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Theorem

(Cummings-Hayut-Magidor-Neeman-S.-Unger, the pandemic years) From large cardinals, we can force the tree property simultaneously at every regular cardinal in the interval $[\aleph_2, \aleph_{\omega^2+2}]$.

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Definition

The **tree property** at κ states that every tree of height κ and levels of size less than κ has an unbounded branch.

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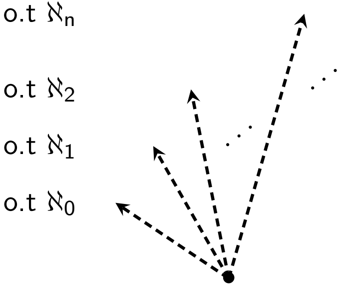

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Here: do it at $[\aleph_2, \aleph_{\omega^2+2}]$ with $\aleph_{\omega^2}$ strong limit.

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Question (Woodin): Can we get the tree property at $\aleph_{\omega+1}$ with not SCH at $\aleph_\omega$?

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By Specker, if κ is a singular strong limit, the failure of SCH at κ is necessary for the tree property at κ^{++}.

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- (S.-Unger, 2018) Can force the tree property at $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$ simultaneously.

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To get the tree property everywhere (or at least at long intervals),

- need many failures of GCH and SCH;
- need large cardinals hypothesis.
- need Prikry type forcing.

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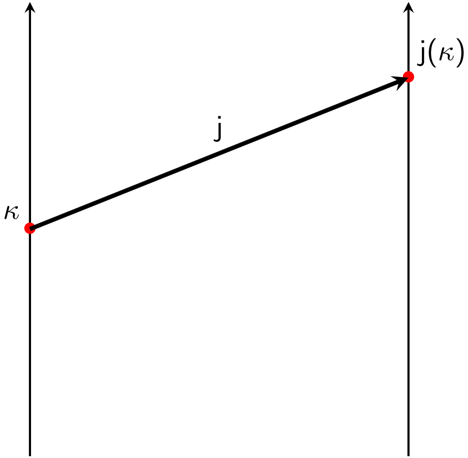

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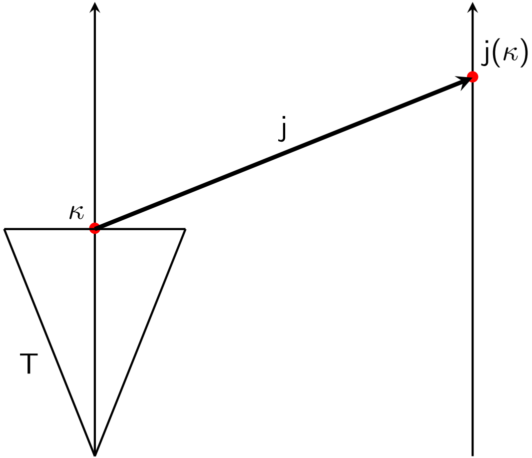

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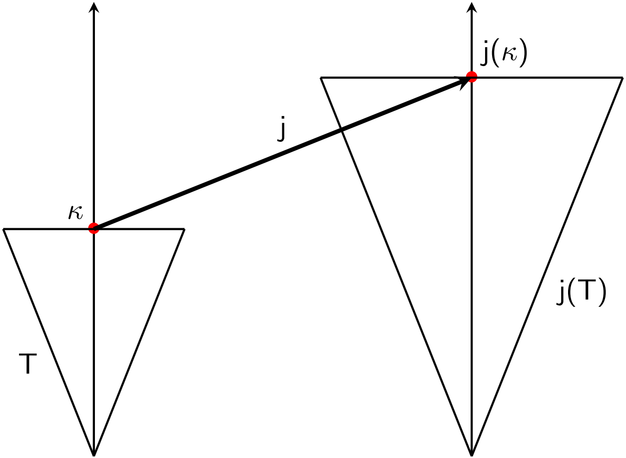

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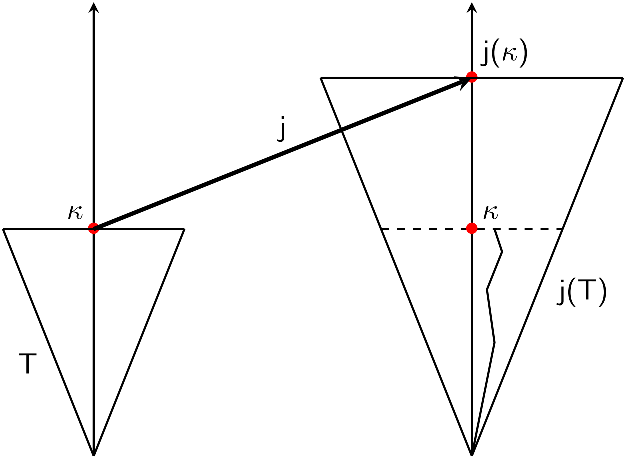

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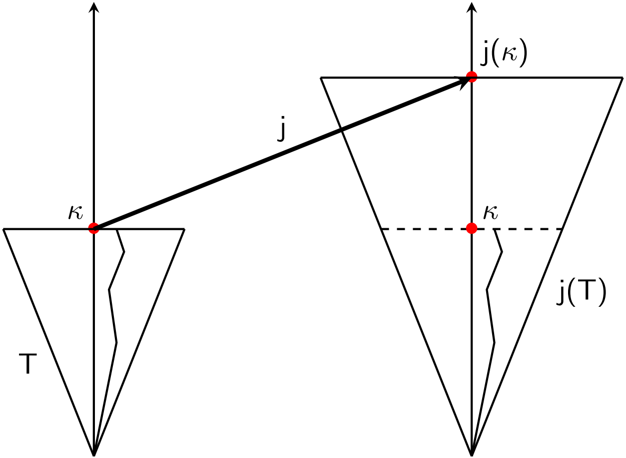

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Prove the tree property still holds at λ in the generic extension.

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General strategy for proving the tree property: lift an elementary embedding; find a branch in the outer model; pull back the branch.

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(Unger) Suppose that there is $\tau < \kappa$ with $2^{\tau} > \kappa$; W is a κ-c.c. extension of V, $\mathbb{P} \in$ V is κ-closed. Then if T $\in$ W is a tree of height κ^+ and levels of size $\leq \kappa$, $\mathbb{P}$ does not add new branches through T.

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(Neeman) From ω-many supercompact cardinals, can force the tree property at each $\aleph_n$, $n \geq 2$ and and $\aleph_{\omega+1}$ simultaneously. Idea: Use a theorem like the above; but need more flexibility in the definition of $\mathbb{C}$ and $\mathbb{L}(\rho)$ to add Cohen subsets of ρ^+ and add reals.

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Lemma

(roughly). Suppose a formerly closed forcing adds a system of branches through a system. Then there is already a branch in V.

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both in the preparation and interleaved with the Prikry.
Start with $\kappa < \lambda^{\mathsf{a}} < \lambda^{\mathsf{b}}$ supercompacts; $\lambda^{\mathsf{a}}_{\omega+3} < \lambda^{\mathsf{b}}$.
Key points:

- prepare the ground model, to arrange the tree property at $[\lambda^{\mathsf{a}}_{\omega+2}, \lambda^{\mathsf{b}}_{\omega})$, *the B-block*;
- use normal measures on $\mathcal{P}_\kappa(\lambda^{\mathsf{b}}_{\mathsf{n}})$, $\mathsf{n} < \omega$ for the diagonal Prikry;
- make $\kappa = \aleph_{\omega^2}$, $\lambda^{\mathsf{b}}_{\omega+1} = \kappa^+$, $\lambda^{\mathsf{b}}_{\omega+2} = \kappa^{++}$, $2^\kappa = \lambda^{\mathsf{b}}_{\omega+3} = \kappa^{+3}$;

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- use the AUS construction from the preparation to get the tree property below κ for the B-block.

Successors of regulars below $\aleph_{\omega^2}$

For "the B-block": Use the AUS construction from the preparation.

Successors of regulars below $\aleph_{\omega^2}$

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For "the A-block": Use the AUS construction from the interleaved components in the Prikry forcing.

Recall, using $\dagger$ we fixed $\rho < \theta$, so that the tree property at $\lambda_{\omega+1}$ holds $V[\mathbb{C}_\lambda][\mathbb{L}_\lambda(\rho)]$ for all λ; θ is the future $\aleph_2$, ρ^+ the future $\aleph_1$.

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Let τ be a Prikry point below κ:

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- For the tree property at $\lambda^{\mathsf{a}}_{\omega+1}$ use † for $\lambda = \lambda^{\mathsf{a}}_0$;
- For the tree property at $\lambda^{\mathsf{b}}_{\omega+1}$ use † for $\lambda = \lambda^{\mathsf{b}}_0$;

Recall, using $\dagger$ we fixed $\rho < \theta$, so that the tree property at $\lambda_{\omega+1}$ holds $V[\mathbb{C}_\lambda][\mathbb{L}_\lambda(\rho)]$ for all λ; θ is the future $\aleph_2$, ρ^+ the future $\aleph_1$. Let τ be a Prikry point below κ:

- For the tree property at $\lambda^{a}_{\omega+1}$ use $\dagger$ for $\lambda = \lambda^{a}_{0}$;
- For the tree property at $\lambda^{b}_{\omega+1}$ use $\dagger$ for $\lambda = \lambda^{b}_{0}$;

In both cases we show that $\mathbb{C}_\lambda$ projects to the relevant part of our posets in a branch preserving quotient.

Further Directions

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3. Test question: can we get the tree property at $\aleph_{\omega_1+1}$ with the failure of SCH at $\aleph_{\omega_1}$?

THANK YOU