Generic gaps are descructible

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Overview

Proposition

The (ω_1, ω_1)-gap in $\mathcal{P}(\omega)$ forced with countable approximations is destructible.

Background on (ω_1, ω_1)-gaps

$G = (A_\alpha, B_\alpha)_{\alpha<\gamma}$, $\gamma \leq \omega_1$ is a *proto-gap* if for each $\alpha < \beta < \gamma$

- $A_\alpha, B_\alpha \subset \omega$,
- $A_\alpha \cup B_\alpha \neq^* \omega$, $A_\alpha \cap B_\alpha = \emptyset$ and
- $A_\alpha \subset^* A_\beta$, $B_\alpha \subset^* B_\beta$.

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Define $\alpha \parallel \beta$ if $A_\alpha \cap B_\beta = \emptyset = B_\alpha \cap A_\beta$; and $\alpha \perp \beta$ if not $\alpha \parallel \beta$.

Pre-gap G is a *gap* if $\forall X \in [\omega_1]^{\omega_1} \exists \alpha, \beta \in X$ such that $\alpha \perp \beta$.
Equivalently $\nexists C \subset \omega$ such that $(\forall \alpha \in \omega_1)\, A_\alpha \subset^* C$ and $C \cap B_\alpha =^* \emptyset$.

A gap G is *destructible* if $\forall X \in [\omega_1]^{\omega_1} \exists \alpha \neq \beta \in X$ such that $\alpha \parallel \beta$.
Equivalently there is a c.c.c. (or just ω_1-preserving) forcing such that G is not a gap in the extension.

A gap is *indestructible* if $(\exists X \in [\omega_1]^{\omega_1} \forall \alpha \neq \beta \in X)\, \alpha \perp \beta$.

Existence of gaps

An (indesctructible) gap exists in ZFC (Hausdorff).

All gaps are indesctructible assuming any of MA, PID, OCA.

How to get a destructible gap:

- Add a Cohen real.
- Add a gap with finite conditions.
- Construct one using $\diamondsuit$ (Dow).

destructible/indesctructible gaps

$$\simeq$$

Suslin/special Aronszajn trees

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Question

What about a gap added by countable approximations?

Forcing a gap with countable conditions

Let $\mathbb{P}$ be the set of countable proto-gaps ordered by end-extension.

Define G as the union of the generic filter.

Claim

$\mathbb{P}$ is σ-closed.

Claim

G is a gap.

Proposition

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Lemma

G is sealing.

Lemma

Sealing $\Rightarrow$ destructible

Let $G = (A_\alpha, B_\alpha)_{\alpha<\gamma}$ be a proto-gap.

For $\alpha < \beta$, $n \in \omega$ define $\alpha \subset_n \beta$ if $(A_\alpha \setminus A_\beta) \cup (B_\alpha \setminus B_\beta) \subseteq n$
For $n \in \omega$, $x \subseteq n$ define $(n, x) \triangleleft \alpha$ if $A_\alpha \cap n \subseteq x$ and $B_\alpha \cap x = \emptyset$.

Set $E \subseteq \omega_1$ is *evading* if $(\forall \beta \in \omega_1 \exists \alpha \in E)\ \alpha \parallel \beta$

Gap G is *sealing* if for each evading $E \subseteq \omega_1$
$\exists \delta \in \omega_1 \exists^\infty n \in \omega \forall x \subseteq n \exists \alpha \in E \cap \delta$ such that $(n, x) \triangleleft \alpha$ and $\alpha \subset_n \delta$.

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Lemma

Sealing $\Rightarrow$ destructible.

Suppose $E \in [\omega_1]^{\omega_1}$ is such that $(\forall \alpha \neq \beta \in E)\ \alpha \perp \beta$.
WLOG assume E is $\subseteq$-maximal with this property $\Rightarrow$ E is evading.
Take $\delta \in \omega_1$ as in the definition of sealing.
Since $|E| = \omega_1$, take $\beta \in E \setminus \delta$.
There is $n \in \omega$, $\alpha \in E \cap \delta$ such that
$\alpha \subset_n \delta \subset_n \beta$ and $(n, n \cap A_\beta) \lhd \alpha \quad \Rightarrow \quad \alpha \parallel \beta$.

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Lemma

The generic gap G is sealing.

Suppose $p \Vdash \dot{E}$ is evading.
Fix $M \prec H(\theta)$ countable elementary submodel; $\dot{E}, p \in M$.
Let $\delta = M \cap \omega_1$.
Extend p in ω-many steps to arrange that

$$\exists^\infty n \in \omega\, \forall x \subseteq n\, \exists \alpha \in E \cap \delta \text{ such that } (n, x) \lhd \alpha \text{ and } \alpha \subset_n \delta.$$

Question

Find a reasonable construction of a destructible gap under $\diamondsuit$ (Dow).