Definable structures in Banach space theory

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This is a joint work with Barnabas Farkas and Sebastian Jachimek.

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- an ideal on ω can be treated as a subset of 2^ω;
- $\mathcal{I}$ is a P-ideal if for each (A_n) from $\mathcal{I}$, there is $A \in \mathcal{I}$ such that $A_n \subseteq^* A$ for every n.

Classical examples of analytic P-ideals.

Example (Summable ideal)

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Example (Density ideal)

$$\mathcal{Z} = \{A \subseteq \omega \colon d(A) = 0\},$$

where

$$d(A) = \lim_{n \to \infty} \frac{|A \cap \{0, \cdots, n\}|}{n+1}.$$

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Theorem (Solecki)

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$$2^\omega \mapsto \mathbb{R}^\omega$$

Continuous version of LSC subnorms.

Consider a function $\varphi\colon \mathcal{P}(\omega) \to [0,\infty]$ such that for each A, $B \subseteq \omega$

- $\varphi(\emptyset) = 0$,
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Such functions are called LSC submeasures.

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$(\pi_A(x))(n) = x(n)$ for $n \in A$ and $(\pi_A(x))(n) = 0$ otherwise.

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Nice norm = *norm* on $\mathbb{R}^\omega$ which may attain infinite values and which is compatible with the topological structure of $\mathbb{R}^\omega$.

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Exercise: What kind of objects are those guys?

Banach spaces

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Theorem (PBN, Farkas)

For every Banach space X with unconditional basis there is a nice norm Φ such that

$$X = \mathrm{Exh}(\Phi).$$

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- $\mathrm{Exh}(\Phi)$ *does not contain an isomorphic copy of* c_0*;*
- $\mathbb{R}^\omega/\mathrm{Exh}(\Phi)$ *is Borel reducible to* $\mathbb{R}^\omega/\ell_\infty$.

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The function

$$\|x\|_{\mathcal{F}} = \sup_{F \in \mathcal{F}} \langle x, F \rangle$$

is a nice norm.

Extreme examples

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$X_{\mathcal{F}}$ is a standard example in the theory of Banach spaces.
E.g. it is a space which is not isomorphic to c_0 but which is c_0-saturated (i.e. each ∞-dimensional closed subspace of $X_{\mathcal{F}}$ contains an isomorphic copy of c_0).

Scattered families

Theorem (Pełczynski, Semadeni, 1959)

A family $\mathcal{F}$ is scattered iff $X_{\mathcal{F}}$ is c_0-saturated.

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The spaces $X_{\mathcal{F}}$ for scattered families $\mathcal{F}$ were studied by Lopez Abad, Todorcevic, Argyros, ...

Examples motivated by analytic P-ideals: trace of null

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$$\mathcal{F} = \{F \subseteq 2^{<\omega} \colon F \text{ is an antichain}\}.$$

- This family induces the trace of null ideal.
- The space $X_{\mathcal{F}}$ is a strange alloy of ℓ_1 and c_0.

Example motivated by analytic P-ideals: Farah's ideal

Example (Farah's family)

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$$\mathcal{F} = \{F \subseteq \omega \colon \forall n\ |F \cap [2^n, 2^{n+1})|/2^n \leq 1/n\}.$$

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- This family induces the Farah's ideal.
- The space $X_{\mathcal{F}}$ is an easy example of a ℓ_1-saturated space (every ∞-dimensional closed subspaces contains ℓ_1) which is not isomorphic to ℓ_1.

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- Every space with the Schur property is ℓ_1-saturated,
- Bourgain showed that there is an ℓ_1-saturated space which does not have Schur property,
- Then several other examples have been constructed.

Farah family of second order

Example (Tldr)

For $N = \{n_1 < n_2 < \dots\} \in [\omega]^\omega$ let $f_N \colon \omega \to \omega$ be defined by $f_N(n_k) = k^{-1}2^{n_k}$ and $f_N(n) = 0$ if $n \notin N$. Denote

$$\mathcal{A}_N = \{A \subseteq \omega \colon |A \cap [2^n, 2^{n+1})| \leq f_N(n)\}.$$

Let

$$\mathcal{F} = \{F \in [\omega]^{<\omega} \colon \exists N \in [\omega]^\omega \ F \subseteq A \text{ for some } A \in \mathcal{A}_N\}.$$

Farah family of second order

Example (Tldr)

For $N = \{n_1 < n_2 < \dots\} \in [\omega]^\omega$ let $f_N\colon \omega \to \omega$ be defined by $f_N(n_k) = k^{-1}2^{n_k}$ and $f_N(n) = 0$ if $n \notin N$. Denote

$$\mathcal{A}_N = \{A \subseteq \omega\colon |A \cap [2^n, 2^{n+1})| \leq f_N(n)\}.$$

Let

$$\mathcal{F} = \{F \in [\omega]^{<\omega}\colon \exists N \in [\omega]^\omega \ F \subseteq A \text{ for some } A \in \mathcal{A}_N\}.$$

- The space $X_{\mathcal{F}}$ is ℓ_1-saturated and does not have Schur property.

How to characterize $X_{\mathcal{F}}$'s which are ℓ_1-saturated?

Conjecture. If $\mathcal{F}$ is perfect, then $X_{\mathcal{F}}$ is ℓ_1-saturated.

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Counterexample. Let (A_n) be a partition of ω into infinite sets. Take the hereditary closure of $\{A_n \colon n \in \omega\}$.

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Counterexample. Let

$$\mathcal{F} = \{F \subseteq \omega \colon \text{ for each } n \text{ but one } |F \cap [2^n, 2^{n+1})|/2^n \leq 1/n\}.$$

How to characterize $X_{\mathcal{F}}$'s which are ℓ_1-saturated?

Conjecture. If $\mathcal{F}$ is perfect in the Ellentuck topology, then there is a c_0-saturated Y and ℓ_1-saturated Z such that

$$X_{\mathcal{F}} = Y \oplus Z.$$