On the Consistency Strength of MM(ω_1)

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Introduction. The forcing axioms MA, PFA and SPFA are known to be relatively consistent (from ZFC in the first case and modulo large cardinals in the other two) by means of forcing iterations which fall in the same class Γ being considered. So, this kind of construction depends on certain preservation criteria.

One of them is the central theorem of Shelah stating that if P_α is a countable support forcing iteration of $\{\dot{Q}_\beta \colon \beta < \alpha\}$ such that every $\dot{Q}_\beta$ is a proper forcing notion in $V^{P_\alpha \restriction \beta}$, then P_α is proper (in particular, P_α does not collapse ω_1). Another one, also due to Shelah, holds in the context of revised countable support forcing iterations and semiproper forcings.

On the other hand, there is no such preservation result for stationary set preserving posets and the classical argument for the consistency of MM goes in a slightly different way: it passes by showing that SPFA implies that every stationary set preserving notion of forcing is semiproper, which in turn implies the equivalence between SPFA and MM.

Let us denote by $\mathrm{PFA}(\omega_1)$ and $\mathrm{MM}(\omega_1)$ the respective restrictions of PFA and MM to posets of cardinality ω_1.

It is well-known that ZFC and $\mathrm{ZFC} + \mathrm{PFA}(\omega_1)$ are equiconsistent, which follows from the fact that under CH, forcings of size ω_1 can be iterated with countable support up to length ω_2 with an ω_2-c.c. forcing iteration.

In the 80's Shelah proved that $\mathsf{ZFC} +$ "there exists a strongly inaccessible cardinal" implies the consistency of $\mathsf{ZFC} + \mathsf{MM}(\omega_1)$. The main theorem of this talk states that Shelah's inaccessible can be taken away from this consistency statement.

Theorem

Assume CH and $2^{\omega_1} = \omega_2$. Then there is a countable support forcing iteration P_{ω_2} of $\{\dot{Q}_\beta : \beta < \omega_2\}$ with the following properties:

1. *Every $\dot{Q}_\beta$ is, in $V^{P_{\omega_2} \restriction \beta}$, a proper poset;*
2. *P_{ω_2} is proper and has the ω_2-chain condition;*
3. *P_{ω_2} forces $MM(\omega_1)$.*

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Consequently, the theories ZFC and $ZFC + MM(\omega_1)$ are equiconsistent.

The iteration described in the above theorem will involve forcing two types of posets:

(1) proper posets of size ω_1, bookkeeping so that all such posets in the final model will have been considered ω_2-many times in the iteration, and

(2) forcing notions which destroy posets which are stationary set preserving but not proper.

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In order to prove that the iteration is ω_2-c.c., we will use a property introduced by Shelah.

Definition

A poset R satisfies the ω_2*-properness isomorphism condition* (ω_2*-p.i.c.* for short) if and only if for every large enough regular cardinal θ, for every well-ordering $<$ of H_θ and for all ordinals $\alpha < \beta < \omega_2$ the following holds: if N_α and N_β are countable elementary submodels of $(H_\theta, \in, <, R)$ such that $\alpha \in N_\alpha$, $\beta \in N_\beta$, $N_\alpha \cap \omega_2 \subset \beta$, $N_\alpha \cap \alpha = N_\beta \cap \beta$, $p \in N_\alpha \cap R$ and $\pi : N_\alpha \to N_\beta$ is an isomorphism satisfying $\pi(\alpha) = \beta$ and $\pi \restriction (N_\alpha \cap N_\beta) = id$, then there exists a master condition q for N_α, extending p and $\pi(p)$, such that

$$q \Vdash_R \pi``(\dot{G} \cap \check{N}_\alpha) = \dot{G} \cap \check{N}_\beta.$$

Every proper poset of size ω_1 has the ω_2-p.i.c., and if CH holds, then every ω_2-p.i.c. poset satisfies the ω_2-chain condition.

Shelah proved that, under the assumption of CH, if P_{ω_2} is a countable support forcing iteration of $\{\dot{Q}_\beta \colon \beta < \omega_2\}$ such that every that every $\dot{Q}_\beta$ has the ω_2-p.i.c. in $V^{P_{\omega_2} \restriction \beta}$, then P_{ω_2} has the ω_2-chain condition.

Therefore, CH implies that P_{ω_2} does not collapse cardinals. In the context of our specific iteration, we will apply this result by taking each $\dot{Q}_\beta$ to be either a name for a proper poset of size ω_1 or a name for a poset Q as described in the next theorem.

Theorem

There exists a proper countably distributive poset Q of cardinality 2^{ω_1} with the ω_2-p.i.c. satisfying that for every poset P of cardinality ω_1, if P is not proper, then

$$\Vdash_Q \check{P} \text{ does not preserve stationary subsets of } \omega_1.$$

With this new ingredient, and assuming CH together with $2^{\omega_1} = \omega_2$, the construction of a countable support forcing iteration P_{ω_2} witnessing our main result is very natural. Since $2^{\omega_1} = \omega_2$, we can fix a function $\Phi : \omega_2 \to H_{\omega_2}$ with the property that $\{\beta \in \omega_2 \,:\, \Phi(\beta) = x\}$ is unbounded in ω_2 for each $x \in H_{\omega_2}$. At stage $\beta < \omega_2$, if $\Phi(\beta)$ is a $P_{\omega_2}\restriction\beta$-name for a proper poset of cardinality ω_1, then let $\dot{Q}_\beta = \Phi(\beta)$. Otherwise, let $\dot{Q}_\beta$ be a $P_{\omega_2}\restriction\beta$-name for a poset Q as above.

In what follows we will prove the existence of such a poset Q.

Definition

In an ω_1-preserving forcing extension $V[G]$, a *continuous V-reflection sequence* is a sequence $\langle \bar{M}_\alpha \colon \alpha \in C \rangle$ such that:

1. $C \subset \omega_1$ is a closed unbounded set;
2. for each $\alpha \in C$, $\bar{M}_\alpha$ is the transitive collapse of some elementary submodel of $(H^V_{\omega_2}, \in)$ such that $\alpha = \omega_1^{\bar{M}_\alpha}$;
3. (*continuity*) for every $\alpha \in C$ and every function $x \colon \alpha^{<\omega} \to \alpha$ in the model $\bar{M}_\alpha$ there is $\gamma \in \alpha$ such that for every ordinal $\delta \in C$ between γ and α, $x \restriction \delta^{<\omega} \in \bar{M}_\delta$ (which implies by (2) above that δ is closed under x);
4. (*reflection*) for every stationary set $S \subset [H^V_{\omega_2}]^\omega$ in V, the set $\{\alpha \in C \colon \bar{M}_\alpha$ is the transitive collapse of some element of $S\} \subset \omega_1$ is stationary.

Q is defined then as the poset for adding a continuous V-reflection sequence by means of countable approximations.

Definition

Q is the set of all pairs $q = \langle a_q, b_q \rangle$ where

1. a_q is a function whose domain is a closed countable subset of ω_1 called the *support of* q, $supp(q)$;
2. for every ordinal $\alpha \in supp(q)$, writing $M = a_q(\alpha)$, we have that M is the transitive collapse of a countable elementary submodel of $(H^V_{\omega_2}, \in)$ such that $\omega_1^M = \alpha$;
3. (continuity) for every $\alpha \in \mathrm{supp}\,(q)$ and every function $x\colon \alpha^{<\omega} \to \alpha$ in the model $a_q(\alpha)$, there is $\gamma \in \alpha$ such that for every ordinal $\delta \in supp(q)$ between γ and α, $x{\restriction}\delta^{<\omega} \in a_q(\delta)$;
4. b_q is a countable set of functions from $\omega_1^{<\omega}$ to ω_1.

We say that $r \leq q$ if $supp(r)$ is an end-extension of $supp(q)$, $a_q \subset a_r$, $b_q \subset b_r$, and for every ordinal $\alpha \in supp(r) \setminus supp(q)$ and every function $x \in b_q$, $x{\restriction}\alpha^{<\omega} \in a_r(\alpha)$.

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Definition

Let M be a countable elementary submodel of $H(\kappa)$, for a large enough regular cardinal κ. Let $g \subset M$ be a filter meeting all open dense subsets of Q in M.

1. Let $a = \bigcup_{s \in g} a_s \cup \{\langle M \cap \omega_1, \bar{M} \rangle\}$, where $\bar{M}$ is the transitive collapse of the model $M \cap H_{\omega_2}$;
2. let b be the set of all functions from $\omega_1^{<\omega}$ to ω_1 belonging to the model M;
3. let $r(M, g) = \langle a, b \rangle$.

Proposition

$r(M, g)$ is a condition in Q extending all conditions in g.

Proof. Write $\alpha = M \cap \omega_1$. Since g is a filter, $\bigcup_{s \in g} a_s$ is a function, and its domain c is a subset of $M \cap \omega_1$ which is closed except perhaps at its supremum. A simple density argument shows that in fact $sup(c) = \alpha$. Thus, to verify that $r(M, g)$ is a condition, it is only necessary to check the continuity of a at α.

Let $y\colon \alpha^{<\omega} \to \alpha$ be any function in the model $\bar{M}$, and let $x \in M$ be the function whose collapse is y. By a density argument, there must be a condition $s \in g$ such that $x \in b_s$. The definition of the ordering on Q then shows that the ordinal $\gamma = \max \operatorname{supp}(s)$ witnesses the continuity condition for α and x.

To verify that for every condition $s \in g$, $r(M, g) \leq s$ holds, it is enough to verify that for every ordinal $\delta \in \mathrm{dom}\,(a) \setminus \operatorname{supp}(s)$ and every $x \in b_s$, it is the case that $x \restriction \delta^{<\omega} \in a(\delta)$. For $\delta \in \alpha$ this is immediately clear from the assumption that g is a filter. If $\delta = \alpha$, then $x \in M$ since $x \in b_s$ and $s \in M$; by the elementarity of M we conclude again that $x \restriction \delta^{<\omega}$ belongs to $a(\alpha)$, since it is the transitive collapse image of the function x.

Corollary

The poset Q satisfies the following properties:

1. *proper;*
2. *countably distributive;*
3. ω_2*-p.i.c.*

Proof.

For (1), let $q \in Q$ and let M be a countable elementary submodel of $H(\kappa)$, for a large enough regular cardinal κ, such that q and Q are in M. Construct a filter $g \subset M \cap Q$ containing the condition q and meeting all dense open subsets of Q which belong to the model M. It is immediate that $r(M, g)$ is a master condition for the model M below q.

For (2), if in addition $\{D_n \colon n \in \omega\}$ is a countable collection of open dense subsets of Q and M is selected in such a way that each D_n is in M, then $r(M, g)$ is a condition below q in the intersection $\bigcap_n D_n$.

□

For (3), suppose that M, N are two isomorphic countable elementary submodels. By the Mostowski collapse lemma, the isomorphism is unique, and we denote it by $\pi \colon M \to N$. Let $q \in M \cap Q$ be an arbitrary condition and let $g \subset M \cap Q$ be a filter having q as an element and meeting all open dense subsets of Q which belong to the model M. It will be enough to show that there is a condition r extending all the elements of the set $g \cup \pi''g$. To find r, write $r(M, g) = \langle a_M, b_M \rangle$ and $r(N, \pi''g) = \langle a_N, b_N \rangle$, and observe that $a_M = a_N$ since the isomorphism π fixes $M \cap \omega_1 = N \cap \omega_1$ and because the two models M and N have the same transitive collapse. So, $r(M, g)$ and $r(N, \pi''g)$ are compatible as witnessed by the common extension $r = \langle a_M, b_M \cup b_N \rangle$ and r works as desired.

Corollary

Let $G \subset Q$ be a generic filter. In the model $V[G]$, let $F = \bigcup\{a\colon \exists r \in G\ a = a_r\}$. Then F is a continuous V-reflection sequence.

Proof.

It is clear that $\mathrm{dom}\,(F)$ is a club subset of ω_1 and that F satisfies the continuity property. Thus, it will be enough to verify the reflection property. For this, return to the ground model, let $q \in Q$ and let $S \subset [H_{\omega_2}]^\omega$ be a stationary set. Let also $\dot{E}$ be a Q-name for a club subset of ω_1. It will be enough to find a condition $r \leq q$ and an ordinal $\alpha \in \mathrm{supp}\,(r)$ such that $a_r(\alpha)$ is the transitive collapse of a model in S and $r \Vdash \check{\alpha} \in \dot{E}$.

$\square$

To this end, use the stationarity of the set S to find a countable elementary submodel M of $H(\kappa)$ for some large enough regular cardinal κ, containing both q and $\dot{E}$ such that $M \cap H_{\omega_2} \in S$. Find a filter $g \subset Q \cap M$ generic over M containing the condition q, and let $r = r(M, g)$ and $\alpha = M \cap \omega_1$. It is clear that $r \leq q$, $r \Vdash \check{\alpha} \in \dot{E}$ since r is a master condition for M, and $a_r(\alpha)$ is a model isomorphic to $M \cap H_{\omega_2} \in S$.

Proposition

Let $V[G]$ be an ω_1-preserving forcing extension in which there exists a continuous V-reflection sequence $\langle \bar{M}_\alpha : \alpha \in C \rangle$.
In V, let P be a forcing of cardinality ω_1 which is not proper.
Then $V[G] \models P$ does not preserve stationary subsets of ω_1.

Proof.

Claim

For every function $x \colon \omega_1^{<\omega} \to \omega_1$ in V, for all but countably many $\alpha \in C$, $x \restriction \alpha^{<\omega} \in \bar{M}_\alpha$ holds.

Proof.

By the reflection property, $S = \{\alpha \in C \colon x \restriction \alpha^{<\omega} \in \bar{M}_\alpha\}$ is stat.

Use the continuity prop. to find a regressive $f \colon S \to \omega_1$ s. t. for every $\alpha \in S$ and every $\delta \in C$ between $f(\alpha)$ and α, $x \restriction \delta^{<\omega} \in \bar{M}_\delta$.

By Fodor, there is $\gamma \in \omega_1$ such that $\{\alpha \in S \colon f(\alpha) = \gamma\}$ is stat.
So, if $\delta \in C$ is above γ, $x \restriction \delta^{<\omega} \in \bar{M}_\delta$. □

Now, let P be a poset of cardinality ω_1 which is not proper, which we may assume has underlying set ω_1. If P collapses ω_1, then it also collapses ω_1 in $V[G]$, and hence is not stationary set preserving. So assume that P preserves ω_1. Note that P can be coded in $(H^V_{\omega_2}, \in)$ by a function $x \colon \omega_1^{<\omega} \to \omega_1$ in V (for example, by the characteristic function of its partial ordering).

By the claim, we may assume that for every $\alpha \in C$, $P{\restriction}\alpha \in \bar{M}_\alpha$. Now, return to V and observe that since P is not proper, by a pigeonhole argument there must be $p \in P$ and a stationary set $S \subset [H_{\omega_2}]^\omega$ s. t. no model in S has a master condition below p.

Move to $V[G]$ and use the reflection prop. to conclude that $T = \{\alpha \in C \colon \bar{M}_\alpha$ is the transitive collapse of some model in $S\}$ is stationary. Let $\dot{E}$ be the P-name for the set $\{\alpha \in C \colon$ the P-generic filter has nonempty intersection with every maximal antichain of $P{\restriction}\alpha$ in the model $\bar{M}_\alpha\}$.
So, in $V[G]$, $p \Vdash_P \dot{E} \cap \check{T} = \emptyset$

Finally, to prove that T does not remain stat. after forcing with P in $V[G]$, it is enough to verify the following

Claim

$\Vdash_P \dot{E}$ is a club , where $\dot{E}$ is a P-name for $\{\alpha \in C :$ the P-generic filter has nonempty intersection with every maximal antichain of $P{\restriction}\alpha$ in the model $\bar{M}_\alpha\}$.

Proof. For the closure, suppose that $q \in P$ forces $\alpha \in C$ to be a lim. point of $\dot{E}$. To show that $q \Vdash \check{\alpha} \in \dot{E}$, let $A \in \bar{M}_\alpha$ be a max. antichain of $P{\restriction}\alpha$ in the model $\bar{M}_\alpha$ and let $r \leq q$; we must find a condition in $A \cap \alpha$ compatible with r. To do this, apply continuity to a suitable function to find $\gamma \in \alpha$ such that for every $\delta \in C$ between γ and α, $A \cap \delta$ is a max. antichain of $P{\restriction}\delta$ in $\bar{M}_\delta$.

Since r forces that α is a limit point of $\dot{E}$, find a condition $s \leq r$ and $\delta \in C$ between γ and α such that $s \Vdash \check{\delta} \in \dot{E}$. So, there is an elem. of $A \cap \delta$ compatible with s, and hence with r.

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Intermezzo prior to the unboundedness

Fix $q \in P$ and $\gamma \in \omega_1$. Back in V, consider the set $U \subset [H_{\omega_2}]^\omega$ of all models which contain γ and have a master cond. below q. Let us prove that U is stationary.

Let $f\colon H_{\omega_2}^{<\omega} \to H_{\omega_2}$. To find $M \in U$ closed under f, let θ be a large enough regular cardinal and let $X = \langle N_\alpha \colon \alpha \in \omega_1 \rangle$ be a continuous increasing tower of ctble. elem. submodels of H_θ containing f as an element, and let $N = \bigcup_\alpha N_\alpha$. Let $H \subset P$ be a generic filter containing q, and consider the models $N_\alpha[H]$ for $\alpha \in \omega_1$ and $N[H]$. Since P is a subset of N, $N[H] \cap V = N$.

The models $\langle N_\alpha[H] \colon \alpha \in \omega_1 \rangle$ form a continuous increasing seq. of ctble. subsets of $N[H]$, so $Y = \langle N_\alpha[H] \cap V \colon \alpha \in \omega_1 \rangle$ is a continuous increasing seq. of ctble. subsets of $N[H] \cap V = N$.

Since ω_1 is preserved passing to $V[H]$, the seq. X and Y must intersect at some point.

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That is, there must be $\alpha \in \omega_1$ such that $N_\alpha[H] \cap V = N_\alpha$. Fix $r \leq q$ in H such that $r \Vdash_P^V N_\alpha[\dot{H}] \cap V = N_\alpha$. Then r is a master condition for $N_\alpha \cap H_{\omega_2}$, and $N_\alpha \cap H_{\omega_2}$ is a model in the set U closed under f.

For the verification of unboundedness of $\dot{E}$, fix again $q \in P$ and $\gamma \in \omega_1$. We proved that, back in V, the set $U \subset [H_{\omega_2}]^\omega$ of all models which contain γ and have a master cond. below q is stationary.

In $V[G]$ again, use the reflection property to find an ordinal $\alpha \in C$ which is greater than γ and such that $\bar{M}_\alpha$ is the transitive collapse of some model M in U. By the definition of U, fix a master condition $r \leq q$ for M. So, r forces that $\check{\alpha} \in \dot{E}$.

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One open question

Given that the forcing notion Q (destroying posets which are stationary set preserving but not proper) is proper but it has cardinality 2^{ω_1}, it is natural to ask whether or not $\mathrm{PFA}(\omega_1)$ implies $\mathrm{MM}(\omega_1)$.

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1) Dobrinen, Natasha; Krueger, John; Marun, Pedro; Mota, Miguel Angel, Zapletal, Jindrich. *On the consistency strength of $MM(\omega_1)$*, Proc. Amer. Math. Soc. 152 (2024), 2229-2237 DOI: https://doi.org/10.1090/proc/16718.

One open question

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