The Gluing Property

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Joint work with Y. Hayut

Question

Let $\mathfrak{M}$ be a mathematical structure of certain kind (e.g., a graph, an Abelian group, etc). Assume that every *small* substructure $\mathfrak{N} \subseteq \mathfrak{M}$ satisfies a property φ.

Is it true that $\mathfrak{M}$ itself satisfies φ?

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This phenomenon is called **compactness**.

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Item (1) yields a counter-example for the "$\aleph_1$-analogue" of Gődel's Compactness Theorem.

$\mathcal{L}_{\kappa,\kappa} \equiv$ Extension of $\mathcal{L}_{\omega,\omega}$ allowing conjunctions and quantifications over lists of size $<\kappa$.

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Theorem (Keisler & Tarski)

The following are equivalent for a cardinal $\kappa > \aleph_0$:

1. κ is strongly compact.
2. $\mathcal{L}_{\kappa,\kappa}$ is compact.
3. For every $\lambda \geq \kappa$ there is an elementary embedding $j\colon V \to M$ such that $\mathrm{crit}(j) = \kappa$, ${}^{\kappa}M \subseteq M$, $j(\kappa) > \lambda$, and there is $s \in M$ such that $j``\lambda \subseteq s$ and $M \models$ "$|s| < j(\kappa)$".

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Question (Mitchell, 1978)

Suppose that κ is a κ-compact cardinal. Is it κ-compact in the inner model $L[\mathcal{U}]$?

Gitik's answer

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If κ is κ-compact then there is an inner model with a Woodin cardinal.

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Theorem (Gitik, 2016)

Suppose that κ is κ-compact. Then, there is an Extender-Based Prikry forcing that is universal for the κ-distributive forcings of size κ.

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Suppose otherwise. Then, the corresponding core model exists and the following is a set:

$$\mathcal{I} := \{i(\kappa) \mid i\colon V \to N,\ {}^{\omega_1}N \subseteq N,\ N \text{ transitive}\}.$$

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$$\langle U_{a_\alpha} \mid \alpha < \beta\rangle \Rightarrow \mathcal{F}_\alpha := \{X \mid \exists \alpha < \beta\, \exists b \subseteq a_\alpha \cap a_\beta\, \exists Y \in U_b\; X = \pi^{-1}_{a_\beta, b}(Y)\} \Rightarrow U_{a_\beta}.$$

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Subtle point: $\mathcal{F}_\alpha$ might not be κ-complete unless the U_{a_α}'s are picked in a coherent way.

The Gluing property

Inspired by Gitik's argument we isolated the following compactness principle:

Definition (Hayut, P., 2021)

Let κ be a measurable cardinal. We say that κ has the λ-**gluing property** if for every sequence of κ-complete ultrafilters on κ, $\langle U_\gamma \mid \gamma < \lambda\rangle$, there is an elementary embedding $j\colon V \to M$, with ${}^\kappa M \subseteq M$, $\operatorname{crit} j = \kappa$ and an increasing sequence of ordinals $\langle \eta_\gamma \mid \gamma < \lambda\rangle$ such that $U_\gamma = \{X \subseteq \kappa \mid \eta_\gamma \in j(X)\}$.

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- The λ-gluing property is essentially saying that the κ-complete filter

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  can be extended to a κ-complete ultrafilter.
- Gitik's argument shows that if κ has the λ-gluing property for every cardinal λ then there is an inner model with a strong cardinal.

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If κ has the λ-filter extension property then it has the (2^λ)-gluing property. In particular, every κ-compact cardinal has the (2^κ)-gluing property.

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Theorem (Hayut, P., 2021)

It is consistent to have a κ-compact cardinal without the $(2^\kappa)^+$-gluing-property.

Cardinals with the ω-Gluing Property

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2. Any level of Π^1_1-subcompactness.
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3. **It has low consistency strength**.

Theorem (Hayut, P., 2022)

Suppose that κ has the ω-gluing property and that there is no inner model for "$\exists\alpha\,(o(\alpha) = \alpha)$". Then $o^{\mathcal{K}}(\kappa) \geq \omega_1$.

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Suppose that κ has the ω-gluing property and that there is no inner model for "$\exists\alpha\,(o(\alpha)=\alpha)$". Then $o^{\mathcal{K}}(\kappa)\geq\omega_1$.

Proof idea for the lower bound

Define a sequence $\langle\mathcal{V}_\alpha\mid\alpha<\omega_1\rangle$ of $\mathcal{K}$-normal and κ-complete measures using the gluing property. The very nature of the gluing property will make the sequence $\langle\mathcal{V}_\alpha\cap\mathcal{K}\mid\alpha<\omega_1\rangle$ to be $\lhd$-increasing. By maximality of $\mathcal{K}$, these latter measures belong to $\mathcal{K}$, which yields $o^{\mathcal{K}}(\kappa)\geq\omega_1$.

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Theorem (Hayut, P., 2022) ($V=\mathcal{K}$)

Suppose that κ is a measurable cardinal with $o(\kappa)=\omega_1$ and that there are no other measurables λ with $o(\lambda)\geq\omega_1$. Then, there is a cardinal-preserving generic extension where κ has the ω-gluing property.

First attempt: ω-gluing property from a strong cardinal

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$$\mathbb{1} \Vdash_{\mathbb{P}_\alpha} \text{“}\ell(\alpha) \text{ is an } \omega\text{-sequence of } \alpha\text{-complete measures on } \alpha.\text{”}$$

in which case $\dot{\mathbb{Q}}_\alpha$ is forced to be the Tree Prikry forcing with respect to the sequence $\ell(\alpha)$.

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In $V[G]$ define the following κ-complete measure on κ^ω:

$$X \in W \;\Leftrightarrow\; X \subseteq \kappa^\omega \wedge \exists p \in G\, \exists \dot{T} \in V^{\mathbb{P}_\kappa}\, (p \cup \{\langle \varnothing, \dot{T}\rangle\} \cup r \Vdash_{\mathbb{P}_\kappa} \dot{b}_\kappa \in j(\dot{X})).$$

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Show that $U_n \leq_{\mathrm{RK}} W$ as witnessed by the evaluation map $e_n\colon \vec{\eta} \mapsto \vec{\eta}(n)$. Once this is done, j_W and $[\mathrm{id}]_W$ witness that $\langle U_n \mid n < \omega\rangle$ can be glued.

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Let $j\colon V \to M$ be a $(\kappa+2)$-strong embedding such that $j(\ell)(\kappa) = \langle \dot{U}_n \mid n < \omega\rangle$.

In $V[G]$ define the following κ-complete measure on κ^ω:

$$X \in W \;\Leftrightarrow\; X \subseteq \kappa^\omega \wedge \exists p \in G\, \exists \dot{T} \in V^{\mathbb{P}_\kappa}\, (p \cup \{\langle \varnothing, \dot{T}\rangle\} \cup r \Vdash_{\mathbb{P}_\kappa} \dot{b}_\kappa \in j(\dot{X})).$$

Show that $U_n \leq_{\mathrm{RK}} W$ as witnessed by the evaluation map $e_n\colon \vec{\eta} \mapsto \vec{\eta}(n)$. Once this is done, j_W and $[\mathrm{id}]_W$ witness that $\langle U_n \mid n < \omega\rangle$ can be glued.

The ω-gluing property from $o(\kappa) = \omega_1$

Work under "$V = \mathcal{K}$"+"There is no inner model for $\exists\kappa\,(o(\kappa) = \kappa^{++})$".

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So let's force it, but without disrupting our control upon the κ-complete measures:

Non-stationary supported Fast Function Forcing

Let κ be an inaccessible cardinal. We denote by $\mathbb{S}$ be the poset consisting on partial functions $s\colon \kappa \to H(\kappa)$ such that

1. $\mathrm{dom}\, s \subseteq \mathrm{Inacc}$,
2. $(\mathrm{dom}\, s) \cap \beta \in \mathrm{NS}_\beta$ for all $\beta \in \mathrm{Inacc} \cap (\kappa + 1)$,
3. and $s(\alpha) \in H(\alpha^+)$ for all $\alpha \in \mathrm{dom}\, s$.

The order of $\mathbb{S}$ is defined naturally as $s \leq t$ iff $s \supseteq t$.

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Let $S \subseteq \mathbb{S}$ be a $\mathcal{K}$-generic filter.

Lemma

Let $\mathcal{U}$ be a κ-complete ultrafilter over κ in $\mathcal{K}[S]$. Then, there are:

1. A finite iteration $\iota \colon \mathcal{K} \to \bar{\mathcal{K}}$ using normal measures in $\mathcal{K}$ with

$$\mathrm{crit}(i_{0,1}) = \kappa < \mathrm{crit}(\iota_{1,2}) = \mu_1 < \cdots < \mathrm{crit}(\iota_{k,k+1}) = \mu_k,$$

2. A function $f \colon \kappa^{k+1} \to \kappa$ in $\mathcal{K}$ such that $[\mathrm{id}]_{\mathcal{U}} = \iota_{k+1}(f)(\mu_0, \ldots, \mu_k)$,
3. and $\langle a_0, \ldots, a_k \rangle \in \prod_{i \leq k} H(\mu_i^+)^{\mathcal{K}}$

such that

$$\mathcal{U} = \{\dot{X}_S \subseteq \kappa \mid \exists p \in S\, (\iota(p) \cup \{\langle \mu_i, a_i \rangle \mid i \leq k\} \Vdash_{\iota(\mathbb{S})} [\mathrm{id}]_{\mathcal{U}} \in \iota(\dot{X})_{\iota(S)})\}.$$

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Any new measure is **coded** by S plus some information from $H(\kappa^+)^{\mathcal{K}}$.

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We would like to force over $\mathcal{K}[S]$ with a **non-stationary supported** iteration of Tree Prikry forcings (guided by $\ell := \bigcup S$) which should give the ω-gluing property of κ.

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Yet another issue

Let $\mathcal{U} \in \mathcal{K}[S][G]$ be a κ-complete measure. Now it is not longer true that $j_{\mathcal{U}} \restriction \mathcal{K}[S]$ is a finite iteration (because $j_{\mathcal{U}}(\mathbb{P}_\kappa)$ introduces many ω-sequences).

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However, there is a way to **"reduce the problem"** to a finite normal iteration of $j_{\mathcal{U}} \restriction \mathcal{K}[S]$, which by the previous lemma is a lifting of some finite normal iteration of measures in $\mathcal{K}$.

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If the above is true then we can code any measure in $\mathcal{K}[S][G]$ as an element of $H(\kappa^+)^{\mathcal{K}}$, hence as a potential value for $i(\ell)(\kappa)$ for $i \colon \mathcal{K} \to \bar{K}$. Then we will use the Tree Prikry forcing with respect to these coded measures to glue all of them.

Lemma (Coding Lemma)

Let $\mathbb{P}_\kappa = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta \mid \alpha < \beta < \kappa \rangle$ be a non-stationary-supported iteration of $\mathcal{U}$-Tree Prikry forcings in $\mathcal{K}[S]$. Assume that, for each $\alpha < \kappa$, the iteration has the following properties:

1. $|\mathbb{P}_\alpha| \leq 2^\alpha$ *and* $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\langle \mathbb{Q}_\alpha, \leq^* \rangle$ *is* α*-closed";*
2. $\mathbb{1} \Vdash_{\mathbb{P}_\alpha}$ "$\forall p, q \in \mathbb{Q}_\alpha$ *compatible* $p \wedge q$ *exists".*

Fix $G \subseteq \mathbb{P}_\kappa$ $\mathcal{K}[S]$-generic. For each κ-complete ultrafilter $\mathcal{U} \in \mathcal{K}[S][G]$ over κ there are

(α) *a finite sub-iteration $\iota\colon \mathcal{K}[S] \to \mathcal{K}^M[\iota(S)]$ of $j_\mathcal{U} \restriction \mathcal{K}[S]$,*

(β) *an ordinal $\bar\epsilon < \iota(\kappa)$ with $\bar\epsilon \in \mathrm{range}(k)$,*

(γ) *$r \in \iota(\mathbb{P}_\kappa)$ with finite support such that $\iota(p) \wedge r$ exists for all $p \in G$*

such that, for each $p \in G$, $p \Vdash^{\mathcal{K}[S]}_{\mathbb{P}_\kappa}$ "$\dot X \in \dot{\mathcal{U}}$" if and only if there is $q \in \iota(\mathbb{P}_\kappa)$ such that

$$(k(q) \in j_\mathcal{U}(G) \;\&\; q \leq^* \iota(p) \wedge r \;\&\; \mathrm{supp}(q) = \mathrm{supp}(\iota(p) \wedge r) \;\&\; q \Vdash_{\iota(\mathbb{P}_\kappa)} \bar\epsilon \in \iota(\dot X)).$$

The ω-gluing property from $o(\kappa) = \omega_1$

Proof sketch

Let $\mathcal{U} = \langle U(\alpha, \zeta) \mid \alpha \leq \kappa,\, \zeta < o^{\mathcal{K}}(\alpha) \rangle$ be the coherent sequence of measures in $\mathcal{K}$ witnessing that $o(\kappa) = \omega_1$.

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$$\mathbb{1} \Vdash_{\mathbb{P}_\alpha} \text{“}\ell(\alpha) \text{ is an } \omega\text{-sequence of } \textbf{codes} \text{ for } \alpha\text{-complete measures on } \alpha.\text{”}$$

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Let $\langle \mathcal{U}_n \mid n < \omega \rangle \in \mathcal{K}[S][G]$ be an ω-sequence of κ-complete measures on κ.

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Take $j_{U(\kappa,\zeta)} \colon \mathcal{K} \to M$. Lift it to $j_{U(\kappa,\zeta)} \colon \mathcal{K}[S] \to M[j(S)]$ in a way that $j(\ell)(\kappa) = \langle c_n \mid n < \omega\rangle$.

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Is there any connection between the gluing property and directedness of the RK-order?

Thank you very much for your attention!