Generating Ultrafilters

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Theorem 1 (Kunen)

It is consistent that there is a non-principal ultrafilter U on ω which is generated by fewer than $\mathfrak{c}$-many sets.

To do that, Kunen iterated the *Mathias forcing relative to an ultrafilter U*, denoted by $\mathbb{M}_U$:

Definition 2

Conditions of $\mathbb{M}_U$ are pairs $(a, A) \in [\omega]^{<\omega} \times U$. The order is defined by $(a, A) \leq (b, B)$ is $b \sqsubseteq a$, $a \setminus b \subseteq B$ and $A \subseteq B$.

$\mathbb{M}_U$ is a ccc forcing which adds a set x which $\subseteq^*$-generates U.

Question (Kunen)

Is it consistent to have a uniform ultrafilter on $\aleph_1$ which is generated by fewer than $2^{\aleph_1}$-many sets?

Kunen's Problem at a measurable cardinal

Question

Is it consistent to have a measurable cardinal κ, carrying a uniform κ-complete ultrafilter which is generated by fewer than 2^κ-many sets?

In an unpublished work, Carlson gave a positive answer from a supercompact.

Definition 3 (Generalized Mathias forcing (aka long Prikry))

Let U be a κ-complete ultrafilter over κ. Conditions are pairs $(a, A) \in [\kappa]^{<\kappa} \times U$, the order is similar to the countable case.

This forcing is κ-closed and κ^+-cc. It adds a set x which $\subseteq^*$-generates U, but the proof that the iteration works has extra layers of complications.

Question

What is the consistency strength of having a uniform κ-complete ultrafilter over $\kappa > \omega$ which is generated by fewer than 2^κ-many sets? Is it more exactly $o(\kappa) = \kappa^{++}$?

The Tukey order

Definition 4 (Tukey [5] '40)

Let $(P, \leq_P), (Q, \leq_Q)$ be two partially ordered (directed) sets. Define

$$(P, \leq_P) \leq_T (Q, \leq_Q) \text{ iff } \exists \text{ a Tukey map } f : P \to Q.$$

Where Tukey means $\forall B \subseteq P$ unbounded, $f[B] \subseteq Q$ is unbounded. Define

$$(P, \leq_P) \equiv_T (Q, \leq_Q) \text{ iff } (P, \leq_P) \leq_T (Q, \leq_Q) \text{ and } (Q, \leq_Q) \leq_T (P, \leq_P).$$

- $\Rightarrow$ We focus on the posets $(U, \supseteq), (U, \supseteq^*)$, where U is an ultrafilter.
- $\Rightarrow$ Throughout this talk, assume that U is a uniform ult over a regular κ.
- $\Rightarrow$ $(U, \supseteq) \leq_T (V, \supseteq)$ iff there is a monotone map $f : V \to U$ such that $Im(f)$ is cofinal in U (i.e. $\forall X \in U \exists Y \in V\ f(Y) \subseteq X$).
- $\Rightarrow$ $U \leq_{RK} V \Rightarrow (U, \supseteq) \leq_T (V, \supseteq)$ (same with $\supseteq^*$).
- $\Rightarrow$ Studied mostly for ultrafilters on ω.

Definition 5 (The Tukey Spectrum (aka the Point Spectrum))

$$Sp_T(U) = \{\lambda \in Reg \mid \lambda \leq_T U\}.$$

Definition 6 (Cohesive Ultrafilters (aka Galvin's Property))

An ultrafilter U is (λ,μ)-cohesive iff $\forall\mathcal{A}\in[U]^\lambda\exists\mathcal{B}\in[\mathcal{A}]^\mu,\ \bigcap\mathcal{B}\in U$.

Theorem 7

Suppose that either $\lambda=\mu=cf(\mu)$ or $\lambda^{<\mu}=\lambda$. TFAE for any ultrafilter U:

1. $(U,\supseteq)$ *is $\leq_T$-above every μ-directed set of cardinality $\leq\lambda$.*
2. $([\lambda]^{<\mu},\subseteq)\leq_T(U,\supseteq)$.
3. U *is not (λ,μ)-cohesive.*

Corollary 8

$Sp_T(U)=\{\lambda\in Reg\mid U$ *is not* (λ,λ)*-cohesive.*

Theorem 9

1. *Assume $\kappa^{<\kappa}=\kappa$, $\forall U$ normal on κ is (κ^+,κ)-cohesive. [Galvin 73']*
2. *Assume $2^\kappa=\kappa^+$, $\forall U$ uniform on κ is not (κ^+,κ^+)-cohesive. [Kanamori 78']*

Gained renewed interest due to their relevance to Prikry-type forcing.

Kanamori's question

Question (Kanamori)

Is it consistent to have a κ-complete ultrafilter U over a measurable cardinal κ which is (κ^+, κ^+)-cohesive?

Question (Kanamori-Reformulated)

Is it consistent to have a κ-complete ultrafilter U over a measurable cardinal κ such that $\kappa^+ \notin Sp_T(U)$?

The results in the next few slides appear in:

Benhamou, T., *On Ultrapowers and Cohesive Ultrafilters*, arXiv:2410.06275 (2024)

The Tukey Spectrum of an ultrafilter

Theorem 10

Let U be an ultrafilter and $\lambda \neq \kappa$ be a regular cardinal. TFAE:

1. $\lambda \in Sp_T(U)$ (i.e. $(U, \supseteq)$ is not (λ, λ)-cohesive).
2. $(U, \supseteq^*)$ is not (λ, λ)-cohesive, i.e. $\exists \langle A_\alpha \mid \alpha < \lambda \rangle \subseteq U$ such that $\forall I \in [\lambda]^\lambda$, $\{A_i \mid i \in I\}$ has no pseudo intersection in U.
3. There is $X \in M_U$ such that[a] $M_U \models j_U'' \lambda \subseteq X$ and for any set I of size λ, $M_U \models j_U(I) \not\subseteq X$.

[a] More precisely, for all $i < \lambda$, $M_U \models j_U(i) \in X$.

Corollary 11

$Sp_T(U) = Sp_T(U, \supseteq^*) \cup \{\kappa\}$.

The higher part of the spectrum

The character of an ultrafilter U is $\mathfrak{ch}(U) = \min\{|\mathcal{B}| \mid \mathcal{B}$ is cofinal in $(U, \supseteq^*)\}$.

Theorem 12

$\mathfrak{ch}(U)$ is an upper bound for $Sp_T(U)$.

Theorem 13

$cf(\mathfrak{ch}(U)) \in Sp_T(U)$. Namely U is not $(cf(\mathfrak{ch}(U)), cf(\mathfrak{ch}(U)))$-cohesive.

This improves Kanamori's theorem to the case where $2^\kappa \geq \kappa^+$.

Corollary 14

If $\mathfrak{ch}(U)$ is regular then $\mathfrak{ch}(U) = \max(Sp_T(U))$.

Question

Is it ZFC provable that $\mathfrak{ch}(U) = \sup(Sp_T(U))$?

A positive answer would give a nice characterization of $\mathfrak{ch}(U)$ via ultrapowers.

The depth spectrum and the lower part

A U-tower of length λ is a seq. $\langle X_i \mid i < \lambda\rangle$ which is $\subseteq^*$-decreasing and there is no $A \in U$ such that $\forall i < \lambda, A \subseteq^* X_i$. Such A is called a U-large pseudo-intersection

Definition 15

$Sp_{Dp}(U) = \{\lambda \in Reg \mid \exists U\text{-tower of length } \lambda\}$.

Proposition 1

$Sp_{Dp}(U) \subseteq Sp_T(U)$

Let $\mathfrak{t}(U) = \min(Sp_{Dp}(U))$
$\mathfrak{p}(U) = \min\{\lambda \mid \exists \mathcal{A} \in [U]^\lambda \text{ with no } U\text{-large pseudo intersection}\}$.

Theorem 16

Let U be a uniform ultrafilter over κ then:

1. $\min(Sp_T(U)) = crit(j_U)$=*the completeness degree of U.*
2. $\min(Sp_T(U, \subseteq^*)) = \mathfrak{p}(U) = \mathfrak{t}(U)$

(Recall $Sp_T(U) = Sp_T(U, \supseteq^*) \cup \{\kappa\}$)

Examples

Tukey top ultrafilters

Definition 17

A κ-complete non-$(\kappa, 2^\kappa)$-cohesive ultrafilter over κ is called κ-*Tukey-top*.

ω-Tukey-top is just Tukey-top. Such ultrafilters are maximal in the Tukey order among all κ-complete ultrafilters, and therefore have maximal complexity.

Proposition 2

If U is κ-Tukey-top then $Sp_T(\kappa) = [\kappa, 2^\kappa] \cap Reg$.

Question

Do κ-Tukey top ultrafilters even exist?

- $\Rightarrow$ There exists a Tukey-top ultrafilter on ω. (Isbell [3] '65, Juhász [4])
- $\Rightarrow$ Consistently yes on a measurable (B.-Gitik [2] '22)
- $\Rightarrow$ Consistently no on a measurable, $L[U]$ (B.-Gitik [1] '21)
- $\Rightarrow$ What about $Sp_{dp}(U)$?

In a Cohen model

In a Cohen model and friends (Easton, Woodin...)

Theorem 18

Assume GCH and let κ *be* λ*-strong for some regular* $\lambda > \kappa$ *and let* $V[G]$ *be the usual generic extension for adding* λ*-many cohen functions to* κ *(with preparation). Then in* $V[G]$, $2^\kappa = \lambda$, *and for any uniform* κ*-complete ultrafilter* U *over* κ*:*

1. $Sp_T(U) = [\kappa, \lambda] \cap Reg$
2. $Sp_{Dp}(U) \subseteq \{\kappa, \kappa^+\}$.

In particular, we see that the Depth and Tukey spectrum are different. In a joint work with Gitik, we showed that this model has a κ-Tukey-top ultrafilter.

Question

What about ultrafilters in the κ*-Sacks model? The* κ*-Miller model?*

Question

Can the spectrum be non-convex?

Simple P_λ-points

An ultrafilter U over κ is a P_λ-point if $(U, \supseteq^*)$ is λ-directed. Equivalently, if $\mathfrak{p}(U) \geq \lambda$. For a regular λ, a simple P_λ-point is an ultrafilter U with a generating set of the form $\langle X_i \mid i < \lambda \rangle$ which is $\subseteq^*$-decreasing.

Corollary 19

U is a simple P_λ-point if and only if $\mathfrak{p}(U) = \lambda = \mathfrak{ch}(U)$ if and only if $Sp_T(U) \setminus \{\kappa\} = \{\lambda\}$.

Theorem 20 (B.-Goldberg 25+)

For κ regular uncountable, if there is a simple P_λ-point then $\lambda = \mathfrak{b}_\kappa = \mathfrak{d}_\kappa = \mathfrak{s}_\kappa = \mathfrak{r}_\kappa = \mathfrak{u}_\kappa$

In particular, there is only one λ such that there is a simple P_λ-point. This is in sharp contrast to ω, where it was recently proven by Brüninger–Mildenberger that it is consistent to have a simple-$P_{\aleph_1}$ and a simple-$P_{\aleph_2}$-point.

Back to the Mathias forcing

Theorem 21

Let $\lambda > \kappa$ be regular. The following are equiconsistent:

1. *There is a κ-complete ultrafilter U such that $\min(Sp_T(U) \setminus \{\kappa\}) \geq \lambda$.*
2. *There is a P_λ-point.*
3. *There is a simple P_λ-point.*

Corollary 22

Starting from a supercompact cardinal, it is consistent to have a κ-complete U such that $\kappa^+ \notin Sp_T(U)$ (A positive answer to Kanamori's question).

Consistency strength

Theorem 23 (B.-Goldberg 25+)

If there is a P_λ-point for $\lambda \geq \kappa^{++}$ then there is an inner model with a 2-strong cardinal.

Theorem 24 (B.-Goldberg 25+)

If $V[G]$ is a generic extension of V where κ is measurable and there is a V-generic set for $\mathbb{M}_U$, $U \in V$ being a κ-complete ultrafilter, then there is an inner model with a μ-measurable.

Rectangular Mathias

The following result is joint with Cummings, Goldberg, Hayut and Poveda:

Theorem 25 (25+)

Assume that κ is indestructible supercompact or $\kappa = \omega$. Then for any $\lambda_1 < \lambda_2$ regular, there is a cofinality preserving generic extension admitting an ultrafilter U generated by a set $\mathcal{B}$ such that $(\mathcal{B}, \supseteq^*) \simeq \lambda_1 \times \lambda_2$. In particular $Sp_T(U) \setminus \{\kappa\} = Sp_{Dp}(U) = \{\lambda_1, \lambda_2\}$.

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Following the AIM forcing, we call such an ultrafilter a simple AIM ultrafilter.

Corollary 26

The spectrum can be a non-convex set.

Proof idea

Force a matrix/rectangular iteration of Mathias forcing, that is, $\langle \mathbb{P}_{<(\alpha,\beta)}, \dot{Q}_{\alpha,\beta} \mid \alpha < \omega_1, \beta < \omega_3 \rangle$. Such that for each α, β,

1. $\mathbb{P}_{<(\alpha,\beta)}$ consists of all finitely supported functions p such that $dom(p) \subseteq \alpha + 1 \times \beta + 1 \setminus \{(\alpha, \beta)\}$, and
2. for each $(\alpha', \beta') < (\alpha, \beta)$, $\Vdash_{\mathbb{P}_{<(\alpha',\beta')}} p(\alpha', \beta') \in \dot{Q}_{\alpha,\beta}$.
3. $\dot{Q}_{\alpha,\beta}$ is a $\mathbb{P}_{<(\alpha,\beta)}$-name for $\mathbb{M}_{\dot{U}_{\alpha,\beta}}$.
4. $\dot{U}_{\alpha,\beta}$ is a $\mathbb{P}_{\alpha,\beta}$-name for a carefully chosen ultrafilter containing $x_{\alpha',\beta'}$- the Mathias real added by $\mathbb{M}_{U_{\alpha',\beta'}}$ for all $(\alpha', \beta') < (\alpha, \beta)$.

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   To see item 4 (and more!) look for our upcoming preprint on arXiv!

Thank you for your attention!

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