Separating Subversion Forcing Principles

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Introduction: What's the plan?

This talk will contain joint work with Gunter Fuchs and Hiroshi Sakai.

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1. Introduce the classes and their forcing axioms.
2. Explain the variations we introduce and their iteration theorems (joint with Fuchs).
3. Discuss new results, particularly how to separate these axioms as well as their connection to old friends like square sequences and reflection principles (joint with Sakai).

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The point is that in this case q forces the embedding $\sigma : \bar{N} \prec N$ to lift to some $\tilde{\sigma} : \bar{N}[\overline{G}] \prec N[G]$.

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A forcing notion $\mathbb{P}$ is *complete* if and only if for all sufficiently large θ there is a cardinal $\tau > \theta$ so that for some $A \subseteq \tau$ we have $H_\theta \subseteq N := L_\tau[A] \models \mathrm{ZFC}^-$ and any $\sigma : \bar{N} \prec N$ where $\bar{N}$ is countable and transitive, and $\mathbb{P} = \sigma(\overline{\mathbb{P}})$ and every $\overline{G} \subseteq \overline{\mathbb{P}}$ generic over $\overline{N}$ there is a q so that if $q \in G$ is $\mathbb{P}$-generic over V then $\sigma``\overline{G} \subseteq G$.

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Again in this case q forces the embedding $\sigma : \bar{N} \prec N$ to lift to some $\tilde{\sigma} : \bar{N}[\overline{G}] \prec N[G]$. Also note that obviously σ-closed implies proper.

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3. *If $\mathbb{P}$ is σ-closed then there are no new countable sets of ordinals.*

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Proof Sketch.

If $S \subseteq \omega_1$ is stationary and $\dot{C}$ is a $\mathbb{P}$-name for a club then find a $\sigma : \overline{N} \prec N$ as in the statement of properness with $\delta = \omega_1^{\overline{N}} \in S$ and $\sigma(\dot{\overline{C}}) = \dot{C}$.

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The two statements are nearly the same so we sketch them both simultaneously. Suppose $\dot{X}$ is a new countable set of ordinals and $\sigma : \overline{N} \prec N$ with $\sigma(\dot{\overline{X}}) = \dot{X}$ as in the definition of properness.

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$\tilde{\sigma}(\overline{X}) = X := \dot{X}^G \subseteq \mathrm{On} \cap \mathrm{range}(\tilde{\sigma}) = \mathrm{On} \cap \mathrm{range}(\sigma) \in V$.

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The generalization implied by the first bullet point above is Subproper forcing (almost).

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Fullness ensures that $\bar{N}$ is not pointwise definable. This is a techincal criterion that won't matter much in this talk except that you will need to take on faith that there enough full models to carry out the arguments discussed here.

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3. *The Hulls Condition*: $\mathrm{Hull}^N(\delta \cup \mathrm{range}(\sigma)) = \mathrm{Hull}^N(\delta \cup \mathrm{range}(\sigma'))$

where $\delta = \delta(\mathbb{P})$ is the weight of $\mathbb{P}$: the least size of a dense subset of $\mathbb{P}$.

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The hulls condition was only used by Jensen in the iteration theorem and later Fuchs and I found one which avoids it. We note that it is unknown whether, up to forcing equivalence, these two definitions are the same. In any case we won't worry so much about it in this talk.

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We can similarly find a "subversion" of σ-closed forcing.

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We can similarly find a "subversion" of σ-closed forcing.

Definition (Jensen, (Fuchs- S.))

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As before, for both subproper and subcomplete (their ∞-versions) the point is that σ' lifts to some $\tilde{\sigma}' : \overline{N}[\overline{G}] \prec N[G]$. Thus, while σ itself might not lift, in the generic extension there is some embedding which does and agrees with σ on any desired finite set.

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- ∞-subcomplete forcings may add new countable sets of ordinals but not new countable subsets of ω_1 (so no new reals). This is because if $\delta = \omega_1^{\overline{N}}$ then it's easy to show we have $\sigma' \upharpoonright \delta = \sigma \upharpoonright \delta$ and the old argument kicks in. We will come back to this momentarily.

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Definition

If Γ is the class of (∞-)subproper forcing notions we denote $\mathsf{FA}(\Gamma)$ by (∞-)SubPFA. Similarly if Γ is the class of (∞-)subcomplete forcing notions we denote $\mathsf{FA}(\Gamma)$ by (∞-)SCFA.

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Using the standard Baumgartner argument Jensen showed.

Theorem

If there is a supercompact cardinal then ∞-SubPFA *and* ∞-SCFA *are consistent. Moreover* ∞-SCFA *is consistent with* $\diamondsuit$.

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Theorem (Fuchs and S.)

If there is a supercompact cardinal, then it's consistent that SCFA *holds alongside* $2^{\aleph_0} = \aleph_2$ *plus any of the following:*

1. *There are Souslin trees.*
2. $\mathfrak{d} = \aleph_1 < cov(\mathcal{N}) = \aleph_2$
3. $\mathrm{MA}_{\aleph_1}(\sigma - linked)$ *holds but* $\mathrm{MA}_{\aleph_1}$ *fails*
4. $cof(\mathcal{N}) = \aleph_1$

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The statement $\square_\lambda$ is the assertion that there is a $\square_\lambda$ sequence.

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Assume SCFA $+ \neg$CH *holds. Then the standard forcing to add a* $\Box_{\aleph_1}$ *sequence preserves* SCFA.

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I want to sketch a proof of this result, but first I want to put it in a broader context. It turns out that there is something interesting going on at the continuum (isn't there always). Below the continuum the "sub" forcing notions look like their non "sub" counterparts. Above, life gets more interesting.

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• This is because for all reals $x \in \overline{N}$ it must be the case that $\sigma(x) = \sigma'(x) = x$ (and being a real is absolute between $\overline{N}$ and V) and moreover, since $N = L_\tau[A]$ there is a definable well order of the universe, and in particular there is a definable bijection of the reals onto κ. We can then apply elementarity to get the claim above.

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• For cardinals μ, denote by ∞-SCFA $\restriction \mu$ the forcing axiom for ∞-subcomplete forcing notions above μ and ∞-SubPFA $\restriction \mu$ the same for subproper.

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\begin{array}{ccccccc}
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\end{array}
$$

The main result I want to sketch in the remaining time is that essentially no arrows are missing from Figure 1 above.

Separating Subversion Forcing Principles

Theorem (Sakai-S.)

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(Except for the trivial $\forall\kappa\neg\square_\kappa \to \forall\kappa \geq 2^{\aleph_0}\neg\square_\kappa$ which did not fit aesthetically into the picture.)

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Assume ∞-SCFA $\restriction \omega_2$ *and let* $\mathbb{P}_0$ *be the standard forcing notion to add a* $\square_{\omega_1}$*-sequence. Then* $\Vdash_{\mathbb{P}_0} \infty\text{-SCFA} \restriction \omega_2$*. In particular* ∞-SCFA $\restriction \aleph_2$ *does not imply* ∞-SCFA $\restriction \aleph_1$*.*

Note this subsumes the previously stated proof since, under $\neg$CH ∞-SCFA $\restriction \omega_2$ is equivalent to ∞-SCFA.

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Proof Sketch.

Concretely $\mathbb{P}_0$ is the forcing notion defined as follows. Conditions $p \in \mathbb{P}_0$ are functions so that the domain of p is $\beta + 1 \cap \mathrm{Lim}$ for some $\beta \in \omega_2 \cap \mathrm{Lim}$ and

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The order is end extension. We remark that a moment's reflection confirms that this poset is σ-closed. □

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Proof Sketch Continued.

By a general forcing axiom preservation theorem of Sean Cox, it suffices to show that if $\dot{\mathbb{Q}}$ is a $\mathbb{P}_0$-name for a ∞-subcomplete forcing above ω_2 and $\dot{\mathbb{T}}_{\dot{G}}$ is the $\mathbb{P}_0$-name for the forcing to thread the generic square sequence with conditions of size $< \aleph_1$ then the three step $\mathbb{P}_0 * \dot{\mathbb{Q}} * \dot{\mathbb{T}}_{\dot{G}}$ is ∞-subcomplete above ω_2.

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Exploiting similar ideas proves the other non-implications in the figure a few slides ago.

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• There remain many open questions regarding these classes but the most pressing and interesting is the following. Does ∞-SCFA imply the continuum is at most $\aleph_2$?

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Thank You!

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