AD and UA

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Outline

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Background theory: ZF + DC.

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Why do models of determinacy resemble canonical inner models of large cardinal axioms?

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Why do models of determinacy resemble canonical inner models of large cardinal axioms?

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Kunen showed that in $L[U]$, there is a unique normal measure, and every measure is a finite power of it.

Solovay showed that under AD, there is a unique normal measure on ω_1, and every measure on ω_1 is a finite power of it.

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If N is a transitive model of set theory and U is an N-ultrafilter, the ultrapower of N by U, using functions in N, is denoted by N_U.

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The ultrapower embedding is denoted by $j_U : N \to N_U$.

The Ultrapower Axiom

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Definition (ZFC)

The *Ultrapower Axiom* states that for all measures U and W, there exist measures $W_* \in V_U$ and $U_* \in V_W$ such that $(V_U)_{W_*} = (V_W)_{U_*}$ and

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The pair (W_*, U_*) is called a *comparison* of (U, W).

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Sketch. Set $W_* = j_U(W)$ and $U_* = U$. Then

$$(V_U)_{W_*} = (V_U)_{j_U(W)} = j_U(V_W) = (V_W)_{U_*}$$

and similarly $j_{W_*} \circ j_U = j_{U_*} \circ j_W$.

The Ketonen order

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Definition

If δ is an ordinal and $U, W \in \upsilon(\delta)$, then U precedes W in the *Ketonen order*, denoted $U <_{\Bbbk} W$, if there are measures $U_\alpha \in \upsilon(\alpha)$, defined for all positive $\alpha < \delta$, such that

$$A \in U \iff \{\alpha < \delta : A \cap \alpha \in U_\alpha\} \in W$$

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Theorem

The Ketonen order on $\upsilon(\delta)$ is a well-founded partial order.

Example: the Ketonen order on normal measures

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Proposition

If $U \in \upsilon(\kappa)$ and W is a normal measure on κ, then $U <_{\Bbbk} W$ if and only if $U \lhd W$.

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As a consequence, the following are equivalent:

- The Ketonen order is linear on normal measures.
- The Mitchell order is linear on normal measures.
- UA holds for normal measures.

The Ketonen order and UA

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Theorem (ZFC)

The following are equivalent:

- *For all ordinals δ, the Ketonen order on $\upsilon(\delta)$ is linear.*
- *The Ultrapower Axiom holds.*

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  Sample theorem (Jackson). Assume AD. If $\alpha < \epsilon_0$, $\aleph_{\alpha+1} \to (\aleph_{\alpha+1})^{\aleph_{\alpha+1}}$ if and only if $\alpha = 0$ or $\alpha = \omega \uparrow\uparrow n$ for some odd number *n*.

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- To extend Jackson's analysis to higher cardinals requires a global classification of measures.

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- Kunen showed that for $\delta < \Theta$, $\upsilon(\delta)$ has a definable well-order.
- The linearity of the Ketonen order can be reformulated as a form of Lipschitz determinacy for measures.
- Many consequences of UA can be established using that $\text{HOD}_x \vDash \text{UA}$ for all $x \in \mathbb{R}$.

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Theorem

Assume $\mathsf{I}_0(\lambda)$*. Then* $L(V_{\lambda+1})$ *satisfies:*

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Theorem

Assume $\mathrm{I}_0(\lambda)$. Then $L(V_{\lambda+1})$ satisfies:

- *Every λ^+-complete filter on an ordinal extends to a λ^+-complete ultrafilter.*
- *For any $\delta < \Theta$, each level of the Ketonen order on $\upsilon(\delta)$ has cardinality less than λ.*

Ultrapowers without choice

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The class of *hereditarily well-orderable sets* is given by

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- If well-founded, N_U can be identified with its collapse.

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- $j_U : N \to N_U$ is cofinal and Σ_1-elementary.
- So N_U is extensional.
- If well-founded, N_U can be identified with its collapse.
- So again $N_U = \bigcup_{A \in N_U} L[A]$.

UA without choice

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Definition

UA(κ) states that for all $U, W \in \upsilon(\kappa)$, there exist $W_* \in \prod_U \upsilon(\kappa)$ and $U_* \in \prod_W \upsilon(\kappa)$ such that $(\mathscr{H}_U)_{W_*} = (\mathscr{H}_W)_{U_*}$ and

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Theorem

The following are equivalent for any cardinal κ:

- *The Ketonen order on $\upsilon(\kappa)$ is linear.*
- UA(κ) *holds.*

The main theorem

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The proof proceeds one aleph at a time.

UA($\aleph_1$)

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Theorem (Solovay)

Assume AD*.*

- *The club filter* $\mathcal{C}_{\omega_1}$ *on* ω_1 *is a normal measure.*
- *Every measure on* ω_1 *is isomorphic to a finite power of* $\mathcal{C}_{\omega_1}$*.*

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Assume AD.

- *The club filter $\mathcal{C}_{\omega_1}$ on ω_1 is a normal measure.*
- *Every measure on ω_1 is isomorphic to a finite power of $\mathcal{C}_{\omega_1}$.*

Proposition

If κ is the least measurable cardinal, the following are equivalent:

- UA(κ) *holds.*
- *There is a unique normal measure on κ, and every measure on κ is isomorphic to a power of it.*

UA($\aleph_2$)

$\mathsf{UA}(\aleph_2)$

Theorem (Martin–Paris, Kunen)

Assume AD.

- *The ω-club and ω_1-club filters on ω_2, denoted $\mathcal{C}_{\omega_2,\omega}$ and $\mathcal{C}_{\omega_2,\omega_1}$, are normal measures.*
- *Every measure on ω_2 is isomorphic to a finite product of $\mathcal{C}_{\omega_1}$, $\mathcal{C}_{\omega_2,\omega}$, and $\mathcal{C}_{\omega_2,\omega_1}$.*

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Note that $\mathcal{C}_{\omega_2,\omega} <_{\Bbbk} \mathcal{C}_{\omega_2,\omega_1}$: $A \subseteq \omega_2$ contains an ω-club iff there is an ω_1-club of $\alpha < \omega_2$ such that $A \cap \alpha$ contains an ω-club.

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Note that $\mathcal{C}_{\omega_2,\omega} <_{\Bbbk} \mathcal{C}_{\omega_2,\omega_1}$: $A \subseteq \omega_2$ contains an ω-club iff there is an ω_1-club of $\alpha < \omega_2$ such that $A \cap \alpha$ contains an ω-club.

Since $\mathcal{C}_{\omega_2,\omega_1}$ is normal, it follows that $\mathcal{C}_{\omega_2,\omega} \lhd \mathcal{C}_{\omega_2,\omega_1}$, and hence UA holds for this pair.

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Assume AD. *For all* $n \geq 3$, *then* ω_n *is singular with cofinality* ω_2.

In particular, ω_3 is not measurable.

Nevertheless, the structure of measures on ω_3 does not reduce to measures on ω_2.

Measures on ω_3

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Definition

Let $\mathcal{F}_{\omega_n}$ be the filter on ω_n generated by the club filter as computed in $\mathscr{H}_{(\mathcal{C}_{\omega_1})^{n-1}}$ and the Fréchet filter on ω_n.

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For $\omega \leq \kappa < \omega_3$, let $\mathcal{F}_{\omega_3,\kappa} = \mathcal{F}_{\omega_3} \restriction \{\alpha \in A : \mathscr{H}_{(\mathcal{C}_{\omega_1})^2} \vDash \mathrm{cf}(\alpha) = \kappa\}$.

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Theorem (Kunen)

Under AD, $\mathcal{F}_{\omega_3,\kappa}$ *is an* ω_2*-complete measure on* ω_3.

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Theorem (Kunen)

Under AD, *$\mathcal{F}_{\omega_3,\kappa}$ is an ω_2-complete measure on ω_3.*

Roughly, every measure on ω_3 is a product of the prime measures $\mathcal{C}_{\omega_1}$, $\mathcal{C}_{\omega_2,\omega}$, $\mathcal{C}_{\omega_2,\omega_1}$, $\mathcal{F}_{\omega_3,\omega}$, $\mathcal{F}_{\omega_3,\omega_1}$, and $\mathcal{F}_{\omega_3,\omega_2}$.

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- $U \leq_{\mathrm{RF}} W$ if there is some $W_* \in \prod_U \upsilon(\kappa)$ such that $\mathscr{H}_W = (\mathscr{H}_U)_{W_*}$ and $j_W = j_{W_*} \circ j_U$.

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Note that $\mathrm{UA}(\kappa)$ is just the statement that $(\upsilon(\kappa), \leq_{\mathrm{RF}})$ is directed.

The Rudin–Frolík order on the primes

$$\begin{tikzcd}
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 & \bullet \arrow[dl] \arrow[ddl] \arrow[dd] & \\
\mathcal{C}_{\omega_1} & & \\
\mathcal{C}_{\omega_2,\omega} & \mathcal{C}_{\omega_2,\omega_1} \arrow[dl] \arrow[d] & \\
\mathcal{F}_{\omega_3,\omega} & \mathcal{F}_{\omega_3,\omega_1} & \mathcal{F}_{\omega_3,\omega_2}
\end{tikzcd}$$

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\end{tikzcd}
$$

It remains to compare $\mathcal{F}_{\omega_3,\omega}$, $\mathcal{F}_{\omega_3,\omega_1}$, and $\mathcal{F}_{\omega_3,\omega_2}$ pairwise, and also to compare $\mathcal{C}_{\omega_2,\omega_1}$ with $\mathcal{F}_{\omega_3,\omega_2}$.

$\mathcal{C}_{\omega_2,\omega_1}$, $\mathcal{F}_{\omega_3,\omega}$, and $\mathcal{F}_{\omega_3,\omega_1}$ lie below $\mathcal{F}_{\omega_3,\omega_2}$

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More precisely, suppose U is $\mathcal{F}_{\omega_3,\omega}$ or $\mathcal{F}_{\omega_3,\omega_1}$. Let $\mathcal{F}_* = j_U(\mathcal{F}_{\omega_3,\omega_2})$. There is $U_* \in \prod_{\mathcal{F}_{\omega_3,\omega_2}} \upsilon(\kappa)$ such that $(\mathscr{H}_U)_{\mathcal{F}_*} = (\mathscr{H}_{\mathcal{F}_{\omega_3,\omega_2}})_{U_*}$ and

$$j_{\mathcal{F}_*} \circ j_U = j_{U_*} \circ j_{\mathcal{F}_{\omega_3,\omega_2}}$$

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$$j_{\mathcal{F}_*} \circ j_U = j_{U_*} \circ j_{\mathcal{F}_{\omega_3,\omega_2}}$$

This is an instance of the *internal relation*, a generalization of the Mitchell order that is a key tool in the theory of UA (under ZFC).

$\mathcal{F}_{\omega_3,\omega}$ vs. $\mathcal{F}_{\omega_3,\omega_1}$

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The internal relation corresponds to weight $n-1$; for example,

$$\mathcal{F}_{\omega_3,\omega} <_2 \mathcal{F}_{\omega_3,\omega_2}, \quad \mathcal{F}_{\omega_3,\omega_1} <_2 \mathcal{F}_{\omega_3,\omega_2}$$

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On the other hand, $\mathcal{F}_{\omega_3,\omega} <_1 \mathcal{F}_{\omega_3,\omega_1}$, reflecting their common Rudin–Frolík predecessor.

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$$\mathcal{C}_{\omega_1} = \mathcal{S}_\varepsilon$$
$$\mathcal{C}_{\omega_2,\omega} = \mathcal{S}_{\varepsilon\omega}, \quad \mathcal{C}_{\omega_2,\omega_1} = \mathcal{S}_{(1)}$$
$$\mathcal{F}_{\omega_3,\omega} = \mathcal{S}_{(1)\omega}, \quad \mathcal{F}_{\omega_3,\omega_1} = \mathcal{S}_{(21)}, \quad \mathcal{F}_{\omega_3,\omega_2} = \mathcal{S}_{(12)}$$

The Rudin–Frolík order up to $\aleph_4$

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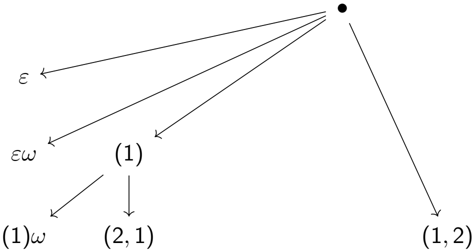

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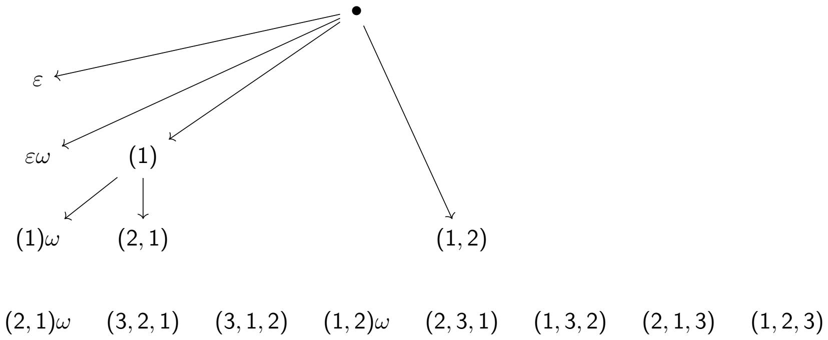

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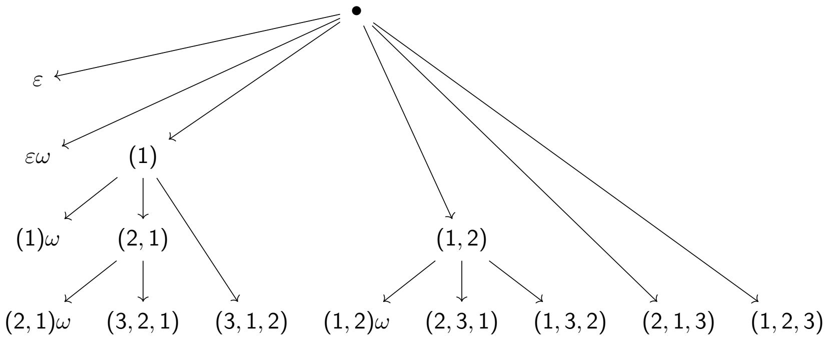

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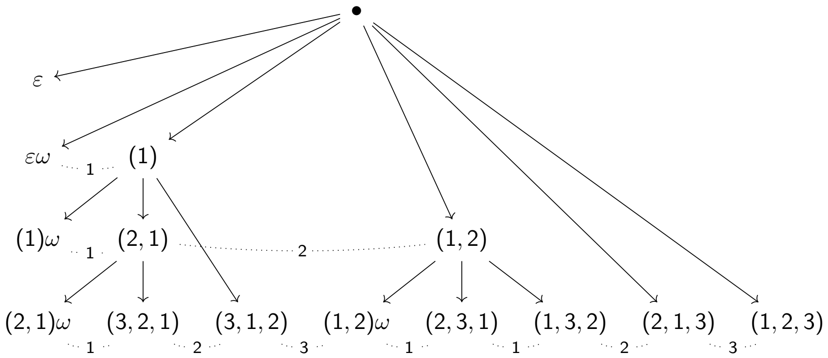

Prime measures on uncountable cofinalities

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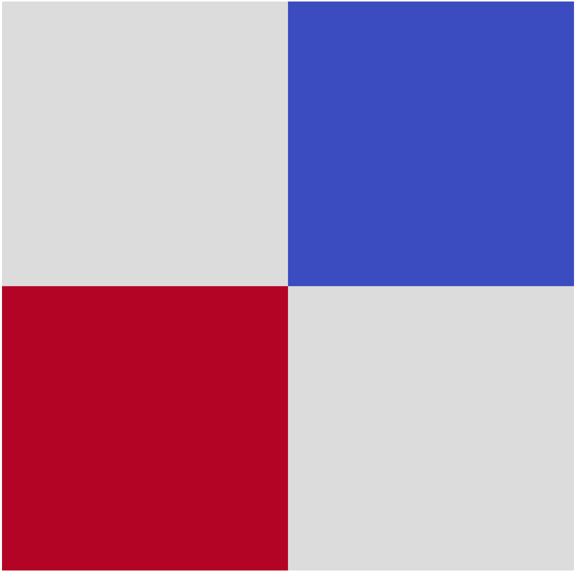

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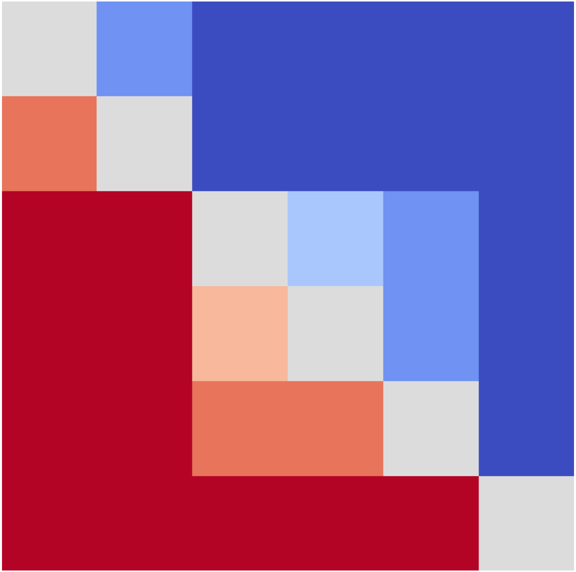

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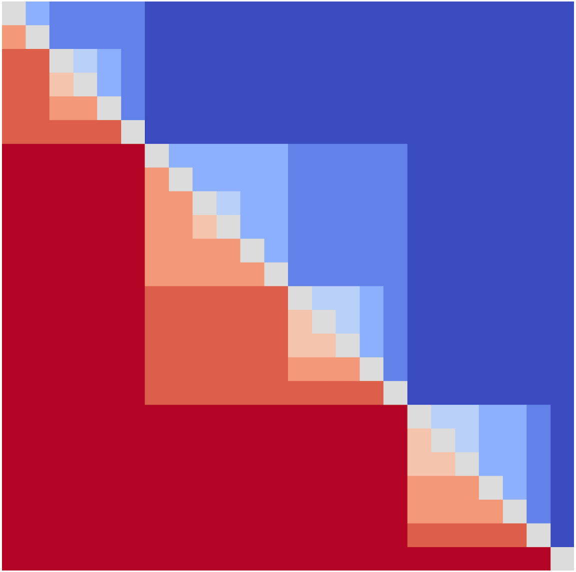

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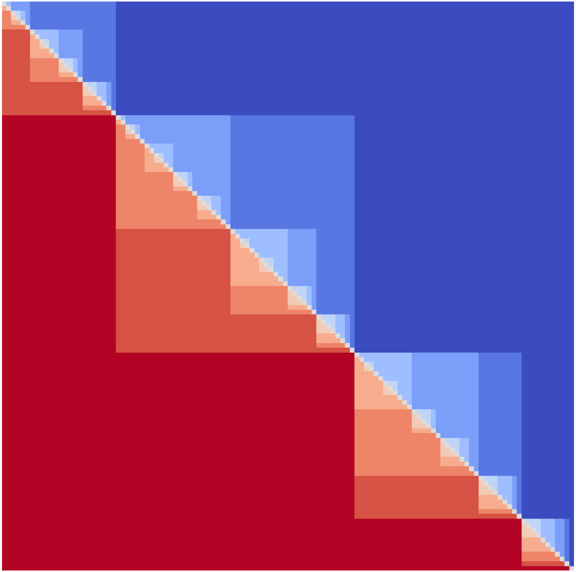

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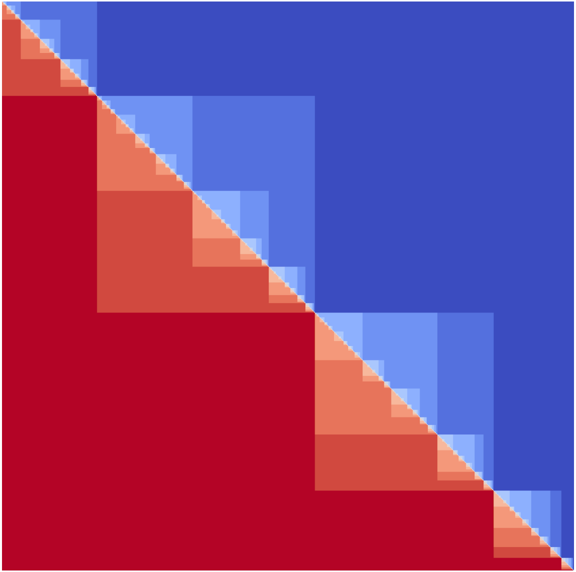

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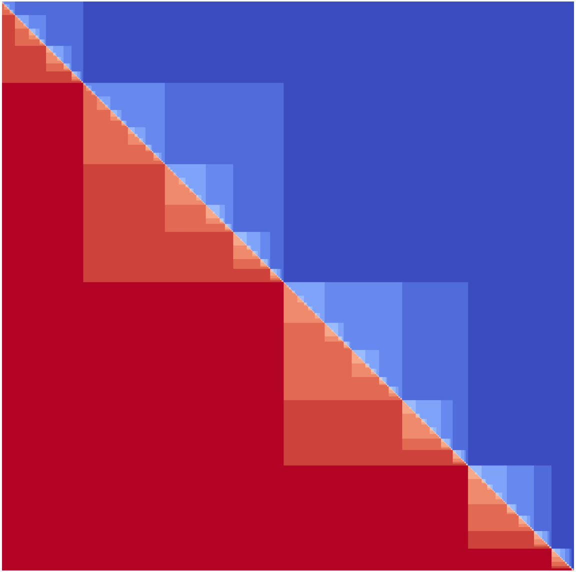

Questions

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What about UA for measures on non-well-orderable sets?

Thanks!