(Short) memory iterations and side conditions

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Strong classical forcing axioms imply $2^{\aleph_0} = \aleph_2$. For example PFA implies this.

On the other hand, Martin's Axiom is of course compatible with arbitrarily large continuum.

Certain forcing axioms in-between PFA and MA are known to be compatible with large continuum. More specifically:

Definition

(A.–Mota) A partial order $\mathcal{Q}$ has the $\aleph_{1.5}$–c.c. iff for every large enough cardinal θ there is a club $D \subseteq [H(\theta)]^{\aleph_0}$ such that for every finite $\mathcal{N} \subseteq D$, if $q \in \mathcal{Q} \cap N$ for some $N \in \mathcal{N}$ of minimal height δ_N within $\mathcal{N}$, then there is some $q^* \leq_{\mathcal{Q}} q$ such that q^* is $(M, \mathcal{Q})$–generic for each $M \in \mathcal{N}$.

(The height of a model N is $\delta_N := N \cap \omega_1$.)

Fact

Every poset with the $\aleph_{1.5}$-c.c. is proper and has the $\aleph_2$-c.c.

Proof.

Properness is trivial (apply definition with $\mathcal{N} = \{N\}$).

$\aleph_2$-c.c.: Suppose $A = \{q_i : i < \lambda\}$ max. antichain, $\lambda \geq \omega_2$. For each i let N_i be model with A, $q_i \in N_i$. Since $\lambda \geq \omega_2$, there are $i_0 \neq i_1$ such that $\delta_{N_{i_0}} = \delta_{N_{i_1}}$ and $q_{i_0} \notin N_{i_1}$. Hence there is $q \leq_{\mathcal{P}} q_{i_0}$ which is $(N_{i_0}, \mathcal{Q})$-generic and $(N_{i_1}, \mathcal{Q})$-generic. But q cannot be $(N_{i_1}, \mathcal{Q})$-generic since $A \in N_{i_1}$ and $q_{i_0} \notin N_{i_1}$. Contradiction.

$\square$

So we have that for each κ:

$$\text{FA}(\{\mathcal{Q} : \mathcal{Q} \text{ is proper and has the } \aleph_2\text{-c.c.}\})_\kappa \Rightarrow \text{MA}^{1.5}_\kappa \Rightarrow \text{MA}_\kappa.$$

Definition

Given a cardinal κ, $\mathrm{MA}^{1.5}_{\kappa}$ is the forcing axiom $\mathrm{FA}_{\kappa}(\aleph_{1.5}\text{-c.c.})$.

Theorem

(A.–Mota) (GCH) Given any infinite cardinal κ, there is a proper and $\aleph_2$-c.c. forcing notion which forces $MA^{1.5}_{\kappa}$.

The forcing notion witnessing this theorem is a “finite-support” iteration $(\mathcal{P}_\alpha : \alpha \leq \kappa)$ with symmetric systems of models with markers (N, ρ) as side conditions.

A condition p in $\mathcal{P}_\alpha$ is of the form $p = (F_p, \Delta_p)$.

- F_p is the *working part*; it is a finite function with $\mathrm{dom}(F_p) \subseteq \alpha$.
- Δ_p is the *side condition*.

Δ_p is a set of models with markers (N, ρ), where $N \preccurlyeq H(\kappa)$ is countable, $\rho \in N \cap (\alpha + 1)$, and

$$\mathrm{dom}(\Delta_p) = \{N : (N, \rho) \in \Delta_p \text{ for some } \rho\}$$

is a symmetric system.

The set of ρ such that $(N, \rho + 1) \in \Delta_p$ is the set of stages at which N is to be seen as ‘active’; any working part at any such ρ has to be forced to be $(N[\dot{G}_\rho], \dot{\mathcal{Q}}_\rho)$-generic, for the relevant $\dot{\mathcal{Q}}_\rho$.

The proof of properness of $\mathcal{P}_\alpha$ proceeds by induction, so we naturally require that

(1) If $(N, \rho) \in \Delta_p$ and $\bar{\rho} \in N \cap \rho$, then also $(N, \bar{\rho}) \in \Delta_p$.

At the limit stage α of the proof of properness for a relevant model N, and for a high enough stage $\alpha_0 \in N \cap \alpha$, working inside $N[\dot{G}_{\alpha_0}]$, we reflect the outside condition p and find a condition $\bar{p} \in N$. $\bar{p}$ will typically have new points $\gamma > \alpha_0$ in its support, and we can easily arrange that $\gamma \notin M$ for any model $M \in N$ coming from Δ_p.

The amalgamating condition will need to provide a condition in $\dot{\mathcal{Q}}_\gamma$ which is $(N[\dot{G}_\gamma], \dot{\mathcal{Q}}_\gamma)$-generic (since of course $(N, \gamma + 1) \in \Delta_p$), but also $(N'[\dot{G}_\gamma], \dot{\mathcal{Q}}_\gamma)$-generic for all other $(N', \gamma + 1) \in \Delta_p$.

By the definition of $\aleph_{1.5}$-c.c., this is in fact possible and the proof goes through.

On PFA(ω_1) and large continuum

Given a cardinal κ, let $\mathsf{PFA}_\kappa(\omega_1)$ be

$$\mathsf{FA}_\kappa(\{\mathcal{Q} : \mathcal{Q} \text{ proper}, |\mathcal{Q}| = \aleph_1\})$$

Let also PFA(ω_1) be $\mathsf{PFA}_{\omega_1}(\omega_1)$.

PFA(ω_1) can be easily forced over any model of GCH by a countable-support iteration of length ω_2.

Question: Is PFA(ω_1) compatible with $2^{\aleph_0} > \aleph_2$?

Note: If we want to force $\mathsf{PFA}(\omega_1)$, the construction with models with markers as side conditions we just sketched does not seem to work. The problem is precisely at the limit stage of the inductive proof of properness:

We may have more than one model N' of height δ_N such that $(N', \gamma + 1) \in \Delta_p$. Even if $N' \cong N$, which is part of the definition of symmetric system, it may be that $N'[\dot{G}_\gamma]$ and $N[\dot{G}_\gamma]$ are forced to be non-isomorphic, and even that

$$\{A \cap \delta_{N'} \,:\, A \in \mathcal{P}(\omega_1)^{N'[\dot{G}_\gamma]}\}$$

and

$$\{A \cap \delta_N \,:\, A \in \mathcal{P}(\omega_1)^{N[\dot{G}_\gamma]}\}$$

are forced to be different. It may then not be possible to extend $\nu \in \delta_N$ to a condition which is $(N[\dot{G}_\gamma], \dot{\mathcal{Q}}_\gamma)$-generic **and** $(N'[\dot{G}_\gamma], \dot{\mathcal{Q}}_\gamma)$-generic.

Theorem

(A.–Golshani) (GCH) For every given κ, there is a proper and $\aleph_2$-c.c. poset forcing $PFA_\kappa(\omega_1)$.

The construction is again a finite-support iteration with systems of models with markers as side conditions, this time combined with the idea of Shelah's "memory iterations".

We circumvent the obstacle I pointed out by performing (0), $(1)^*$, and (2) below.

(0) Given a stage $\alpha < \kappa$, our book-keeping function $\phi : \kappa \longrightarrow H(\kappa)$ may feed us a sequence $\langle A_i : i < \omega_1 \rangle$ of antichains of $\mathcal{P}_\alpha$, *each* of size $\leq \aleph_1$, such that $\bigcup_{i<\omega_1} \{i\} \times A_i$ is an object of interest (a suitable proper forcing on ω_1). We then associate to α sets $\overline{\mathcal{U}}^\alpha, \mathcal{U}^\alpha \in [\alpha]^{\leq \aleph_1}$ given by

- $\overline{\mathcal{U}}^\alpha = \bigcup \{\mathrm{dom}(F_p) : p \in A_i^\alpha, i < \omega_1\} \cup \bigcup \{N \cap \rho : (N, \rho) \in \Delta_p, p \in A_i^\alpha, i < \omega_1\}$ and
- $\mathcal{U}^\alpha = \overline{\mathcal{U}}^\alpha \cup \bigcup \{\mathcal{U}^\beta : \beta \in \overline{\mathcal{U}}^\alpha\}$.

(1)* We drop the requirement (1). [This was: If $(N, \rho) \in \Delta_p$ and $\bar{\rho} \in N \cap \rho$, then also $(N, \bar{\rho}) \in \Delta_p$.]

However, we insist that for any $(N, \alpha + 1) \in \Delta_p$, $(N, \rho) \in \Delta_p$ for every $\rho \in \mathcal{U}^\alpha \cap N$.

(2) Given α, we define $\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha$ as the suborder of $\mathcal{P}_\alpha$ consisting of those $p \in \mathcal{P}_\alpha$ such that

- $\mathrm{dom}(F_p) \subseteq \mathcal{U}^\alpha$ and
- $\rho \in \mathcal{U}^\alpha$ for all $(N, \rho) \in \Delta_p$.

(This is a complete suborder of $\mathcal{P}_\alpha$.)

We then require that

$$p \restriction \mathcal{U}^\alpha = (F_p \restriction \mathcal{U}^\alpha, \{(M, \rho) \in \Delta_p, \rho \in \mathcal{U}^\alpha\}) \in \mathcal{P}_\alpha \restriction \mathcal{U}^\alpha$$

forces, in $\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha$, that $F_p(\nu)$ is $(N[\dot{G}_{\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha}], \dot{\mathcal{Q}}_\alpha)$-generic for all $(N, \alpha + 1) \in \Delta_p$.

[$\dot{\mathcal{Q}}_\alpha$ is the forcing coded by $\bigcup_{i<\omega_1} \{i\} \times A_i$, and is in fact a $\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha$-name.]

The crucial symmetry fact: If $(N_0, \alpha+1), (N_1, \alpha+1) \in \Delta_p$, $\delta_{N_0} = \delta_{N_1}$, $s \in (\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha) \cap N_0$, and $p \restriction \mathcal{U}^\alpha \leq_{\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha} s$, then

$$p \restriction \mathcal{U}^\alpha \Vdash_{\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha} \Psi_{N_0,N_1}(s) \in \dot{G}_{\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha}$$

(where $\Psi_{N_0,N_1} : N_0 \longrightarrow N_1$ is the isomorphism between these models).

As a result, $p \restriction \mathcal{U}^\alpha$ forces that

$$\{A \cap \delta_{N_0} \,:\, A \in \mathcal{P}(\omega_1)^{N_0[\dot{G}_{\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha}]}\} =$$

$$\{A \cap \delta_{N_1} \,:\, A \in \mathcal{P}(\omega_1)^{N_1[\dot{G}_{\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha}]}\}$$

and so $p \restriction \mathcal{U}^\alpha$ forces that if a condition is $(N_0[\dot{G}_{\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha}], \dot{\mathcal{Q}}_\alpha)$-generic, then it is also $(N_1[\dot{G}_{\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha}], \dot{\mathcal{Q}}_\alpha)$-generic. $\square$

Incidentally:

A standard argument using the above property shows that $\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha$ forces CH:

Suppose $\dot{r}_\xi$ are $\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha$-names for reals, $\xi < \omega_2$, and $p \Vdash_{\mathcal{P}_\alpha \restriction \mathcal{U}^\alpha} \dot{r}_{\xi_0} \neq \dot{r}_{\xi_1}$ for all $\xi_0 \neq \xi_1$. Find suitable models N_ξ containing p, $\dot{r}_\xi$.

By CH, find $\xi_0 \neq \xi_1$ such that

$$(N_{\xi_0}; \in, \dot{r}_{\xi_0}, \ldots) \cong (N_{\xi_1}; \in, \dot{r}_{\xi_1}, \ldots)$$

and the isomorphism fixes $N_{\xi_0} \cap N_{\xi_1}$.

Extend the side condition Δ_p of p by adding (N_{ξ_0}, ρ_0) and (N_{ξ_1}, ρ_1) for all $\rho_0 \in N_{\xi_0} \cap \mathcal{U}^\alpha$ and $\rho_1 \in N_{\xi_1} \cap \mathcal{U}^\alpha$.

The resulting condition then forces $\dot{r}_{\xi_0} = \dot{r}_{\xi_1}$ by the crucial symmetry fact. Contradiction. $\square$

An extension for Prikry-type proper forcing

Let us say that $\mathcal{Q}$ is a *Prikry-type* partial order in case there is a set $\mathrm{Res}(\mathcal{Q})$ such that:

(1) $\mathcal{Q}$ is a partial order with conditions being ordered pairs (s, A) with $A \in \mathrm{Res}(\mathcal{Q})$;

(2) for all $A_0, A_1 \in \mathrm{Res}(\mathcal{Q})$, $A_0 \cap A_1 \in \mathrm{Res}(\mathcal{Q})$;

(3) for every $(s, A_0) \in \mathcal{Q}$ and every $A_1 \in \mathrm{Res}(\mathcal{Q})$, if $A_1 \subseteq A_0$, then (s, A_1) is a condition in $\mathcal{Q}$ extending (s, A_0).

In the above situation we will sometimes refer to s as the *stem* of (s, A) and to A as its *reservoir*. Given a set X, we will say that a Prikry-type partial order $\mathcal{Q}$ has *stems in* X if for all $(s, A) \in \mathcal{Q}$, $s \in X$.

- Prikry forcing on a measurable cardinal κ is isomorphic to a Prikry-type forcing with stems in κ.
- Mathias forcing is not isomorphic to a Prikry-type forcing with stems in ω (the reservoirs are not closed under intersections).
- The natural forcing $\mathcal{Q}$ for shooting a club diagonalising the club filter is isomorphic to a Prikry-type forcing with stems in $2^{\aleph_0}$: $\mathcal{Q}$ is the partial order of pairs (x, C), where
  (a) x is a closed countable subset of ω_1 and
  (b) C is a club of ω_1,

  and where (x_1, C_1) extends (x_0, C_0) if
  (i) x_1 is an end-extension of x_0,
  (ii) $C_1 \subseteq C_0$, and
  (iii) $x_1 \setminus x_0 \subseteq C_0$.

A forcing axiom-like principle: ω_1-Prikry-type BPFA^{++} from NS_{ω_1}-correct ground models of CH

An inner model M is NS_{ω_1}*-correct* iff for every $S \in \mathcal{P}(\omega_1)^M$, if $M \models S$ is stationary, then S is stationary.

Local^{++} CH: Every set in $H(\omega_2)$ is in some NS_{ω_1}-correct ground model satisfying CH.

Note: Local^{++} CH fails if NS_{ω_1} is $\aleph_1$-dense. (Thanks to Andreas Lietz for pointing this out.)

Definition

ω_1-Prikry-type BPFA^{++} from NS_{ω_1}-correct ground models of CH, CH-Pr_{ω_1}-BPFA^{++}, is the conjunction of the following two statements.

(1) Local^{++} CH

(2) Suppose $\varphi(x, y)$ is a Σ_0 formula in the language of set theory with a unary predicate. Suppose for every $a \in H(\omega_2)$ and every NS_{ω_1}-correct ground model M such that $M \models \mathrm{CH}$, if $a \in M$, then in M it holds that there is a proper Prikry-type forcing notion $\mathcal{Q}$ with stems in ω_1 such that $\mathcal{Q}$ forces $(H(\omega_2); \in, \mathrm{NS}_{\omega_1}) \models \exists y \varphi(a, y)$.

Then $(H(\omega_2); \in, \mathrm{NS}_{\omega_1}) \models \forall x \exists y \varphi(x, y)$.

A variant of the construction for the first theorem yields the following:

Theorem

(A.–Golshani) Assume GCH. Let $\kappa \geq \aleph_2$ be a regular cardinal. Then there is a proper partial order $\mathcal{P}$ with the $\aleph_2$-c.c. and forcing the following statements.

(1) $2^{\aleph_0} = \kappa$

(2) $PFA(\aleph_1)_{<\kappa}$

(3) $CH\text{-}\mathrm{Pr}_{\omega_1}\text{-}BPFA^{++}$

This variant incorporates decorations at every stage α. This makes sure that for every $\alpha < \kappa$,

$$\{\delta_M \,:\, (M, \alpha+1) \in \Delta_p \text{ for some } p \in \dot{G}\}$$

is forced to be a club of ω_1. This ensures that every stationary subset in any relevant intermediate inner model of CH is NS_{ω_1}-correct.

Consequences of CH-Pr_{ω_1}-BPFA^{++}

(1) Baumgartner's Axiom for $\aleph_1$-dense sets of reals

(2) OCA($\aleph_1$) (i.e., Todorčević's Open Colouring Axiom for sets of reals of size $\aleph_1$)

(3) Measuring

(4) The P-ideal Dichotomy for $\aleph_1$-generated ideals on ω_1

(5) $\mathrm{OCA}_{\mathrm{ARS}}$

Hence, all these statements are simultaneously compatible with arbitrarily large continuum.

Gilton-Neeman build a model of $\mathrm{OCA}_{\mathrm{ARS}}$ in which $2^{\aleph_0} = \aleph_3$. Their method doesn't work to yield models of $2^{\aleph_0} > \aleph_3$ and they ask for the consistency of $\mathrm{OCA}_{\mathrm{ARS}} + 2^{\aleph_0} > \aleph_3$. Our results answer their question.

All these statements follow in fact from the weaker form of $\mathsf{CH}\text{-}\mathsf{Pr}_{\omega_1}\text{-}\mathsf{BPFA}^{++}$ (call it $\mathsf{CH}\text{-}\mathsf{Pr}_{\omega_1}\text{-}\mathsf{BPFA}$) in which we

- wave the requirement that the relevant ground models be NS_{ω_1}-correct and
- replace $(H(\omega_2); \in, \mathsf{NS}_{\omega_1})$ with $(H(\omega_2); \in)$.

Theorem

(Dobrinen-Krueger-Marun-Mota-Zapletal) There is a proper poset forcing $MM(\omega_1)$ *(= Martin's Maximum for posets of size* $\aleph_1$*).*

In their paper, DKMMZ ask if $\mathrm{MM}(\omega_1)$ is consistent with large continuum.

Fact

CH-Pr_{ω_1}-$BPFA^{++}$ *implies* $MM(\omega_1)$*.*

Hence, by our second theorem, the answer to the DKMMZ question is Yes.

Recall: For all κ,
$\mathrm{FA}(\{\mathcal{Q} : \mathcal{Q} \text{ is proper and has the } \aleph_2\text{-c.c.}\})_\kappa \Rightarrow \mathrm{MA}^{1.5}_\kappa \Longrightarrow \mathrm{MA}_\kappa$.

A natural question at this point:

Question: Is $\mathrm{FA}(\{\mathcal{Q} : \mathcal{Q} \text{ is proper and has the } \aleph_2\text{-c.c.}\})_{\aleph_1}$ compatible with $2^{\aleph_0} > \aleph_2$? Is $\mathrm{FA}(\{\mathcal{Q} : \mathcal{Q} \text{ is proper and has the } \aleph_2\text{-c.c.}\})_{\aleph_2}$ consistent?

These questions are still open. On the other hand:

Let $\mathrm{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$ denote
$\mathrm{FA}(\{\mathcal{Q} : \mathcal{Q}$ stationary set preserving and $\aleph_2$-c.c.$\})_{\aleph_2}$.

Theorem

(A.-Tananimit) $\mathrm{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$ is false.

This theorem improves an earlier theorem of Shelah showing that there is no naive high analogue of MM:

For every regular cardinal $\kappa \geq \omega_2$, $\mathrm{FA}(\{\mathcal{Q} :$
$\mathcal{Q}$ preserves all stat. subsets of all uncountable regular $\mu \leq \kappa\})_\kappa$
is false.

Structure of the proof:

(1) We prove that a certain consistent strengthening $\mathrm{MA}^{1.5}_{\aleph_2}$(stratified) of $\mathrm{MA}^{1.5}_{\aleph_2}$ implies $\square_{\omega_1,\omega_1}$.

(2) Assuming $\mathrm{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$, and using the fact that $\square_{\omega_1,\omega_1}$ holds by (1), we show that a certain form of uniformization holds which in turn implies CH. This is a contradiction since $2^{\aleph_0} \geq \aleph_3$.

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Definition

A family $\mathcal{N}$ of countable elementary submodels is *stratified* if $\mathrm{ot}(N_0 \cap \mathrm{Ord}) < \delta_{N_1}$ for all $N_0, N_1 \in \mathcal{N}$ such that $\delta_{N_0} < \delta_{N_1}$.

Fact

Every poset with the $\aleph_{1.5}$-c.c. with respect to finite stratified families is proper and has the $\aleph_2$-c.c.

Proof.

Same as for the $\aleph_{1.5}$-c.c. $\square$

So we have: For all κ,

$\mathrm{FA}(\{\mathbb{P} : \mathbb{P} \text{ is proper and has the } \aleph_2\text{-c.c.}\})_\kappa \Rightarrow$
$\mathrm{MA}^{1.5}_\kappa(\text{stratified}) \Rightarrow \mathrm{MA}^{1.5}_\kappa \Longrightarrow \mathrm{MA}_\kappa$.

Theorem

(essentially A.–Mota) (GCH) Given any infinite cardinal κ, there is a proper and $\aleph_2$-c.c. forcing notion which forces $MA^{1.5}_{\kappa}$(stratified).

$\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) and $\square_{\omega_1,\omega_1}$

Recall: $(\mathcal{C}_\alpha : \alpha \in \mathrm{Lim}(\omega_2))$ is a $\square_{\omega_1,\omega_1}$-sequence iff for all $\alpha \in \mathrm{Lim}(\omega_2)$,

- $\mathcal{C}_\alpha$ is a set of clubs of α of order type at most ω_1,
- $|\mathcal{C}_\alpha| \leq \aleph_1$, and
- for every $C \in \mathcal{C}_\alpha$ and every limit point β of C, $C_\alpha \cap \beta \in \mathcal{C}_\beta$.

Theorem

(A.–Tananimit) $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) implies $\square_{\omega_1,\omega_1}$.

Proof sketch of Theorem: Apply axiom to the forcing consisting of pairs $p = (h^p, \mathcal{N}^p)$, where:

(1) h^p is a finite approximation to a $\square_{\omega_1,\omega_1}$-sequence $\langle (C^\alpha_\nu)_{\nu<\omega_1} \,:\, \alpha \in \mathrm{Lim}(\omega_2)\rangle$ together with an index function i^p specifying, for $\alpha \in \mathrm{Lim}(\omega_2)$, $\nu < \omega_1$, and $\beta \in \mathrm{Lim}(C^\alpha_\nu)$, some $\bar\nu \in \omega_1$ such that $C^\alpha_\nu \cap \beta = C^\beta_{\bar\nu}$;

(2) $\mathcal{N}^p$ is a finite stratified family of countable $N \preccurlyeq (H(\omega_2); \in, \vec{e})$ for some fixed sequence $\vec{e} = (e_\alpha \,:\, \alpha < \omega_2)$, where $e_\alpha : |\alpha| \longrightarrow \alpha$ is a surjection for all α. (All countable $N \preccurlyeq H(\omega_2)$ being considered are sufficiently closed in this sense.)

(3) For every $N \in \mathcal{N}^p$ and every $\alpha \in \mathrm{dom}(h^p) \in [\mathrm{Lim}(\omega_2)]^{<\omega}$ such that $\alpha \in N$,
    (a) N is closed under the approximating functions h^p_α from h^p,
    (b) if $\mathrm{cf}(\alpha) = \omega_1$, then $\delta_N \in \mathrm{dom}(h^p_\alpha)$ and $h^p_\alpha(\delta_N) = \sup(N \cap \alpha)$, and
    (c) N is closed under the index function at α.

$\square$

A consistent uniformization principle implying CH

Theorem

(Shelah) Suppose for every $F : S^{\omega_2}_{\omega_1} \longrightarrow 2$ there is a function $G : \omega_2 \longrightarrow 2$ and clubs $D_\alpha \subseteq \alpha$ (for $\alpha \in S^{\omega_2}_{\omega_1}$) such that for every $\alpha \in S^{\omega_2}_{\omega_1}$ and every $\xi \in D_\alpha$, $G(\xi) = F(\alpha)$. Then CH holds.

($S^{\omega_2}_{\omega_1} = \{\alpha < \omega_2 \ : \ \mathrm{cf}(\alpha) = \omega_1\}$.)

Proof: The hypothesis clearly implies the following stronger statement:

For every $F : S^{\omega_2}_{\omega_1} \longrightarrow {}^{\omega}2$ there is a function $G : \omega_2 \longrightarrow {}^{\omega}2$ and clubs $D_\alpha \subseteq \alpha$ (for $\alpha \in S^{\omega_2}_{\omega_1}$) such that for every $\alpha \in S^{\omega_2}_{\omega_1}$ and every $\xi \in D_\alpha$, $G(\xi) = F(\alpha)$.

Hence, if $2^{\aleph_0} \geq \aleph_2$, the following also holds:

For every $F : S^{\omega_2}_{\omega_1} \longrightarrow \omega_2$ there is a function $G : \omega_2 \longrightarrow \omega_2$ and clubs $D_\alpha \subseteq \alpha$ (for $\alpha \in S^{\omega_2}_{\omega_1}$) such that for every $\alpha \in S^{\omega_2}_{\omega_1}$ and every $\xi \in D_\alpha$, $G(\xi) = F(\alpha)$.

Now let F be the identity on $S^{\omega_2}_{\omega_1}$. Apply the hypothesis and get uniformizing function G. Let $\alpha \in S^{\omega_2}_{\omega_1}$ be such that $G``\alpha \subseteq \alpha$. But then there is no club $D_\alpha \subseteq \alpha$ such that $G(\xi) = \alpha$ for all $\xi \in D_\alpha$. Contradiction. $\square$

Proof sketch of the inconsistency of $\mathrm{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$

Assume $\mathrm{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$. Then $2^{\aleph_0} \geq \aleph_3$ and so there is a function $F : S^{\omega_2}_{\omega_1} \longrightarrow 2$ which cannot be club-uniformized; i.e., there is no $G : \omega_2 \longrightarrow 2$, together with clubs $D_\alpha \subseteq \alpha$ (for $\alpha \in S^{\omega_2}_{\omega_1}$), such that for every $\alpha \in S^{\omega_2}_{\omega_1}$ and every $\xi \in D_\alpha$, $G(\xi) = F(\alpha)$. We will show that there is such a G after all, which is a contradiction.

By $\mathrm{MA}^{1.5}_{\aleph_2}(\text{stratified})$, there is a $\square_{\omega_1,\omega_1}$-sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \,:\, \alpha \in \mathrm{Lim}(\omega_2)\rangle$.

Let $\mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}}$ be the class of countable models N such that $N \cap \omega_2 = \bigcup_{\gamma \in C} e_\gamma``\delta_N$ for some $C \in \mathcal{C}_\alpha$, where $\alpha = \sup(N \cap \omega_2)$ (for some fixed sequence $\vec{e} = (e_\gamma \,:\, \gamma < \omega_2, \gamma \neq 0)$ of surjections $e_\gamma : \omega_1 \longrightarrow \gamma$).

For every cardinal $\theta > \omega_1$, $\mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}} \cap H(\theta)$ is a projective stationary subset of $[H(\theta)]^{\aleph_0}$.

Hence, we can define a natural stationary set preserving poset $\mathcal{Q}$, using side conditions from $\mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}}$, for adding a uniformizing function for F. The coherence of $\vec{\mathcal{C}}$ and the fact that the models in the side condition come from $\mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}}$ is used in the proof of $\aleph_2$-c.c.

A conjecture

Conjecture: BFA($\{\mathcal{Q} : \mathcal{Q}\ \omega$–proper$\}$) implies $2^{\aleph_0} = \aleph_2$.