PFA and Derived Models

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Joint work with Nam Trang

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If κ is limit of Woodin cardinals and G generic for the Levy collapse $Col(\omega, < \kappa)$ define the *symmetric reals* $\mathbb{R}^* = \bigcup_{\alpha<\kappa} \mathbb{R}^{V[G \restriction \alpha]}$

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What might a "bigger" model of AD look like?

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For $A \subseteq \mathbb{R}$, $\Theta(A) = sup\{\alpha \in On : \text{ exists surjection } f : \mathbb{R} \to \alpha$ which is $OD(A, x)$ for some $x \in \mathbb{R}\}$.

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We set

1. $\Theta_0 = \Theta(\emptyset)$,
2. $\Theta_{\alpha+1} = \Theta(A)$ for any A such that $w(A) = \theta_\alpha$, and
3. $\Theta_\lambda = sup_{\alpha<\lambda}\Theta_\alpha$ for λ a limit ordinal.

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- In $L(\mathbb{R}^*)$, every set of reals was $OD(x)$ for some $x \in \mathbb{R}^*$, so $\Theta_0 = \Theta$ and Θ_1 is not defined.

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Woodin developed a method of producing models of *AD*:

Definition

If κ is limit of Woodin cardinals and G generic for the Levy collapse $Col(\omega, < \kappa)$ define

1. the *symmetric reals* $\mathbb{R}^* = \bigcup_{\alpha<\kappa} \mathbb{R}^{V[G\restriction\alpha]}$
2. the *symmetric extension* $V(\mathbb{R}^*) = HOD^{V[G]}_{V,\mathbb{R}^*}$
3. the *derived model* $D(V, \kappa) = L(\mathcal{A}, \mathbb{R}^*)$, where $\mathcal{A} = \{A \subseteq \mathbb{R}^* : A \in V(\mathbb{R}^*) \wedge L(A, \mathbb{R}^*) \models AD^+\}$

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2. $V = L(P(\mathbb{R}^*))$

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Let κ be a singular cardinal with $cof(\kappa) = \omega$.

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Definition

A *covering matrix* for κ^+ is a sequence $(K_{\alpha,i} : \alpha < \kappa^+, i < \omega)$ s.t.

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A covering matrix for κ^+ is *coherent* if for any $\alpha < \beta < \kappa^+$

1. $(\forall i)(\exists j) K_{\alpha,i} \subseteq K_{\beta,j} \cap \alpha$
2. $(\forall j)(\exists i) K_{\beta,j} \cap \alpha \subseteq K_{\alpha,i}$.

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Theorem (Viale)

- $V = L \implies$ *is a coherent covering matrix for κ^+.*
- $PFA \implies$ *is no coherent covering matrix for κ^+.*

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Wilson used Viale's Theorem to show:

Theorem (Wilson)

(*PFA*) *If* κ *is a limit of Woodin cardinals and cof*$(\kappa) = \omega$, *then* $\Theta_0^{D(V,\kappa)} < \kappa^+$.

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Proof.

Let $\langle \kappa_i : i < \omega \rangle$ be cofinal in κ.
Suppose have $OD^{D(V,\kappa)}$ surjections $f_\alpha : \mathbb{R}^* \to \alpha$.

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Note $\mathbb{R}^* = \bigcup_{i<\omega} \mathbb{R}^{V[G \restriction \kappa_i]}$.
So $\alpha = \bigcup_{i<\omega} f_\alpha[\mathbb{R}^{V[G \restriction \kappa_i]}]$.
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Let $K_{\alpha,i} = f_\alpha[\mathbb{R}^{V[G\restriction\kappa_i]}]$.

$\langle K_{\alpha,i} : \alpha < \kappa^+, i < \omega \rangle$ is a covering matrix for κ^+ in V.
Use $D(V,\kappa) \models CC_{\mathbb{R}}$ to check $\langle K_{\alpha,i} \rangle$ is coherent. $\square$

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Lemma

Suppose κ is a limit of Woodin cardinals and $D(V,\kappa) \models AD_{\mathbb{R}}$. Then $\Theta^{D(V,\kappa)} < \kappa^+$.

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Proof idea.

By the Lemma, we may assume $\Theta = \Theta_{\gamma+1}$.
Have surjections $f_\alpha : \mathbb{R}^* \to \alpha$ which are ordinal definable from a set A of Wadge rank Θ_γ in $D(V,\kappa)$.
If A has a nice enough name in V, then $f_\alpha[\mathbb{R}^{V[G \restriction \kappa_i]}] \in V$ and Wilson's proof applies. $\square$

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Is the assumption $cof(\kappa) = \omega$ necessary?

Wilson's Conjecture

Conjecture (Wilson)

(PFA) If κ is a limit of Woodin cardinals, then

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Remark

The 2nd part of Wilson's conjecture implies the 1st.

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We prove the first part of the conjecture with the additional assumption that the derived model satisfies "mouse capturing" (MC).

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Theorem (L., Trang)

Suppose $V \models \neg\Box_\kappa$, κ *is a limit of Woodin cardinals and* $D(V,\kappa) \models MC$. *Then* $\Theta_0^{D(V,\kappa)} < \kappa^+$.

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Proof.

- If $\Theta_0^{D(V,\kappa)} = \kappa^+$, then $\Theta_0^{D(V,\kappa)} = \Theta^{D(V,\kappa)}$.
- Any derived model satisfies $V = L(P(\mathbb{R}))$, so $D(V,\kappa) \models V = L(P_{\Theta_0}(\mathbb{R}^*))$.

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- Let τ be a $Col(\omega, <\kappa)$-name for $\mathbb{R}^*$.
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Theorem (L.,Trang)

Suppose $V \models \neg\square_\kappa$, κ is a limit of Woodin cardinals and $D(V,\kappa) \models MC$. Then $\Theta_0^{D(V,\kappa)} < \kappa^+$.

Proof.

- If $\Theta_0^{D(V,\kappa)} = \kappa^+$, then $\Theta_0^{D(V,\kappa)} = \Theta^{D(V,\kappa)}$.
- Any derived model satisfies $V = L(P(\mathbb{R}))$, so $D(V,\kappa) \models V = L(P_{\Theta_0}(\mathbb{R}^*))$.
- Then MC implies $D(V,\kappa) \models P(\mathbb{R}^*) = Lp(\mathbb{R}^*) \cap P(\mathbb{R}^*)$.
- So $Lp(\mathbb{R}^*)$ has height κ^+.

- Let τ be a $Col(\omega, <\kappa)$-name for $\mathbb{R}^*$.
- Then $Lp(\tau)$ has height κ^+.
- By arguments of Schimmerling/Trang/Zeman, can construct a $\square_\kappa$ sequence from $Lp(\tau)$.

$\square$

Extending Wilson's Conjecture

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Theorem (L.,Trang)

Suppose $V \models \neg\square_\kappa$, κ *is a limit of Woodin cardinals and* $D(V,\kappa) \models$ "$\Theta_{\alpha+1}$ *exists*" + "*there is a hod pair* (P,Σ) *below* $AD_{\mathbb{R}}$ + 'Θ *is regular' such that* $P_{\Theta_{\alpha+1}}(\mathbb{R}^*) = Lp^{\Sigma}(\mathbb{R}^*) \cap P(\mathbb{R}^*)$, Σ *is fullness-preserving, and has branch condensation.*" *Then* $\Theta_{\alpha+1}^{D(V,\kappa)} < \kappa^+$.

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Conjecture

Suppose $V \models \neg\Box_\kappa$ and κ is a limit of Woodin cardinals. Then $\Theta^{D(V,\kappa)} < \kappa^+$.

Extending Wilson's Conjecture

Definition

$\square_\kappa^*$ (weak square) is the statement that there is a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ s.t.

1. $\mathcal{C}_\alpha$ is nonempty, $|\mathcal{C}_\alpha| \leq \kappa$, and each $C \in \mathcal{C}_\alpha \subset \alpha$ is a club.
2. For all $C \in \mathcal{C}_\alpha$ and all $\beta \in acc(C)$, $C \cap \beta \in \mathcal{C}_\beta$.
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Suppose $V \models \neg\square^*_\kappa$, κ is a regular limit of Woodin cardinals, and there is a least branch hod pair (P, Σ) in $D(V, \kappa)$ such that $D(V, \kappa) = L(Lp^\Sigma(\mathbb{R}^*))$. Then $\Theta^{D(V,\kappa)} < \kappa^+$.

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A tree T is $< \kappa$-*absolutely complemented* if is tree U such that for any $\lambda < \kappa$ and any $Col(\omega, \lambda)$-generic G, $\rho[T] = (\rho[U])^c$.

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Definition

The "old" derived model is $olD(V, \kappa) = L(Hom^*, \mathbb{R}^*)$, where $Hom^* = \{\rho[T] \cap \mathbb{R}^* : (\exists \gamma < \kappa) T \in V[G \restriction \gamma] \wedge V[G \restriction \gamma] \models T \text{ is } < \kappa - \text{absolutely complemented}\}$

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$Hom^*=$ the Suslin-co-Suslin sets of $olD(V,\kappa)$ (or $D(V,\kappa)$)

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Suppose $\Theta^{olD(V,\kappa)} = \kappa^+$.

- There is a $Col(\omega, <\kappa)$-name π for $(Hom^*, \mathbb{R}^*)$ and a code $CODE$ for π such that $CODE \subset H_\kappa^V$.

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- Let $M = L(H_\kappa^V, CODE)$.

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- Let $M = L(H^V_\kappa, CODE)$.
- $M[G] \supseteq olD(V,\kappa)$, so $\Theta^{M[G]} = \kappa^+$.

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- Then $\Theta^M = \kappa^+$.

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- Let $M = L(H_\kappa^V, CODE)$.
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- Then $\Theta^M = \kappa^+$.

$M \subseteq V$, and by arguments of Schimmerling/Trang/Zeman can build a $\Box_\kappa$-sequence in V. $\Box$