}(s)) \in T^*$,

  where each d_i is the minimal subset of d such that $\theta_i(s, x_i, \vec{a}_i) \in d_i$.

Here, e.g., φ_d^* for $\xi \in d$ is obtained by inductively replacing each subformula $\exists x \psi(x)$ of φ by
$\exists x\, (\, \exists \beta \in \mathrm{Ord}\Psi(x, \beta, J^T_{\sup T}, \eta_1, \dots, \eta_n, s) \wedge \psi(x)\,)$

Key lemma continued

Moreover, the following are equivalent

- $(M, E, S, (P')_{\xi+1}) \models \varphi([\theta_1(s, x, \vec{a}_1)], \ldots, [\theta_m(s, x, \vec{a}_m)])$,
- $\mathrm{aa}^+ s\, \varphi^*_\emptyset(f_{\theta_1(s,x,\vec{a}_1)}(s), \ldots, f_{\theta_m(s,x,\vec{a}_m)}(s)) \in T^*$.

M_d next admissible of the aa^+-ultrapower

Corollary

Suppose $\mathbf{M}$ *and* $\mathbf{M}_d$*, where* $d = \{\eta_1, \ldots, \eta_n\} \neq \emptyset$*, are well-founded and* $\bar{\mathbf{M}} = (J_\beta^{\bar{T}}, \in, \bar{T}, \bar{T}^*, (\bar{P})_{\xi+1})$ *and* $\bar{\mathbf{M}}_d = (\bar{M}_d, \in, \bar{T}_d, (\bar{T}_d)^*, \bar{P}^d_{\eta_1}, \ldots, \bar{P}^d_{\eta_n})$ *are their transitive collapses. Then* $\bar{M}_d$*, the domain of* $\bar{\mathbf{M}}_d$*, is the next admissible set of* $(J_\beta^{\bar{T}}, \in, \bar{T}, \bar{P}_{\eta_1}, \ldots, \bar{P}_{\eta_n})$*.*

Moreover, for any good first-order φ *in vocabulary* τ_d^- *and any* $[\theta_1(s, x, \vec{a}_1)]_d, \ldots, [\theta_m(s, x, \vec{a}_m)]_d \in M_d$*, we have*

$$\begin{aligned}
&\varphi([\theta_1(s, x, \vec{a}_1)]_d, \ldots, [\theta_m(s, x, \vec{a}_m)]_d) \in (\bar{T})^*_d \\
\Leftrightarrow\, &\bar{\mathbf{M}}_d^- \models \varphi([\theta_1(s, x, \vec{a}_1)]_d, \ldots, [\theta_m(s, x, \vec{a}_m)]_d).
\end{aligned}$$

Ultrapower is an $\mathtt{aa}^+$-premouse

Lemma

If the ultrapower $\mathbf{M} = (M, \in, S, S^*, (P')_{\xi+1})$ *of an* $\mathtt{aa}^+$*-premouse* $\mathbf{J}^T_\alpha = (J^T_\alpha, \in, T, T^*, (P)_\xi)$ *is well-founded, then its transitive collapse* $\bar{\mathbf{M}} = (J^{\bar{T}}_\beta, \in, \bar{T}, \bar{T}^*, (\bar{P})_{\xi+1})$ *is also an* $\mathtt{aa}^+$*-premouse.*
Moreover, the ultrapower embedding j *(composed with the collapse) is an* $\mathtt{aa}^+$*-embedding.*

Iterated aa$^+$-ultrapowers

Definition

Suppose $\mathbf{J}^T_\gamma = (J^T_\gamma, \in, T, T^*, (P^0)_0)$ is an aa$^+$-premouse with vocabulary τ_0 and δ is an ordinal. The models $\mathbf{M}_\alpha = (M_\alpha, E_\alpha, T_\alpha, T^*_\alpha, (P^\alpha)_\alpha)$, $\alpha < \delta$, and the directed system

$$\langle (M_\alpha, E_\alpha, T_\alpha, T^*_\alpha, (P^\alpha)_\alpha), j_{\alpha\beta} : \alpha < \beta < \delta \rangle,$$

called the aa$^+$-*iteration* of $\mathbf{J}^T_\alpha$ of length δ, are defined as follows:

- $(M_0, E_0, T_0, T^*_0, (P^0)_0) = \mathbf{J}^T_\gamma$
- The vocabulary of $(M_\alpha, E_\alpha, T_\alpha, T^*_\alpha, (P^\alpha)_\alpha)$ is τ_α.
- The $j_{\alpha\beta}$ form a commuting system of aa$^+$-embeddings

  $$j_{\alpha\beta} : (M_\alpha, E_\alpha, T_\alpha, T^*_\alpha, (P^\alpha)_\alpha) \to (M_\beta, E_\beta, T_\beta, T^*_\beta, (P^\beta)_\beta) \restriction \tau_\alpha.$$

- For successor $\alpha + 1$ we let

  $$(M_{\alpha+1}, E_{\alpha+1}, T_{\alpha+1}, T^*_{\alpha+1}, (P^{\alpha+1})_{\alpha+1}) = \mathrm{Ult}(M_\alpha, E_\alpha, T_\alpha, T^*_\alpha, (P^\alpha)_\alpha).$$

  The embedding $j_{\alpha,\alpha+1}$ is the aa$^+$-ultrapower embedding.

Definition continued

- For limit ν, we let $(M_\nu, E_\nu, T_\nu, T^*_\nu, (P^\nu)_\nu)$ be the direct limit of the directed system

$$\langle (M_\alpha, E_\alpha, T_\alpha, T^*_\alpha, (P^\alpha)_\alpha), j_{\alpha\beta} : \alpha < \beta < \nu \rangle.$$

The embeddings $j_{\alpha\nu}$ for $\alpha < \nu$ are the direct limit embeddings.

The models $\mathbf{M}_\alpha$ are called iterates of $\mathbf{J}^T_\gamma$.

Iterated aa$^+$-ultrapowers

Definition

Suppose α is an ordinal and $d \in [\alpha+1]^{<\omega}$.

1. $(M^\alpha_\emptyset)'$ is the set of $\varphi(s,x,\vec{a})$, where $\vec{a} \in M_\alpha$, such that aa$^+ s\,\varphi(s,x,\vec{y})$ is good, and aa$^+ s\,\exists x(x \in J^T_{\sup T} \wedge \varphi(s,x,\vec{a})) \in T^*_\alpha$.
2. If $\alpha \in d = \{\eta_1,\ldots,\eta_n,\alpha\}$, then $(M^\alpha_d)'$ is the set of $\varphi(s,x,\vec{a})$, where $\vec{a} \in M_\alpha$, such that aa$^+$ $\varphi(s,x,\vec{a}) \in (M^\alpha_e)'$ for some $e \subsetneq d$ or
$$\text{aa}^+ s\,\exists x \exists \beta \in \mathrm{Ord}\,(\,\Gamma(\beta,x,\eta_1,\ldots,\eta_n,s) \wedge \varphi(s,x,\vec{a})\,) \in T^*_\alpha.$$
3. If $\alpha \notin d = \{\eta_1,\ldots,\eta_n\}$, then $(M^\alpha_d)'$, is the set of those good $\varphi(s,x,\vec{a})$, where $\vec{a} \in M_\alpha$, such that
$$\text{aa}^+ s\,\exists x \exists \beta \in \mathrm{Ord}\,(\,\Gamma(\beta,x,\eta_1,\ldots,\eta_n) \wedge \varphi(s,x,\vec{a})\,) \in T^*_\alpha.$$
4. For members of $(M^\alpha_d)'$, $\sim^\alpha_d$ is defined as $\sim_d$ was defined in Definition 10, only replacing T^* with T^*_α. The equivalence class of $\varphi(s,x,\vec{a})$ under $\sim^\alpha_d$ is denoted by $[\varphi(s,x,\vec{a})]^\alpha_d$.

Iterated aa$^+$-ultrapowers

Definition

Suppose $\mathbf{J}_\gamma^T = (J_\gamma^T, \in, T, T^*, (P^0)_0)$ is an aa$^+$-premouse with vocabulary τ_0, and the iteration of $\mathbf{J}_\gamma^T$ of length δ is as in Definition 17. For any $\alpha < \beta < \delta$ and nonempty finite $d \subset \alpha$, the models $\mathbf{M}_d^\alpha = (M_d^\alpha, E_d^\alpha, T_d^\alpha, (T_d^\alpha)^*, P_{d,\eta_1}^\alpha, \ldots, P_{d,\eta_n}^\alpha)$, where $d = \{\eta_1, \ldots, \eta_n\}$, and the maps $j_d^{\alpha\beta} : \mathbf{M}_d^\alpha \to \mathbf{M}_d^\beta$ are defined inductively as follows:

1. If $\alpha = \lambda + 1$, then $\mathbf{M}_d^\alpha$ is defined using T_λ^* and the equivalence classes $[\varphi(s, x, \vec{a})]_d^\lambda$ as $\mathbf{M}_d$ was defined in 13 using T^* and $[\varphi(s, x, \vec{a})]_d$.
2. If $\alpha = \lambda + 1$, $j_d^{\alpha,\alpha+1} (= j_d^{\lambda+1,\lambda+2})$ is defined by
$$j_d^{\alpha,\alpha+1}([\varphi(s, x, \vec{a})]_d^\lambda) = [\varphi(\mathbf{P}_\lambda, x, j_{\lambda,\alpha}(\vec{a}))]_d^\alpha.$$
3. Suppose α is a limit, and $\mathbf{M}_d^\lambda$ and $j_d^{\lambda\eta}$ have been defined for all $\lambda < \eta < \alpha$ such that $d \subset \lambda$, and $\langle \mathbf{M}_d^\lambda, j_d^{\lambda\eta} : d \subset \lambda < \eta < \alpha \rangle$ is a directed system. Then $\mathbf{M}_d^\alpha$ is a direct limit of the system $\langle \mathbf{M}_d^\gamma, j_d^{\lambda\eta} : d \subset \lambda < \eta < \alpha \rangle$ and $j_d^{\lambda\alpha}$, $\lambda < \alpha$, are the direct limit maps.

Definition continued

- Suppose α is a limit, $\mathbf{M}_d^\lambda$ has been defined for each $\lambda \leq \alpha$ such that $d \subset \lambda$, and $j_d^{\lambda\eta}$ has been defined for all $d \subset \lambda < \eta \leq \alpha$. Then $j_d^{\alpha,\alpha+1}$ is defined as follows. Suppose $y \in M_d^\alpha$ is such that $y = j_d^{\lambda+1,\alpha}([\varphi(s,x,\vec{a})]_d^\lambda)$ for some $[\varphi(s,x,\vec{a})]_d^\lambda \in M_d^{\lambda+1}$. Then we define
$$j_d^{\alpha,\alpha+1}(y) = [\varphi(\mathbf{P}_\lambda, x, j_{\lambda\alpha}(\vec{a}))]_d^\alpha.$$

Iterated aa$^+$-ultrapowers

Lemma

Suppose $\langle (M_\alpha, E_\alpha, T_\alpha, T^*_\alpha, (P^\alpha)_\alpha), j_{\alpha\beta} : \alpha < \beta < \delta\rangle$ is the aa$^+$*-iteration of $(M_0, \in, T_0, T^*_0)$ of length δ and for all $\alpha < \delta$ and nonempty $d \in [\alpha]^{<\omega}$ the models $\mathbf{M}^\alpha_d = (M^\alpha_d, E^\alpha_d, T^\alpha_d, (T^\alpha_d)^*, P^\alpha_{d,\eta_1}, \ldots, P^\alpha_{d,\eta_n})$ are as in the preceding definition. Suppose that for all $\alpha < \delta$ and all nonempty $d \in [\alpha]^{<\omega}$, M_α and M^α_d are well-founded. Then the following hold:*

- *For all $\alpha < \delta$, $\mathbf{M}_\alpha$ is (isomorphic to) an* aa$^+$*-premouse.*
- *For all $\alpha < \beta < \delta$, $j_{\alpha\beta}$ is an* aa$^+$*-embedding.*
- *For all $\alpha < \delta$ and $d = \{\eta_1, \ldots, \eta_n\} \in [\alpha]^{<\omega}$, M^α_d is the next admissible of $(M_\alpha, \in, T_\alpha, P^\alpha_{\eta_1}, \ldots, P^\alpha_{\eta_n})$.*
- *For all $\alpha < \delta$, $d \in [\alpha]^{<\omega}$, all good first-order φ in vocabulary τ^-_d, and all $\vec{a} \in M^\alpha_d$, $\varphi(\vec{a}) \in (T^\alpha_d)^*$ if and only if $(\mathbf{M}^\alpha_d)^- \models \varphi(\vec{a})$.*
- *For all $\alpha < \beta < \delta$, $j^{\alpha\beta}_d$ is an* aa$^+$*-embedding.*

Frame Title

Lemma

Suppose $\langle (M_\alpha, E_\alpha, T_\alpha, T^*_\alpha, (P^\alpha)_\alpha), j_{\alpha\beta} : \alpha < \beta < \delta \rangle$ *is the* aa^+*-iteration of an* aa^+*-mouse* $(M_0, \in, T_0, T^*_0)$ *of length* δ*, and for* $d \subset \alpha < \beta < \delta$ *the models* $\mathbf{M}^\alpha_d = (M^\alpha_d, E^\alpha_d, T^\alpha_d, (T^\alpha_d)^*, P^\alpha_{d,\eta_1}, \ldots, P^\alpha_{d,\eta_n})$ *and maps* $j^{\alpha\beta}_d : \mathbf{M}^\alpha_d \to \mathbf{M}^\beta_d$*, are as in Definition 19. Suppose further that* κ *is a regular cardinal such that* $|M_0| < \kappa$ *and* $\kappa < \delta$*.*
Then for any $d \in [\kappa]^{<\omega}$*, any good formula* φ *in vocabulary* τ^-_d*, and any* $\vec{a} \in M^\kappa_d$*,*

$$\varphi(\vec{a}) \in (T^\kappa_d)^* \Leftrightarrow ((\mathbf{M}^\kappa_d)^-; M_\kappa) \models^+ \varphi_\kappa(\vec{a}),$$

where φ_κ *is obtained from* φ *by replacing each quantifier* $\mathrm{aa}^+ s$ *by* $\mathrm{aa}^+_\kappa\, s$*.*

The main result

Theorem

Suppose $\langle (M_\alpha, E_\alpha, T_\alpha, T^*_\alpha, (P^\alpha)_\alpha), j_{\alpha\beta} : \alpha < \beta < \delta\rangle$ *is the* aa^+*-iteration of an* aa^+*-mouse* $(M_0, \in, T_0, T^*_0)$ *of length* δ*. If* $\kappa < \delta$ *is a regular cardinal such that* $|M_0| < \kappa$*, then the domain* $M_\kappa = J^{T_\kappa}_{\beta_\kappa}$ *of the* κ*-iterate is the level* $J^{\mathrm{Tr}^\kappa}_{\beta_\kappa}$ *of* $C(\mathrm{aa}^+_\kappa)$*. In particular, if* M_0 *is countable, then* M_{ω_1} *is the level* $J^{\mathrm{Tr}}_{\beta_{\omega_1}}$ *of* $C(\mathrm{aa}^+)$*.*

Thank you!

References

[1] Gabriel Goldberg and John Steel. The structure of C(aa). Preprint, 2024.

[2] Juliette Kennedy, Menachem Magidor, and Jouko Väänänen. Inner models from extended logics: Part 1. *J. Math. Log.*, 21(2):Paper No. 2150012, 53, 2021.

[3] Juliette Kennedy, Menachem Magidor, and Jouko Väänänen. Inner models from extended logics: Part 2, 2024. `https://arxiv.org/abs/2007.10766`.