Beyond the reach of forcing [1]

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[1] Joint work with Peter Koellner

Categorical models of Set Theory

Suppose $\mathcal{M} = (M, E)$ is a model of ZFC.

- The model $\mathcal{M}$ is **wellfounded** if there is no infinite sequence
$$a_1, \ldots, a_n, \ldots$$
of elements of M such that $a_{i+1} \, E \, a_i$ for all i.
- The logic for wellfounded models of ZFC is β-logic.

Definition

Suppose φ is a sentence in the language $\mathcal{L}_{\mathrm{ST}}$ of Set Theory.

1. $\mathrm{ZFC} + \varphi$ is β-**satisfiable** if there is a wellfounded model $\mathcal{M}$ such that $\mathcal{M} \models \varphi$.
2. $\mathrm{ZFC} + \varphi \models_\beta \psi$ if for all wellfounded models $\mathcal{M} \models \mathrm{ZFC}$, if $\mathcal{M} \models \varphi$ then $\mathcal{M} \models \psi$.
   - This is **logical implication** in β-logic
     - for models of ZFC.
3. $\mathrm{ZFC} + \varphi$ is β-**categorical** if for all wellfounded models $\mathcal{M}, \mathcal{N}$ of $\mathrm{ZFC} + \varphi$,
$$\mathcal{M} \cong \mathcal{N}.$$

An easy example

Theorem (Gödel)

Suppose N is a transitive set and that

$$N \models \mathrm{ZFC}.$$

Let $\alpha = \mathrm{Ord}^N = N \cap \mathrm{Ord}$. Then

1. $L_\alpha \subseteq N$ *and* $L_\alpha \models \mathrm{ZFC}$.
2. $N \models$ "$V = L$" *if and only if* $N = L_\alpha$.

Let φ_L be the sentence of $\mathcal{L}_{\mathrm{ST}}$ which expresses

- $V = L$.
- ZFC is not β-satisfiable.

Corollary

Suppose that ZFC is β-satisfiable. Then $\mathrm{ZFC} + \varphi_L$ is β-satisfiable and β-categorical.

- The unique transitive model of $\mathrm{ZFC} + \varphi_L$ is L_α where α is the least ordinal such that $L_\alpha \models \mathrm{ZFC}$.

An interesting question

Question

Suppose that $\mathrm{ZFC} + \varphi$ is β-satisfiable and β-categorical.

- Must $\mathrm{ZFC} + \varphi \models_\beta$ "$V = L$"?

Theorem (Stanley:1984)

Assume ZFC *is* β*-satisfiable. Then there is a* Π^1_2*- formula* $\psi(x_0)$ *and* $x \in \mathbb{R}$ *such that the following hold where* α *is the least ordinal such that* $L_\alpha \models \mathrm{ZFC}$.

1. $L_\alpha(x) \models \mathrm{ZFC} + \psi[x]$.
2. *Suppose* $y \in \mathbb{R}$, $x \neq y$, *and that* $L_\alpha(y) \models \mathrm{ZFC} + \psi[y]$.
   - *Then* $L_\alpha(x, y) \not\models \mathrm{ZFC}$.

- (2) implies that $\mathrm{ZFC} + \varphi \not\models_\beta$ "$V = L$", where $\varphi = (\exists x_0 \psi)$.
- Stanley's Theorem answers the question but only with a weak form of $\mathrm{ZFC} + \varphi$ is β-categorical.
  - Stanley's proof using the machinery of class forcing.

Theorem (after Vopěnka)

Suppose M is a countable transitive set such that

$$M \models \mathrm{ZFC} + \varphi.$$

Then one of the following hold.

1. $M \models$ "$V = \mathrm{HOD}$".
2. *There is an uncountable set of countable transitive sets N such that*
   - $N \models \mathrm{ZFC} + \varphi$ *and* $\mathrm{Ord}^N = \mathrm{Ord}^M$.

Theorem (after H. Friedman)

Suppose M is a countable transitive set such that

$$M \models \mathrm{ZFC} + \varphi.$$

Then one of the following hold.

1. $M \models$ "$0^\#$ does not exist".
2. *There is an uncountable set of countable transitive sets N such that*
   - $N \models \mathrm{ZFC} + \varphi$ *and* $\mathrm{Ord}^N = \mathrm{Ord}^M$.

Corollary

Suppose $\mathrm{ZFC} + \varphi$ *is* β*-categorical.*

- *Then* $\mathrm{ZFC} + \varphi \models_\beta$ "$V = \mathrm{HOD}$".

Corollary

Suppose $\mathrm{ZFC} + \varphi$ *is* β*-categorical.*

- *Then* $\mathrm{ZFC} + \varphi \models_\beta$ "$0^\#$ does not exist".

- Assume $V = L$. Then $V = \mathrm{HOD}$ and $0^\#$ does not exist.
  - These are each fundamental consequences of $V = L$.

This suggests that if $\mathrm{ZFC} + \varphi$ *is* β*-satisfiable and* β*-categorical then*

$$\mathrm{ZFC} + \varphi \models_\beta \text{"}V = L\text{"}.$$

$\mathcal{L}_{\mathrm{ST}}(c)$: Expanding the formal language $\mathcal{L}_{\mathrm{ST}}$ of Set Theory with a constant

- Suppose $\varphi \in \mathcal{L}_{\mathrm{ST}}(c)$ and M is a transitive model such that
$$M \models \mathrm{ZFC} + \varphi.$$
Then c_M is the interpretation of c.
  - So in essence M is a transitive set with a distinguished element.

Definition

Suppose $\varphi \in \mathcal{L}_{\mathrm{ST}}(c)$. Then $\mathrm{ZFC} + \varphi$ is β-**categorical** if for all transitive models N, M of $\mathrm{ZFC} + \varphi$,

- if $c_N = c_M$

then $N = M$.

- This is really just β-categorical modulo interpretation of c.

Strong hypotheses in Set Theory

Definition

A set $A \subset \mathbb{R}$ is a Σ^1_2 set if the set A can be defined in the structure

$$(V_{\omega+1}, \in)$$

by a Σ_2-formula without parameters.

Definition

Σ^1_2-Determinacy is the axiom which asserts that every Σ^1_2 set $A \subseteq \mathbb{R}$ is determined.

Theorem (Martin, Steel:1985)

Assume that there is a Woodin cardinal with a measurable cardinal above. Then Σ^1_2-Determinacy holds.

- Thus if there is a Woodin cardinal with a measurable cardinal above, then the following holds.
  - Σ^1_2-Determinacy + $x^{\#}$ exists for all $x \in \mathbb{R}$.

  This is the hypothesis which we will use.

Theorem (Σ^1_2-Determinacy + $x^\#$ exists for all $x \in \mathbb{R}$)

There exists a sentence $\varphi \in \mathcal{L}_{\mathrm{ST}}(c)$ such that the following hold.

1. $\mathrm{ZFC} + \varphi$ *is β-satisfiable and β-categorical.*
2. $\mathrm{ZFC} + \varphi \models_\beta$ "$V \neq L(c)$".
3. *For a Turing cone of $x \in \mathbb{R}$, there is transitive model*
$$M \models \mathrm{ZFC} + \varphi$$
such that $c_M = x$.

- The conclusions (1) and (2) are absolute to L.
- The conclusion (3) is not absolute to L.

Question

Is there a sentence $\varphi \in \mathcal{L}_{\mathrm{ST}}(c)$ such that (1)–(3) hold but for **all** $x \in \mathbb{R}$?

- This is downward absolute to L.

β-categorical in height

How about more models? Does this change anything?

Definition

Suppose $\varphi \in \mathcal{L}_{\mathrm{ST}}$. Then $\mathrm{ZFC} + \varphi$ is β-**categorical in height** if for all countable transitive sets N, M, if

- $M \models \mathrm{ZFC} + \varphi$ and $N \models \mathrm{ZFC} + \varphi$,
- $\mathrm{Ord}^N = \mathrm{Ord}^M$;

Then $M = N$.

Lemma

Suppose $\varphi \in \mathcal{L}_{\mathrm{ST}}$. Then the following are equivalent.

1. $\mathrm{ZFC} + \varphi$ *is β-categorical in height.*
2. $V[G] \models$ "$\mathrm{ZFC} + \varphi$ is β-categorical in height", *for every generic extension $V[G]$ of V.*

Generalizing the basic question

Lemma

Suppose $\mathrm{ZFC} + \varphi$ *is* β*-categorical in height. Then for every transitive model*

$$M \models \mathrm{ZFC} + \varphi,$$

necessarily $M \in L$.

Corollary

Suppose $\mathrm{ZFC} + \varphi$ *is* β*-categorical in height and that* M *is an uncountable transitive model of* $\mathrm{ZFC} + \varphi$.

- *Then* $M \models$ "$\mathbb{R} \subset L$".

Question

Suppose that $\mathrm{ZFC} + \varphi$ is β-categorical in height and that there is a proper class of transitive models of $\mathrm{ZFC} + \varphi$. Must there exist a transitive model M of $\mathrm{ZFC} + \varphi$ such that

$$M \models \text{“}V = L\text{”?}$$

β-categorical in height for $\mathcal{L}_{\mathrm{ST}}(c)$

Definition

Suppose $\varphi \in \mathcal{L}_{\mathrm{ST}}(c)$. Then

- $\mathrm{ZFC} + \varphi$ is **β-categorical in height**

if for all countable transitive models N, M, if

- $M \models \mathrm{ZFC} + \varphi$ and $N \models \mathrm{ZFC} + \varphi$,
- $(c_N, \mathrm{Ord}^N) = (c_M, \mathrm{Ord}^M)$;

Then $M = N$.

Lemma

Suppose $\varphi \in \mathcal{L}_{\mathrm{ST}}(c)$. Then the following are equivalent.

1. $\mathrm{ZFC} + \varphi$ *is β-categorical in height.*
2. $V[G] \models$ "$\mathrm{ZFC} + \varphi$ is β-categorical in height", *for every generic extension $V[G]$ of V.*

Theorem (Σ^1_2-Determinacy + $x^\#$ exists for all $x \in \mathbb{R}$)

There exists $\varphi \in \mathcal{L}_{\mathrm{ST}}(c)$ such that the following hold.

1. $\mathrm{ZFC} + \varphi$ *is* β*-satisfiable and* $\mathrm{ZFC} + \varphi$ *is* β*-categorical in height.*
2. $\mathrm{ZFC} + \varphi \models_\beta$ "$V \neq L(c)$".
3. *For a Turing cone of $x \in \mathbb{R}$, there is a proper class of transitive models*
$$M \models \mathrm{ZFC} + \varphi$$
such that $c_M = x$.

Question

Is there a sentence $\varphi \in \mathcal{L}_{\mathrm{ST}}(c)$ such that (1)–(3) hold but for **all** $x \in \mathbb{R}$?

- This is downward absolute to L.

What is the maximum possibility on heights?

Definition

Suppose that α is an ordinal and that $L_\alpha \models \mathrm{ZFC}$. Then α is **fragile** if α is collapsed in L_η where η is the least admissible ordinal above α.

Theorem (after H. Friedman)

Suppose M is a countable transitive set such that

$$M \models \mathrm{ZFC} + \varphi + \text{“}V \neq L\text{”}.$$

Then one of the following hold.

1. Ord^M *is fragile.*
2. *There is an uncountable set of countable transitive sets N such that $N \models \mathrm{ZFC} + \varphi$ and such that $\mathrm{Ord}^N = \mathrm{Ord}^M$.*

Corollary

Suppose $\mathrm{ZFC} + \varphi$ is β-categorical in height and that M is a transitive model of $\mathrm{ZFC} + \varphi + \text{“}V \neq L\text{”}$. Then Ord^M is fragile.

Complicated fragility

Suppose γ is a cardinal and that $L_\gamma \models \mathrm{ZFC}$.

- γ is not fragile.

Fix an ordinal η such that $\gamma << \eta < \xi$ where ξ is the least admissible ordinal above γ.

- Let X be the set of all $p \in L_\eta$ such that p is definable in
$$(L_\eta, \in).$$

Then $L_\alpha \models \mathrm{ZFC}$ and α is fragile, where α is the ordertype of $X \cap \gamma$.

- But α is **not collapsed** in $L_{\bar{\eta}}$ where $L_{\bar{\eta}}$ is the Mostowski collapse of X.

- There are very complicated examples of fragile ordinals α where one cannot determine if α is fragile by any simple inspection of even the entire first order theory of L_α with parameters.

The necessary weakening of β-categorical in height

- The existence of complicated examples of fragile ordinals α makes finding a sentence φ such that
  - $\mathrm{ZFC} + \varphi$ is categorical in height

  and has a model of height α for such α, which is not L_α, look very challenging.

But there is another more serious obstruction.

Lemma (Overspill)

Suppose $\mathrm{ZFC} + \varphi$ has a transitive model of height α for every countable ordinal α such that $L_\alpha \models \mathrm{ZFC}$ and such that α is fragile.

- *Then $\mathrm{ZFC} + \varphi$ has a model of height α for every countable ordinal α such that $L_\alpha \models \mathrm{ZFC}$.*

- Thus to find examples φ such that $\mathrm{ZFC} + \varphi$ has a unique model of every possible ordinal height, we must weaken the notion that $\mathrm{ZFC} + \varphi$ is β-categorical in height.

$\mathcal{L}_{\mathrm{ST}}(c, d)$: Expanding $\mathcal{L}_{\mathrm{ST}}$ with two constants

- Suppose $\varphi \in \mathcal{L}_{\mathrm{ST}}(c, d)$ and M is a transitive model such that
  $$M \models \mathrm{ZFC} + \varphi.$$
  Then c_M is the interpretation of c and d_M is the interpretation of d.

Definition

Suppose $M \models \mathrm{ZFC} + \varphi$ and that M is transitive. Then:

- The model M is **categorical in height for** $\mathrm{ZFC} + \varphi$,

if for all transitive models $N \models \mathrm{ZFC} + \varphi$, if

- $(c_M, d_M, \mathrm{Ord}^M) = (c_N, d_N, \mathrm{Ord}^N)$

then $N = M$ (and if M is uncountable, then this must hold after collapsing M to be countable).

- $\mathrm{ZFC} + \varphi$ is β-categorical in height if and only if **every** transitive model of $\mathrm{ZFC} + \varphi$ is categorical in height for $\mathrm{ZFC} + \varphi$.

Relativizing fragility

Definition

Suppose $x \in \mathbb{R}$, α is an ordinal, and that $L_\alpha(x) \models \mathrm{ZFC}$.

- Then α is x-**fragile** if α is collapsed in $L_\eta(x)$ where $\eta > \alpha$ is the least ordinal which is x-admissible.

Theorem

Suppose $\varphi \in \mathcal{L}_{\mathrm{ST}}(c, d)$, $x \in \mathbb{R}$, and that M is a transitive model of $\mathrm{ZFC} + \varphi$ such that the following hold.

- *M is categorical in height for $\mathrm{ZFC} + \varphi$.*
- *$c_M = x$ and $d_M < \mathrm{Ord}^M$.*
- *$M \models$ "$V \neq L(c)$".*

Then Ord^M is x-fragile and $M \in L[x]$.

The main theorem

Theorem (Σ^1_2-Determinacy + $x^\#$ exists for all $x \in \mathbb{R}$)

There exists $\varphi \in \mathcal{L}_{\mathrm{ST}}(c, d)$ such that the following hold.

1. $\mathrm{ZFC} + \varphi$ *is β-satisfiable.*
2. $\mathrm{ZFC} + \varphi \models_\beta$ "$V \neq L(c)$".
3. *For a Turing cone of $x \in \mathbb{R}$, for all ordinals α, if*
   - $L_\alpha(x) \models \mathrm{ZFC}$ *and α is x-fragile,*

   then there is a transitive model $M \models \mathrm{ZFC} + \varphi$ of height α such that $c_M = x$, $d_M < \mathrm{Ord}^M$, and such that
   - *M is categorical in height for $\mathrm{ZFC} + \varphi$.*

- The conclusions (1) and (2) are absolute to L.
- Conclusion (3) is not absolute to L.

(Technical aside)

For a Turing cone of x: $\mathrm{ZFC} + \varphi$ actually has a model of height α for **all** α such that $L_\alpha(x) \models \mathrm{ZFC}$. Does this follow from (1)–(3)?

- The proof of the main theorem uses methods from the Inner Model Program.

Mitchell-Steel inner models and $\mathbb{M}_1^{\#}$

Theorem (Scott:1961)

Assume $V = L$. Then there are no measurable cardinals.

- The Inner Model Program seeks to construct enlargements of L in which large cardinals can exist.
  - These enlargements are **core models**.
  - The stronger the large cardinal notion the harder the problem.

The solution at the level of one Woodin cardinal is given by the Mitchell-Steel core models for exactly one Woodin cardinal.

- These inner models are of the form $L(E)$ where $E \subset \mathrm{Ord}$.
  - But unlike the inner model L, these inner models are **not** unique.
- The main theorem of Mitchell and Steel, et al, is that at the level of exactly one Woodin cardinal:
  - If $L(E)$ and $L(F)$ are *iterable* Michell-Steel models for the existence of one Woodin cardinal then

$$L(E)^{\#} \equiv L(F)^{\#}$$

- This defines $\mathbb{M}_1^{\#}$ which is a real number (just like $0^{\#}$).

A convergence of strong hypotheses

Theorem

The following are equivalent.

1. Σ^1_2-Determinacy + $Z^\#$ *exists for all* $Z \subset \mathrm{Ord}$.
2. *For all* $Z \subset \mathrm{Ord}$, $Z^\#$ *exists, and there is an inner model* N *such that*
    - $\mathrm{Ord} \subset N$
    - $N \models \mathrm{ZFC} +$ "There is a Woodin cardinal".
3. *There is an iterable inner model* N *such that*
    - $\mathrm{Ord} \subset N$
    - $N \models \mathrm{ZFC} +$ "There is a Woodin cardinal".
4. $\mathbb{M}_1^\#$ *exists.*

To simplify things, we focus on the hypothesis:

- Σ^1_2-Determinacy + $Z^\#$ exists for all $Z \subset \mathrm{Ord}$.

$\mathbb{M}_1$-like models

Theorem (Jensen,Steel:2004)

Suppose that for all sets $E \subset \mathrm{Ord}$,

$$L(E) \models \text{“There is no Woodin cardinal”}$$

Then there is a maximal approximation $\mathbb{K}$ to $\mathbb{M}_1$ and $\mathbb{K}$ is iterable.

- Assume $\mathbb{M}_1^{\#}$ exists. Let δ be the Woodin cardinal of $\mathbb{M}_1$.
  - For all sets $E \in \mathbb{M}_1$, if $E \subset \delta$ and $\sup(E) < \delta$ then
    - $\mathbb{M}_1 \cap V_\delta \models \mathrm{ZFC} +$ “There are no Woodin cardinals in $L(E)$”.
  - (Jensen,Steel) $\mathbb{M}_1 \cap V_\delta \models$ “$V = \mathbb{K}$”.

Definition

Suppose that M is a transitive set, $\delta \in M$, and

- $M \models \mathrm{ZFC} +$ “δ is a Woodin cardinal”.

Then M is an $\mathbb{M}_1$-**like** model if

- $M \models$ “$V = L(V_\delta)$”
- $M \cap V_\delta \models$ “$V = \mathbb{K}$”.

Iterable $\mathbb{M}_1$-like models and genericity

Theorem (Genericity Theorem)

Suppose that M is an iterable $\mathbb{M}_1$-like model and that δ is the Woodin cardinal of M. Then there is an iteration tree T_M on M of length δ with limit model N_T such that the following hold.

1. T_M *is definable in* M.
2. N_T *is an* $\mathbb{M}_1$*-like model.*
3. $N_T \subset M$ *and* M *is a generic extension of* N_T.
4. δ *is the Woodin cardinal of* N_T.

- The formula which defines T_M is independent of M.

Since M is iterable, T_M has a cofinal wellfounded branch b which yields an elementary embedding

$$j_b : M \to N_T$$

The branch b cannot be in M since M is a generic extension of N_T.

The extender algebra and genericity iterations

Theorem

Suppose that M is an $\mathbb{M}_1$-like model, δ is the Woodin cardinal of M, and that M is iterable. Then there is a Boolean $\mathbb{B}_M$ such that the following hold.

1. $M \models$ "$\mathbb{B}_M$ is a complete Boolean algebra".
2. $M \models$ "$\mathbb{B}_M$ satisfies the δ chain condition".
3. $|\mathbb{B}_M|^M = \delta$.
4. *For each $Z \subset \mathrm{Ord}$, there is an iteration embedding*

   $$j \colon M \to M_Z$$

   such that Z is M_Z-generic for $j(\mathbb{B}_M)$.
   - $M_Z \in L(Z, M)$.
     - *But $Z \in M$ does not in general imply $M_Z \subseteq M$.*

- $\mathbb{B}_M$ is the *extender algebra* of M.

The view within $L_\alpha(x)$ and the key formula $\Psi_{\mathbb{M}_1}$

Theorem (Σ^1_2-Determinacy + $Z^\#$ exists for all $Z \subset \mathrm{Ord}$)

Suppose $x \in \mathbb{R}$, $\mathbb{M}_1^\#$ is recursive in x, and that $L_\alpha(x) \models \mathrm{ZFC}$.

- *Then for every uncountable limit cardinal γ of $L_\alpha(x)$ there exists an $\mathbb{M}_1$-like model M_γ such that the following hold.*
  1. *$M_\gamma \subset L_\alpha(x)$ and $\alpha = \mathrm{Ord}^{M_\gamma}$.*
  2. *The Woodin cardinal of M_γ is $(\gamma^+)^{L_\alpha(x)}$.*
  3. *Every $y \in L_\alpha(x) \cap \mathbb{R}$ is M_γ-generic for $\mathbb{B}_{M_\gamma}$.*
  4. *M_γ is uniformly definable in $L_\alpha(x)$ from γ by the formula $\Psi_{\mathbb{M}_1}(x_0, x_1)$.*
     - *More precisely for all x, α, γ, and for all $N \in L_\alpha[x]$,*
       $$L_\alpha[x] \models \Psi_{\mathbb{M}_1}[\gamma, N]$$
       if and only if $N = M_\gamma$.
  5. *M_γ is iterable.*

- The conditions (1)–(4) are first order conditions in $L_\alpha(x)$.
- The condition (5) is **not** a first order condition in $L_\alpha(x)$.

How the formula $\Psi_{\mathbb{M}_1}$ works

- Fix $x \in \mathbb{R}$ such that $\mathbb{M}_1^\#$ is recursive in x.
- Fix an ordinal α such that $L_\alpha(x) \models \mathrm{ZFC}$.
- Fix an uncountable limit cardinal γ of $L_\alpha(x)$.

Let $\mathcal{M}$ be the set of all $\mathbb{M}_1$-like models M such that

- $M \subset L_\alpha(x)$ and $\mathrm{Ord}^M = \alpha$.
- $\delta < \gamma$ where δ is the Woodin cardinal of M.

(Key points)

1. $|\mathcal{M}| = \gamma$ in $L_\alpha(x)$.
2. There are iterable $\mathbb{M}_1$-like models in $\mathcal{M}$.
   - since $\mathbb{M}_1^\#$ is recursive in x.

The model M_γ is obtained in two steps of simultaneous iterations.

Step I: Jointly compare all the models in $\mathcal{M}$.

- This produces an $\mathbb{M}_1$-like model $\hat{M}$ which must be iterable.
  - The Woodin cardinal of $\hat{M}$ must be above γ (it is γ^+).

Step II: Iterate $\hat{M}$ to make y generic for the extender algebra, for every $y \in \mathbb{R} \cap L_\alpha[x]$. This produces M_γ.

The first technical theorem (about the formula $\Psi_{\mathbb{M}_1}$)

Theorem (Σ^1_2-Determinacy + $Z^\#$ exists for all $Z \subset \mathrm{Ord}$)

Suppose $x \in \mathbb{R}$ and that $\mathbb{M}_1^\#$ is recursive in x. Suppose α is an ordinal and that

$$L_\alpha(x) \models \mathrm{ZFC}.$$

Then all sufficiently large $\gamma < \alpha$, there is a cofinal wellfounded branch b of T_{M_γ} such that the following hold where δ is the Woodin cardinal of M_γ and where M_γ is as defined in $L_\alpha(x)$.

1. *δ is a Woodin cardinal in $M_\gamma[b]$,*
2. *For all $\eta < \delta$, $\mathcal{P}(\eta) \cap M_\gamma = \mathcal{P}(\eta) \cap M_\gamma[b]$.*

- Key point: By (1) and (2), it follows that $\mathbb{B}_{M_\gamma}$ is a complete Boolean algebra in $M_\gamma[b]$.
  - Therefore x is $M_\gamma[b]$-generic for $\mathbb{B}_{M_\gamma}$, since x is M_γ-generic for $\mathbb{B}_{M_\gamma}$, and so
    - $L_\alpha(x, b) \models \mathrm{ZFC}$.

The sentence φ

(What the sentence φ asserts)

- $c \in \mathbb{R}$ and in $L(c)$, M_γ exists for all uncountable cardinals γ.
  - More precisely, the definition $\Psi_{\mathbb{M}_1}$ works within $L(c)$ at all γ
    - to achieve (1)–(4).
- d is an uncountable limit cardinal of $L(c)$.
- There is a cofinal wellfounded branch b of T_{M_γ} where $\gamma = d$ and M_γ is as defined in $L(x)$ where $x = c$, such that
  - $V = L(c, b)$

  and such that the following hold where δ is the Woodin cardinal of M_γ.

  1. δ is a Woodin cardinal in $M_\gamma[b]$,
  2. For all $\eta < \delta$, $\mathcal{P}(\eta) \cap M_\gamma = \mathcal{P}(\eta) \cap M_\gamma[b]$

- If b_1 and b_2 are distinct wellfounded branches of T_{M_γ} then
  - $L_\alpha(x, b_1, b_2) \not\models \mathrm{ZFC}$.

  But this does not imply $\mathrm{ZFC} + \varphi$ has at most one model of height α. (Just as in Stanley's theorem).

The second technical theorem (about the formula $\Psi_{\mathbb{M}_1}$)

Theorem (Σ^1_2-Determinacy + $Z^\#$ exists for all $Z \subset \mathrm{Ord}$)

Suppose $x \in \mathbb{R}$ and that $\mathbb{M}_1^\#$ is recursive in x. Then the following are equivalent for all ordinals α such that

$$L_\alpha(x) \models \mathrm{ZFC}$$

and such that α is countable.

1. *α is x-fragile.*
2. *For all sufficiently large $\gamma < \alpha$, there is at most one cofinal wellfounded branch of T_{M_γ} where M_γ is as defined in $L_\alpha(x)$.*

- (2) is best possible. More precisely, there exist x, α, and γ such that the following hold.
  - $\mathbb{M}_1^\#$ is recursive in x.
  - α is x-fragile and γ is an uncountable limit cardinal of $L_\alpha(x)$.
  - There are uncountably many cofinal wellfounded branches of T_{M_γ} where M_γ is as defined in $L_\alpha(x)$.
- This is why for the main theorem we need the second constant d.

Open questions

1. Is there a sentence φ of $\mathcal{L}_{\mathrm{ST}}(c)$ for which the main theorem holds? (i.e.; Can one eliminate the second constant d?)
2. Is there a sentence φ of $\mathcal{L}_{\mathrm{ST}}(c, d)$ for which the main theorem holds but for all $x \in \mathbb{R}$?
   - Downward absolute to L.
3. Is there a sentence φ of $\mathcal{L}_{\mathrm{ST}}(c)$ for which the main theorem holds but for all $x \in \mathbb{R}$?
   - Downward absolute to L.

Suppose that $\varphi \in \mathcal{L}_{\mathrm{ST}}(c)$, $\mathrm{ZFC} + \varphi$ is β-satisfiable, and that $\mathrm{ZFC} + \varphi$ is β-categorical.

- Must $\mathrm{ZFC} + \varphi \models_\beta$ "If $c \in \mathbb{R}$ then CH holds"?

Back to the original question

Question

Suppose that $\mathrm{ZFC} + \varphi$ is β-satisfiable and β-categorical.

- Must $\mathrm{ZFC} \models_\beta V = L$?

- Fixing $x \in \mathbb{R}$ and adding a constant for x, the answer is no, for a Turing cone of x.

Thus if the answer is yes, then the proof **cannot** relativize to a real.

- There is no known nontrivial example of this.

On the other hand, if the answer is no, then a completely new method for building transitive models of ZFC is needed.

- A technique which is:
  - Beyond forcing.
  - Beyond methods from the Inner Model Program.