Independence and singular cardinals

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Section 1

Independent families at uncountable cardinals

Basic definitions

Definition

Assume that κ is a regular cardinal and χ is an infinite cardinal. Let $\mathcal{A}$ be a family of subsets of χ such that $|\mathcal{A}| \geq \kappa$:

- *We denote by $\mathrm{BF}_\kappa(\mathcal{A})$ the family of partial functions $\{h : \mathcal{A} \to 2 : |\mathrm{dom}(h)| < \kappa\}$ and call it the family of bounded functions on $\mathcal{A}$.*

Definition

- *Given $h \in \mathsf{BF}_\kappa(\mathcal{A})$, we define*

$$\mathcal{A}^h = \bigcap\{A^{h(A)} : A \in \mathcal{A} \cap \mathsf{dom}(h)\},$$

where $A^{h(A)} = A$ if $h(A) = 0$ and $A^{h(A)} = \chi \backslash A$ otherwise. We call $\mathcal{A}^h$ the Boolean combination of $\mathcal{A}$ associated to h and we refer to $\{\mathcal{A}^h : h \in \mathsf{BF}_\kappa(\mathcal{A})\}$ as the family of generalized boolean combinations of the family $\mathcal{A}$.

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Independent families

Definition

Let κ be a regular cardinal. A family $\mathcal{A} \subseteq \mathcal{P}(\chi)$ such that $|\mathcal{A}| \geq \kappa$ is called κ-independent if for for every $h \in \mathsf{BF}_\kappa(\mathcal{A})$, the set $\mathcal{A}^h$ has size χ.

A κ-independent family $\mathcal{A}$ is said to be maximal κ-independent if it is not properly contained in another κ-independent family. We call the cardinal κ the degree of independence of the family $\mathcal{A}$.

The issue with existence

- Analogously to the classical case ($\chi = \kappa = \omega$) it is possible to construct κ-independent families of size 2^κ (under some assumptions on κ).
- However, it is not possible to use Zorn's lemma to prove the existence of maximal κ-independent families, if κ is uncountable.

The following result of Kunen provides necessary conditions for the existence of maximal κ-independent families in the general context when κ is a regular uncountable cardinal.

Kunen's Theorem

Theorem (See Theorem 1 in [Kun83])

Suppose that κ is regular and uncountable and χ is any infinite cardinal. Also assume that there is a maximal κ-independent family $\mathcal{A} \subseteq \mathcal{P}(\chi)$, with $|\mathcal{A}| \geq \kappa$. Then:

1. $2^{<\kappa} = \kappa$ *and,*
2. *there is a* Γ *with* $\sup\{(2^{\alpha})^{+} : \alpha < \kappa\} \leq \Gamma \leq \min\{\chi, 2^{\kappa}\}$ *such that, there is a non-trivial* κ^{+}*-saturated* Γ*-complete ideal over* Γ*.*

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Sufficient conditions

Lemma

Suppose κ is regular, $2^{<\kappa} = \kappa$, $\kappa \leq \chi$ and $\mathcal{I}$ is a κ^+-saturated χ-complete ideal over χ such that $\mathcal{B}(\mathsf{Fn}_\kappa(2^\chi, 2))$ isomorphic to $\mathcal{P}(\chi)/\mathcal{I}$.
Then, there is a maximal κ-independent family of subsets of χ.

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A consistency result

Theorem (Kunen)

If there is a measurable cardinal, then there is a maximal σ-independent family $\mathcal{A} \subseteq \mathcal{P}(2^{\omega_1})$.

The proof

- Start with a measurable cardinal κ in a ground model V where CH holds.
- Let $\mathcal{U}$ be a normal measure witnessing the measurability of κ.
- We construct a model in which CH still holds and if $\kappa = 2^{\aleph_1}$, there is an ω_2-saturated, κ-complete ideal $\mathcal{J}$ over κ such that the Boolean algebras $\mathcal{P}(\kappa)/\mathcal{J}$ and $\mathcal{B}(\mathrm{Fn}_{\omega_1}(2^\kappa, 2))$ are isomorphic.

Sufficient conditions

- Let $\mathbb{P}$ be $\mathsf{Fn}_{\omega_1}(\kappa, 2)$ and let G to be a $\mathbb{P}$-generic filter over V. In $V[G]$, $\kappa = 2^{\aleph_1}$ and we can define the following collection of subsets of κ:

$$\mathcal{J} = \{X \subseteq \kappa : \exists Y \in \mathcal{U}(X \cap Y = \emptyset)\}$$

- $\mathcal{J}$ is, in turn a κ-complete ω_2-saturated ideal because $\mathbb{P}$ has the ω_2-cc and so $\mathcal{J}$ is ω_2-saturated and κ-complete in $V[G]$.

The rest of the argument aims to construct an isomorphism between the Boolean algebras $\mathcal{P}(\kappa)/\mathcal{I}$ and $\mathcal{B}(\mathrm{Fn}_{\omega_1}(2^\kappa, 2))$ in $V[G]$.

- Let $j : V \to M = \mathrm{Ult}(V, \mathcal{U})$ be the ultrapower embedding associated to $\mathcal{U}$, i.e. j is elementary, $\mathrm{crit}(j) = \kappa$.
- Let $\kappa^* = j(\kappa) > \kappa$, then $2^\kappa < \kappa^* < (2^\kappa)^+$ and the posets $\mathrm{Fn}_{\omega_1}(2^\kappa, 2)$ and $\mathrm{Fn}_{\omega_1}(\kappa^* \backslash \kappa, 2)$ are isomorphic.

The isomorphism

Let's define the isomorphism $\Gamma : \mathcal{P}(\kappa)/\mathcal{I} \to \mathcal{B}(\mathsf{Fn}_{\omega_1}(\kappa^*\backslash\kappa, 2))$ in $V[G]$ as follows: Given $[X] \in (\mathcal{P}(\kappa)/\mathcal{I})^{V[G]}$, and let $\dot{X}$ be a $\mathbb{P}$-name for the set X. We define the function as follows:

$$\Gamma([X]) := \bigvee\{q \in \mathsf{Fn}_{\omega_1}(\kappa^*\backslash\kappa, 2) : \exists p \in G(p \cup q \Vdash \check{\kappa} \in j(\dot{X}))\}.$$

Two more consistency results

Corollary

Assume κ is strongly compact in V. Then in $V[G]$, where G is $\mathbb{P}$-generic (for $\mathbb{P} = \mathrm{Fn}_{\omega_1}(\kappa, 2)$ like in the theorem above) for every cardinal $\chi \geq \kappa$ such that $\mathrm{cf}(\chi) \geq \kappa$ there is a maximal σ-independent family of subsets of χ.

Theorem

Let δ be a regular cardinal such that $2^{<\delta} = \delta$ and κ be a measurable cardinal above it. Then there is a generic extension in which there is a maximal δ-independent family $\mathcal{A} \subseteq \mathcal{P}(2^\delta)$.

Section 2

A word on the regular case

Countable independence degree

If we assume $\kappa = \omega$ the existence of maximal κ-independent families at a cardinal χ is a straightforward consequence of Zorn's lemma. The following is a result of Fischer and myself regarding these families.

Theorem (See [FM20])

Let χ be a measurable cardinal and let $2^\chi = \chi^+$. Then there is a generic extension in which there is a maximal ω-independent family of subsets of χ, which remains maximal after the χ-support product of δ-many copies of χ-Sacks forcing.

On uncountable independence degree

Also, Eskew and Fischer have studied the concept of independence for regular cardinals. In [EF21] they prove in particular that if $\mathfrak{i}(\kappa)$ is the minimum size of a maximal κ-independent family of subsets of κ. Then, it is consistent that $\kappa^+ < \mathfrak{i}(\kappa) < 2^\kappa$.

They also studied the spectrum of maximal κ-independent families at χ and gave a wide set of results involving it.

Section 3

The singular case

Framework

Now, we want to study the concept of independence in the case when λ is a singular cardinal of cofinality $\kappa < \lambda$.

Look at the definition of ▸ Independence and notice, there is no a priori restriction about lifting it to the context of a singular.

Hausdorff's example at $\aleph_\omega$

Let

$$\mathcal{C} = \{(a, A) : a \in [\lambda]^{<\omega}, A \subseteq \mathcal{P}(a)\}$$

and note $|\mathcal{C}| = \aleph_\omega$.

For $X \subseteq \lambda$ define

$$\mathcal{Y}_X = \{(a, A) \in \mathcal{C} : X \cap a \in A\}.$$

Then, $\mathcal{A} = \{\mathcal{Y}_X : X \subseteq \lambda\} \subseteq \mathcal{P}(\mathcal{C}) \simeq \mathcal{P}(\aleph_\omega)$ is ω-independent (or σ-independent).

Given $X_0, X_1, \ldots X_i$ and $Z_0, Z_1, \ldots Z_j$ for $i, j < \omega$, if $a \in [\lambda]^{<\omega}$ is such that $X_l \cap a \neq X_{l'} \cap a \neq Z_n \cap a \neq Z_{n'} \cap a$ for all $l, l' \leq i$ and $n, n' \leq j$. Then $a \in \bigcap_{l \leq i} \mathcal{Y}_{X_l} \cap \bigcap_{l \leq j} \lambda \backslash \mathcal{Y}_{Z_j}$.

- Notice that $\mathcal{A}$ is not ω_1-independent: If $X_0 \subseteq X_1 \subseteq \ldots X_n \subseteq \ldots$ is cofinal in λ. Take $(a, A) \in \bigcap_{i \text{ even}} \mathcal{Y}_{X_i} \cap \bigcap_{i \text{ odd}} \lambda \backslash \mathcal{Y}_{X_i}$. Since the sequence of the X_n's is cofinal there is a $n_a \in \omega$ (we can take it minimal) such that $a \subseteq X_{n_a}$, but then for all $i \geq n_a$, $a \cap X_i = a$ which leads to a contradiction.

More simple properties

The former is a general behavior:

Proposition

Let λ be a singular cardinal of cofinality $\kappa < \lambda$. Suppose that $\mathcal{A}$ is a κ-independent family of subsets of λ, then $\mathcal{A}$ is <u>not</u> κ^+-independent.

Proposition

Suppose λ is a strong limit singular cardinal with cf$(\lambda) = \kappa$. *Then there is a κ-independent family of subsets of λ of size 2^λ.*

Maximality

Now we turn to maximality and the issue of the existence of maximal independent families at singular cardinals. From now on, we assume that λ is a singular cardinal of cofinality $\kappa < \lambda$.

First we establish that a κ-independent family $\mathcal{A} \subseteq [\lambda]^\lambda$ is **maximal** if for all $X \in [\lambda]^\lambda$ there is a bounded function $\mathrm{BF}_\kappa(\mathcal{A})$ such that either $\mathcal{A}^h \setminus X$ or $\mathcal{A}^h \cap X$ is bounded in λ (i.e. of size $< \lambda$).

Cases

- Let's consider the case where λ is singular of countable cofinality. In this case existence of a maximal ω-independent family (or just *independent*) of subsets of λ can be proven using Zorn's lemma.
- In the case of λ singular of cofinality $\kappa > \omega$ we have the following: if there exists $\mathcal{A} \subseteq [\lambda]^\lambda$ a maximal κ-independent family, then Kunen's Theorem implies that $2^{<\kappa} = \kappa$ and that there is an ordinal Γ with $\sup\{(2^\alpha)^+ : \alpha < \lambda\} \leq \Gamma \leq \min\{\lambda, 2^\kappa\}$ such that, there is a non-trivial κ^+-saturated Γ-complete ideal over Γ.

Main results

The next result guarantees the existence of a maximal κ-independent family at a singular cardinal λ of cofinality κ.

Lemma

Assume that λ is a singular cardinal of cofinality $\kappa > \omega$ which is a limit of the discrete sequence of regular cardinals $(\lambda_\alpha : \alpha < \kappa)$ and that $\delta \leq \kappa < \lambda_0$ is a successor cardinal $\delta = \mu^+$. If for each $\alpha < \kappa$ there is a dense maximal δ-independent family $\mathcal{A}_\alpha \subseteq [\lambda_\alpha]^{\lambda_\alpha}$ and also there is a maximal δ-independent family $\mathcal{C}$ of subsets of κ. Then, there is a maximal δ-independent family $\mathcal{B} \subseteq [\lambda]^\lambda$. [a]

[a] One can prove an analog of this theorem in which instead of densely maximal, we just have *maximal*

Theorem

Assume that λ is a singular cardinal of cofinality κ which is a limit of a discrete sequence of cardinals $(\lambda_\alpha : \alpha < \kappa)$. Let also $(\delta_\alpha : \alpha < \mu)$ be a sequence of cardinals with limit $\mu \leq \kappa$ where μ is a regular cardinal. Suppose also that for each $\alpha < \kappa$, there is a maximal δ_α^+-independent family $\mathcal{A}_\alpha \subseteq [\lambda_\alpha]^{\lambda_\alpha}$ and $\kappa < \lambda_0$ is regular such that there is a maximal μ-independent family of subsets of κ. Then, there is a maximal μ-independent family $\mathcal{B} \subseteq [\lambda]^\lambda$.

Theorem

Start with a ground model V in which GCH *holds. Suppose that λ is a singular cardinal of cofinality $\kappa > \omega$ which is a limit of supercompact cardinals $(\lambda_\alpha : \alpha < \kappa)$. Let also $(\delta_\alpha : \alpha < \kappa)$ be a sequence of regular successor cardinals converging to κ so that $\delta_\alpha^{<\delta_\alpha} = \delta_\alpha$. Then there is a generic extension of V in which:*

$$V^{\mathbb{P}} \models \text{There is a maximal } \kappa\text{-independent family of subsets of } \lambda.$$

Sizes of independent families

Let λ be a singular cardinal of cofinality $\kappa < \lambda$, let's define:

$$\mathfrak{i}(\lambda) = \{|\mathcal{A}|\colon \mathcal{A} \subseteq [\lambda]^\lambda \text{ such that } \mathcal{A} \text{ is maximal } \kappa\text{-independent}\}$$

Lemma

The following inequalities hold:

1. $\mathfrak{i}_\kappa(\lambda)^{<\kappa} \geq \lambda^+$.
2. $\mathfrak{i}_\kappa(\lambda)^{<\kappa} \geq \mathfrak{r}(\lambda)$.

More ZFC properties

Proposition

If for all $\gamma < \mathfrak{d}(\lambda)$ *we get* $\gamma^{<\kappa} < \mathfrak{d}(\lambda)$ *then* $\mathfrak{i}_\kappa(\lambda) \geq \mathfrak{d}(\lambda)$.

Theorem (Shelah)

- *If* λ *is strong limit singular, then* $\mathfrak{d}(\lambda) = 2^\lambda$.
- *If* λ *is singular and* $\alpha < \lambda$ *then* $|\alpha|^{\mathrm{cf}(\lambda)} < \lambda$ *then* $\mathfrak{d}(\lambda) = 2^\lambda = \mathfrak{e}(\lambda)$.

Questions

- Given a singular strong limit cardinal λ, is it the case that $\mathfrak{i}(\lambda) = 2^\lambda$?
- Looking at the construction of the maximal δ-independent family at a singular from Theorem 2. Regardless of the sizes of the intermediate families $\mathcal{A}_\alpha$ for $\alpha < \kappa$, the size of the final family $\mathcal{B}$ is always 2^λ. Is it possible to refine the family $\mathcal{B}$ in such a way that its size is dependent on the size of the intermediate families?

Thanks!

Regular ultrafilters

Definition

Let κ, λ, μ be cardinals so that $\kappa \leq \mu$. An ultrafilter $\mathcal{D}$ over λ is (κ, μ)-regular if there is a family $\{X_\alpha : \alpha < \mu\}$ of elements of $\mathcal{D}$ so that every subset S of μ of cardinality κ, $\bigcap_{\alpha \in S} X_\alpha = \emptyset$. We call the sequence $\{X_\alpha : \alpha < \mu\}$ a regularizing sequence for $\mathcal{U}$.

Classical results

Theorem

If $\kappa > \omega$ is a regular cardinal, then κ is strongly compact if and only if for every $\lambda > \kappa$ there is a κ-complete, (κ, λ)-regular ultrafilter.

Theorem (Ketonen)

If $\kappa > \omega$ is strongly compact, then for every regular $\lambda > \kappa$ there is a uniform κ-complete, (κ, λ)-regular ultrafilter over λ.

Proposition

Any κ-complete, (κ, λ)—regular ultrafilter is $(\kappa, \lambda^{<\kappa})$-regular.

Theorem (Solovay)

If $\lambda > \kappa$ is a regular and κ is strongly compact then $\lambda^{<\kappa} = \lambda$. Hence, the singular cardinal hypothesis holds at any singular strong limit cardinal $> \kappa$ of cofinality κ.

Regular maximal independent families

Definition

Let $\mathcal{A}$ be a maximal κ-independent family at some cardinal χ, we say that $\mathcal{A}$ is regular if for every $g \in \mathrm{BF}_{\kappa^+}(\mathcal{A})$ with $|\mathrm{dom}(g)| = \kappa$, the corresponding boolean combination $\mathcal{A}^g = \emptyset$.

Lemma

Let $\mathcal{A}$ be a dense maximal independent family at a singular cardinal λ of cofinality κ such that for all $g \in \mathrm{BF}_{\kappa^+}$, there is $g' \in \mathrm{BF}_{\kappa^+}(\mathcal{A})$ such that $\emptyset = \mathcal{A}^{g'} =^* \mathcal{A}^g$.

Question

Are all maximal κ-independent families regular, or in other words: Given an uncountable cardinal κ. Is there a cardinal χ such that there are no regular maximal κ-independent families of subsets of χ?

References I

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- Vera Fischer and Diana Carolina Montoya, *Higher independence*, Submitted, 2020.
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