Self-distributivity and Borel reducibility

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Joint work with Sheila Miller and Filippo Calderoni.

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Given j, k both elementary embeddings $V_\lambda \to V_\lambda$, we can define

$$j * k = \bigcup_{\alpha<\lambda} j(k \restriction V_\alpha) = \bigcup_{\alpha<\lambda} j(\{(x, k(x)) : x \in V_\alpha\}).$$

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For any $j, k, \ell\colon V_\lambda \to V_\lambda$,

$$j * (k * \ell) = (j * k) * (j * \ell).$$

Call an algebraic structure with a binary relation $*$ satisfying $a * (b * c) = (a * b) * (a * c)$ an *LD-system*. (Algebraists also call such a thing a *shelf*.)

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Theorem (Laver)

$\mathcal{E}_j$ is the free LD-system on the single generator j.

Set theory in classifiability

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Any reasonably definable map will be Borel. So if you have a classification programme looking for a map

$$\{\text{codes for objects}\}/\sim_{\text{obj}} \longrightarrow \{\text{codes for invariants}\}/\sim_{\text{inv}}$$

but descriptive set theory tells you there is no such Borel map, then there can be no such classification.

Definition

Let X and Y be Polish spaces (separable completely metrizable spaces, e.g. $\mathbb{R}$), let E be an equivalence relation on X, and let F be an equivalence relation on Y. We say that E is *Borel reducible* to F, and write

$$E \leq_B F$$

if there is a Borel function $f : X \to Y$ such that

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Natural Question

Where does isomorphism of LD-systems fit in the Borel reducibility partial order?

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isomorphism relations on first order classes of countable structures.

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So then an n-ary relation is then encoded by a subset of $\mathbb{N}^n$, ie, an element of $2^{\mathbb{N}^n}$.

So for example the space of all (directed) graphs is encoded as $2^{\mathbb{N}^2}$ — essentially, Cantor space. Likewise, the space of LD-systems is a subspace of $2^{\mathbb{N}^3}$, and any first order class of structures is similarly encoded in a Polish space.

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We say such a class $\mathcal{C}$ is *Borel complete* if its isomorphism relation is maximal: for every other first order class of countable structures $\mathcal{D}$,

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- Graphs
- Groups (Mekler)
- Rooted trees (Friedman & Stanley)
- Linear orders (Friedman & Stanley)
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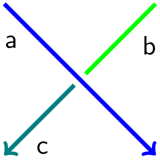
More refined question

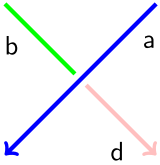
Is the class of countable LD-systems Borel complete?

Back to algebra

Take an oriented knot diagram. We define an algebraic structure with two binary operations $*$ and $*'$, a generator for each arc of the diagram, and a relation for each crossing, as follows:

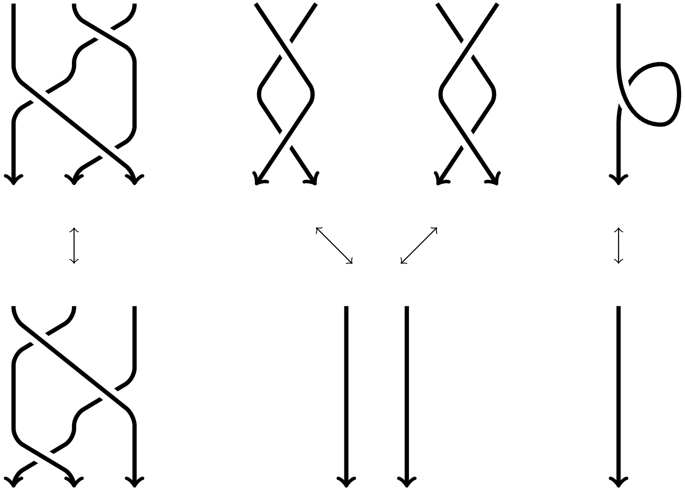


a$*$b=c

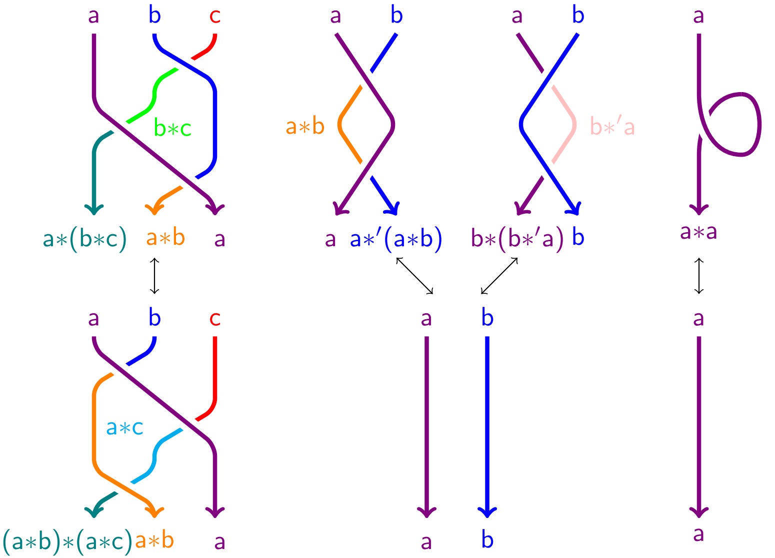


a$*'$b=d

The Reidemeister moves



Respecting the Reidemeister moves



Definitions

A *quandle* is a set with two binary operations $*$ and $*'$ such that

1. $\forall a \forall b \forall c [a * (b * c) = (a * b) * (a * c)]$
2. $\forall a \forall b [a * a *' b = a *' a * b = b]$
3. $\forall a [a * a = a]$.

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Another example

Any group with the operation of conjugation $(a * b = aba^{-1})$ is a quandle.

Quandles as knot invariants

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Heuristically, it seemed hard to determine whether two quandles are isomorphic.

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The class of quandles is Borel complete. Consequently, the class of LD-systems is Borel complete.

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Proof

We construct a mapping Q taking (directed, irreflexive) graphs to quandles such that

$$\Gamma \cong_{Graphs} \Gamma' \qquad \text{iff} \qquad Q(\Gamma) \cong_{Quandles} Q(\Gamma').$$

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Since the class of graphs is known to be Borel complete, this implies that the class of quandles is Borel complete.

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Then the operation $*$ on X given by

$$x * y = \begin{cases} y & \text{if } [x]_\tau \in \theta([y]_\tau) \\ \tau y & \text{if } [x]_\tau \notin \theta([y]_\tau) \end{cases}$$

makes $(X, *)$ a quandle, the *dynamical quandle derived from* (X, τ) *with respect to* θ.

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- $X = V \times 2$
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Then we define $Q(\Gamma)$ to be the dynamical quandle derived from (X, τ) with respect to θ.

Clearly if $\Gamma \cong \Gamma'$ then $Q(\Gamma) \cong Q(\Gamma')$.

Interesting part: if there is a quandle isomorphism $f : Q(\Gamma) \to Q(\Gamma')$, why must there be a graph isomorphism $\Gamma \to \Gamma'$?

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The isomorphism f need not arise from a graph isomorphism. Nevertheless, given f can we construct an isomorphism $\varphi : \Gamma \to \Gamma'$?

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Then the "twinning" of (v, j) with $(v, 1 - j)$ is witnessed by the action of (u, i). Since f is a quandle isomorphism, the action of $f(u, i)$ on $f(v, j)$ is also nontrivial, and so takes $f(v, j)$ it to *its* twin. So the first component of $f(v, j)$ is independent of $j \in \{0, 1\}$, and we take this to be $\varphi(v)$.

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These definitions of $\varphi(v)$ combine to produce a graph isomorphism from Γ to Γ'. $\square$

Going further

There are various strengthenings of Borel completeness that one can consider.

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Theorem (A.B.-T., F. Calderoni, S. Miller)

The embeddability relation on countable quandles is a complete $\mathbf{\Sigma}^1_1$ *quasiorder, and further, is an invariantly universal* $\mathbf{\Sigma}^1_1$ *quasiorder.*

Category theory

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Call the analogue of Borel completeness (i.e. maximality for classes of first order structures) in this context *functorial universality*.

Then the class of graphs is functorially universal, but those of rooted trees, linear orders and Boolean algebras are not.

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The only rigid quandles are the empty quandle and the singelton: for trivial quandles (where every left-multiplication operation is the identity), any permutation is an automorphism, and for non-trivial quandles, there is a left-multiplication automorphism that is not the identity!

On the other hand, there are many distinct rigid graphs. So there can be no functorial Borel reduction from graphs to quandles. □