On the limits of properness

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Forcing axioms

Forcing axioms are principles asserting, for some class Γ of forcing notions, that sufficiently generic filters for all forcings in Γ exist. These are principles asserting that the universe is

saturated, in some restricted sense, relative to generic extensions by forcings in Γ.

Given a class Γ of forcing notions and a cardinal κ,

$$\mathsf{FA}_\kappa(\Gamma)$$

is the statement that for every $\mathcal{P} \in \Gamma$ and every collection $\{D_i \,:\, i < \kappa\}$ of dense subsets of $\mathcal{P}$ there is a filter $G \subseteq \mathcal{P}$ such that $G \cap D_i \neq \emptyset$ for all $i < \kappa$.

If we replace dense sets D_i by maximal antichains A_i of $\mathcal{P}$, we obtain an equivalent principle.

Examples:

- MA_κ is FA_κ(c.c.c.).
- MA is $\mathrm{FA}_{<2^{\aleph_0}}$(c.c.c.).
- PFA is FA_{ω_1}(proper).
- MM is $\mathrm{FA}_{\omega_1}(\{\mathcal{P} : \mathcal{P}$ preserves stationary subsets of $\omega_1\})$.

(Solovay-Tenenbaum) c.c.c. forcing is preserved by finite-support iterations. Hence, for every cardinal κ we can build a finite-support iteration forcing MA_κ. In the resulting model, $2^{\aleph_0} > \kappa$ (by $\mathrm{FA}_\kappa(\{\mathrm{Cohen}\})$).

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A forcing $\mathcal{P}$ is proper iff for every large enough θ there is a club E of $[H_\theta]^{\aleph_0}$ such that for every $N \in E$ and every $p \in N \cap \mathcal{P}$ there is some $q \leq_{\mathcal{P}} p$ which is $(N, \mathcal{P})$-*generic* (i.e., given any dense set $D \subseteq \mathcal{P}$, $D \in N$, every $q' \leq_{\mathcal{P}} q$ is $\leq_{\mathcal{P}}$-compatible with some $r \in D \cap N$).

$\mathcal{P}$ c.c.c. $\Rightarrow$ $\mathcal{P}$ is proper $\Rightarrow$ $\mathcal{P}$ preserves stationary subsets of ω_1 $\Rightarrow$ $\mathcal{P}$ preserves ω_1

(Shelah) Properness is preserved by countable-support iterations. Hence, if κ is a supercompact cardinal, then there is a countable-support iteration $(\mathcal{P}_\alpha : \alpha \leq \kappa)$, $\mathcal{P}_\kappa \subseteq V_\kappa$, such that $\mathcal{P}_\kappa$ forces PFA (Baumgartmer, Shelah).

All V-cardinals λ such that $\omega_1 < \lambda < \kappa$ are collapsed to ω_1 along the iteration. Hence, $2^{\aleph_0} = \aleph_2$ holds in the end.

In fact, if $(\mathcal{P}_\alpha : \alpha \leq \kappa)$ is **any** countable-support iteration and $\mathrm{cf}(\kappa) \geq \omega_1$, then for every $\alpha < \kappa$, $\mathcal{P}_\kappa$ collapses $\mathbb{R}^{V^{\mathcal{P}_\alpha}}$ to ω_1. Hence, $\mathcal{P}_\kappa$ forces $2^{\aleph_0} \leq \aleph_2$.

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In fact, if $(\mathcal{P}_\alpha : \alpha \leq \kappa)$ is **any** countable-support iteration and $\mathrm{cf}(\kappa) \geq \omega_1$, then for every $\alpha < \kappa$, $\mathcal{P}_\kappa$ collapses $\mathbb{R}^{V^{\mathcal{P}_\alpha}}$ to ω_1. Hence, $\mathcal{P}_\kappa$ forces $2^{\aleph_0} \leq \aleph_2$.

Bounded forcing axioms

We obtain *bounded forcing axioms* by restricting the sizes of the antichains being considered. For example:

Given a cardinal κ,

$$\mathrm{BFA}_{\kappa}(\Gamma)$$

says that for every $\mathcal{P} \in \Gamma$ and every family $\{A_i : i < \kappa\}$ of maximal antichains of $\mathcal{P}$, if $|A_i| \leq \kappa$ for all $i < \kappa$, then there is a filter $G \subseteq \mathcal{P}$ such that $G \cap A_i \neq \emptyset$ for all i.

Bounded forcing axioms can be cast as natural principles of generic absoluteness:

(Bagaria) $\mathrm{BFA}_{\kappa}(\Gamma)$ holds iff $H(\kappa^+)^V \preccurlyeq_{\Sigma_1} H(\kappa^+)^{V[G]}$ for every $\mathcal{P} \in \Gamma$ and every $\mathcal{P}$-generic G.

Not surprisingly, PFA implies $2^{\aleph_0} = \aleph_2$ (Todorčević, Veličković).

Note, incidentally, that FA_{ω_2}(proper) is inconsistent ($\mathrm{Coll}(\omega_1, \omega_2)$ is proper).

In fact:

(Moore) BFA_{ω_1}(proper), a.k.a. BPFA, implies $2^{\aleph_0} = \aleph_2$. In fact, if BPFA holds, then given any ladder system $\vec{C}$ on ω_1 and any partition $\vec{S}$ of ω_1 into $\aleph_1$-many stationary sets, there is a well-order of $H(\omega_2)$ in length ω_2 Σ_1-definable over $H(\omega_2)$ with $\vec{C}$ and $\vec{S}$ as parameters.

The proof of the above also shows that the Mapping Reflection Principle (MRP) implies the same conclusion.

Given $\alpha < \omega_1$, a forcing $\mathcal{P}$ is α-proper if for every large enough θ there is a club E of $[H(\theta)]^{\aleph_0}$ such that for every $\in$-chain $(N_i)_{i<\alpha}$ of models in E and every $p \in N_0$ there is $q \leq_{\mathcal{P}} p$ which is $(N_i, \mathcal{P})$-generic for all $i < \alpha$.

The proofs of Todorčević and Veličković show that in fact $\mathsf{FA}_{\omega_1}(<\omega_1\text{-closed} * \text{c.c.c.})$ (and so, in particular, $\mathsf{FA}_{\omega_1}(<\omega_1\text{-proper})$) implies $2^{\aleph_0} = \aleph_2$. On the other hand, all known proofs of $\mathsf{BPFA} \Rightarrow 2^{\aleph_0} = \aleph_2$ involve proper forcings which are badly non-ω-proper.

Question

Does $\mathsf{BFA}_{\omega_1}(\omega\text{-proper})$ imply $2^{\aleph_0} = \aleph_2$?

The answer should be Yes, but proving that would seem to require completely new coding techniques beyond the reach of MRP.

Martin's Maximum^{++} and $(*)$

A forcing $\mathcal{P}$ is *semiproper* iff for every large enough θ, for club-many $N \in [H(\theta)]^{\aleph_0}$, if $p \in N \cap \mathcal{P}$ then there is $q \leq_{\mathcal{P}} p$ such that q is $(N, \mathcal{P})$*-semigeneric* (i.e., for every $\mathcal{P}$-name for an ordinal in ω_1 such that $\tau \in N$, $q \Vdash_{\mathcal{P}} \tau \in N$).

$\mathcal{P}$ c.c.c. $\Rightarrow$ $\mathcal{P}$ is proper $\Rightarrow$ $\mathcal{P}$ is semiproper $\Rightarrow$ $\mathcal{P}$ preserves stationary subsets of ω_1 $\Rightarrow$ $\mathcal{P}$ preserves ω_1

(Shelah) Semiproper forcing is preserved by *revised* countable support iterations.

(Foreman-Magidor-Shelah) *Martin's Maximum* (MM) is $\mathrm{FA}_{\omega_1}(\{\mathcal{P} : \mathcal{P}$ preserves stationary subsets of $\omega_1\})$.

(1) MM is a maximal forcing axiom: If $\mathbb{P}$ does not preserve stationary subsets of ω_1, then $\mathrm{FA}_{\omega_1}(\{\mathbb{P}\})$ fails.
(2) If κ is a supercompact cardinal, then there is a *revised* countable support iteration $(\mathcal{P}_\alpha : \alpha \leq \kappa)$ such that $\mathcal{P}_\kappa \subseteq V_\kappa$ and $\mathcal{P}_\kappa$ forces $\mathrm{FA}_{\omega_1}(\text{semiproper})$, a.k.a. SPFA.
(3) (Shelah) SPFA $\iff$ MM

In fact, the standard iteration for forcing MM gives rise to a model of MM^{++}.

MM^{++} is the following strong form of MM: For every $\mathbb{P}$ preserving stationary subsets of ω_1, every $\{D_i : i < \omega_1\}$ consisting of dense subsets of $\mathbb{P}$, and every $\{\tau_i : i < \omega_1\}$ consisting of $\mathbb{P}$–names for stationary subsets of ω_1 there is a filter $G \subseteq \mathbb{P}$ such that

- $G \cap D_i \neq \emptyset$ for each $i < \omega_1$, and
- $\{\nu < \omega_1 : (\exists p \in G)\, p \Vdash_{\mathbb{P}} \nu \in \tau_i\}$ is a stationary subset of ω_1 for each $i < \omega_1$.

Woodin's $\mathbb{P}_{\max}$ axiom $(*)$ says:

(1) $\mathsf{AD}^{L(\mathbb{R})}$

(2) $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$.

(Woodin) $(*)$ is a very appealing axiom:

(1) (Π_2-**maximality**) $(*)$ + large cardinals implies that $(H(\omega_2); \in, \mathrm{NS}_{\omega_1})$ satisfies all forcible Π_2 sentences over $(H(\omega_2); \in, \mathrm{NS}_{\omega_1})$.

(2) (**Minimality**) $(*)$ implies that $L(\mathcal{P}(\omega_1))$ is of the form $L(\mathbb{R})[A]$ for any $A \subseteq \omega_1$ such that $\omega_1 = \omega_1^{L[A]}$.

(3) (**Completeness**) $(*)$ + large cardinals provides a complete theory for $L(\mathcal{P}(\omega_1))$ modulo set-forcing; i.e., any two models of $(*)$ obtained by forcing agree on the theory of $L(\mathcal{P}(\omega_1))$.

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Theorem

(A.–Schindler) MM^{++} implies $(*)$.

This result makes $(*)$ into a really nice axiom as it shows that $(*)$ is compatible with all consistent large cardinals assuming there is a supercompact cardinal (in fact significantly less is enough).

Moreover, it unifies $(*)$ with strong *classical* forcing axioms (at the level of ω_1).

MM^{++} and completeness for $H(\omega_3)$

The completeness provided by $(*)$ for the theory of $H(\omega_2)$ certainly doesn't extend to $H(\omega_3)$: Force $\square_{\omega_1}$ by $<\omega_2$-distributive forcing, hence preserving $(*)$.

How about MM^{++}? Does MM^{++} provide a complete theory, modulo forcing, for $H(\omega_3)$?

The answer should of course be No, but it's not so straightforward to find examples:

- (Todorčević) PFA implies $\neg\square_{\omega_1}$.
- (Sakai) MM implies partial square on $S^{\omega_2}_{\omega_1}$.
- PFA implies $2^{\aleph_1} = \aleph_2$, so it implies $\diamondsuit(S^{\omega_2}_{\omega})$ (Shelah).
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Given a cardinal κ of uncountable cofinality and a stationary set $S \subseteq \kappa$, *Strong Club Guessing at S*, SCG(S), is the following statement:

There is a sequence $(C_\delta : \delta \in S)$ such that

- for every $\delta \in S$, C_δ is a club of δ, and
- for every club $D \subseteq \kappa$ there are club-many $\delta \in D$ such that $C_\delta \setminus \alpha \subseteq D$ for some $\alpha < \delta$.

Theorem

$\mathrm{Add}(\omega_2, \omega_3)$ *forces* $\neg\, SCG(S)$ *for every stationary* $S \subseteq S^{\omega_2}_{\omega}$*.*
Hence, if MM^{++} *holds, then forcing with* $\mathrm{Add}(\omega_2, \omega_3)$ *yields a model of* $MM^{++} + \neg\, SCG(S)$ *for every stationary* $S \subseteq S^{\omega_2}_{\omega}$*.*

Given a stationary $S \subseteq S^{\omega_2}_{\omega}$, there is a natural stationary-set-preserving iteration $\mathcal{Q}(S)$ for forcing SCG(S): $\mathcal{Q}(S) = \mathcal{Q}_0 * \dot{\mathcal{Q}}_1$, where

- $\mathcal{Q}_0$ is the forcing for adding a club-sequence $(\dot{C}_\delta \,:\, \delta \in S)$ by initial segments and
- $\dot{\mathcal{Q}}_1$ is a $\aleph_1$-support iteration of length ω_3 in which, at each stage α, we shoot a club through

$$\{\delta \in \omega_2 \,:\, \delta \in S \Rightarrow \dot{C}_\delta \setminus \alpha \subseteq \dot{D}_\alpha \text{ for some } \alpha < \delta\},$$

  where $\dot{D}_\alpha$ is a club of ω_2 given by some fixed book-keeping function.

Theorem

Let κ be a supercompact cardinal, and let $\mathcal{P}$ be the standard RCS-iteration of length κ forcing MM^{++}. Let $S = (S^{\omega_2}_{\omega})^V$. Then $\mathcal{P} * \dot{\mathcal{Q}}(S)$ forces MM^{++}+$SCG(S)$.

Question: Is there any forcible Σ_2 axiom A deciding the theory of $H(\omega_3)$ modulo forcing?

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Limitations on completeness

Theorem

(Woodin) Suppose the Ω conjecture and the AD^+-conjecture are true in all set-generic extensions. Then there is no Ω consistent axiom A such that A provides, modulo forcing, a complete theory for Σ^2_3 sentences.

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Also:

Theorem

(Woodin) Assume the Ω Conjecture holds and there is a proper class of Woodin cardinals. Then there is no Ω-consistent axiom A such that

(1) *A implies $MM^{++}(\aleph_2)$ and*

(2) *A provides, modulo forcing, a complete theory for $\underset{\sim}{\Sigma}^2_1$ sentences.*

(The reason is that, under $\mathrm{MM}^{++}(\aleph_2)$, the proof relation $\vdash_\Omega$ becomes lightface Σ^2_1-definable.)

Compare this with the well-known result, due to Woodin, that if there is a proper class of measurable Woodin cardinals, then CH provides, modulo forcing, a complete theory for $\underset{\sim}{\Sigma}^2_1$ sentences.

The following is an important open question in this area.

Question Is there any reasonable large cardinal hypothesis relative to which $\diamondsuit$ is maximal for Σ^2_2 sentences consistent with CH modulo forcing? I.e., is it true that if $\diamondsuit$ holds and σ is a Σ^2_2 sentence such that σ + CH is forcible, then σ is true?

If so, then $\diamondsuit$ would be complete, modulo forcing, for the Σ^2_2 theory; i.e., any two forcing extensions satisfying $\diamondsuit$ agree on Σ^2_2 sentences.

In this vein, Woodin proved the following.

Theorem

(Woodin) Suppose there is a proper class of measurable Woodin cardinals. Then there is a forcing extension satisfying all Σ^2_2 sentences σ such that $\sigma + CH$ is forcible.

If one could force this form of Σ^2_2 maximality without adding reals, then one would need to redesign the inner model programme at the level of the relevant large cardinal assumption.

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High Π_2 maximality?

Π_2 forcing maximality for the theory $H(\omega_3)$ is false, at least in the presence of a Mahlo cardinal:

Both $\square_{\omega_1}$ and $\neg\square_{\omega_1}$ can be forced, and $\square_{\omega_1}$ is $\Sigma_1(\omega_2)$ over $H(\omega_3)$.

Question

Does ZFC prove that Π_2 forcing maximality for the theory $H(\omega_3)$ is false? Does it in fact prove that there is a $\Sigma_1(\omega_2)$ sentence σ such that both σ and $\neg\sigma$ are forcible?

A vague question:

Question: Can there (still) be any reasonable successful analogue of MM^{++} for $H(\omega_3)$ or higher up?

Such an analogue of MM^{++}, if it extends $\mathsf{FA}_{\omega_2}(\{\mathsf{Cohen}\})$, should presumably imply $2^{\aleph_0} = \aleph_3$.

Alternatively, we could instead focus, in the context of CH, on interesting classes Γ of countably closed forcings.

Fat high forcing axioms: Extending Martin's Axiom

Definition

(A.–Mota) A partial order $\mathcal{P}$ has the $\aleph_{1.5}$–c.c. iff for every large enough cardinal θ there is a club $D \subseteq [H(\theta)]^{\aleph_0}$ such that for every finite $\mathcal{N} \subseteq D$, if $p \in \mathcal{P} \cap N$ for some $N \in \mathcal{N}$ of minimal height within $\mathcal{N}$, then there is some $q \leq_{\mathcal{P}} p$ such that q is $(M, \mathcal{P})$–generic for each $M \in \mathcal{N}$. (The height of a model N is $\delta_N := N \cap \omega_1$.)

Definition

Given a cardinal κ, $\mathrm{MA}^{1.5}_{\kappa}$ is the forcing axiom $\mathrm{FA}_{\kappa}(\aleph_{1.5}\text{-c.c.})$.

Theorem

(A.–Mota) (GCH) Given any infinite cardinal κ, there is a cardinal-preserving forcing notion which forces $MA^{1.5}_{\kappa}$.

The forcing notion witnessing this theorem is a "finite-support" iteration with symmetric systems of models as side conditions.

The proof of the theorem gives in fact consistency of slightly stronger forcing axioms than $\mathsf{MA}^{1.5}_{\kappa}$:

For a given nice property of finite families of countable models, say that $\mathcal{P}$ has the $\aleph_{1.5}$*-c.c. with respect to finite families with nice property* if for every large enough θ there is a club $D \subseteq [H(\theta)]^{\aleph_0}$ such that for every finite $\mathcal{N} \subseteq D$ with nice property and every $p \in \mathcal{P} \cap N$, where N has minimal height in $\mathcal{N}$, there is some $q \leq p$ such that q is $(M, \mathcal{P})$-generic for each $M \in \mathcal{N}$.

We then define $\mathsf{MA}^{1.5}_{\kappa}$(nice property) accordingly.

Definition

A family $\mathcal{N}$ of countable elementary submodels is *stratified* if $\mathtt{ot}(N_0 \cap \mathrm{Ord}) < \delta_{N_1}$ for all $N_0, N_1 \in \mathcal{N}$ such that $\delta_{N_0} < \delta_{N_1}$.

Note: $\mathrm{MA}_\kappa^{1.5}(\text{stratified}) \Rightarrow \mathrm{MA}_\kappa^{1.5} \Longrightarrow \mathrm{MA}_\kappa$.

Fact

Every poset with the $\aleph_{1.5}$-c.c. with respect to finite stratified families is proper and has the $\aleph_2$-c.c.

Proof.

Properness is trivial ($\{N\}$ is stratified).

$\aleph_2$-c.c.: Suppose $A = \{p_i : i < \lambda\}$ max. antichain, $\lambda \geq \omega_2$. For each i let N_i be model with A, $p_i \in N_i$. Since $\lambda \geq \omega_2$, there are $i_0 \neq i_1$ such that $\delta_{N_{i_0}} = \delta_{N_{i_1}}$ and $p_{i_0} \notin N_{i_1}$. $\{N_{i_0}, N_{i_1}\}$ is stratified. Hence there is $q \leq_{\mathcal{P}} p_{i_0}$ which is $(N_{i_0}, \mathcal{P})$-generic and $(N_{i_1}, \mathcal{P})$-generic. But q cannot be $(N_{i_1}, \mathcal{P})$-generic since $A \in N_{i_1}$ and $p_{i_0} \notin N_{i_1}$. Contradiction. $\square$

So we have:

$\mathrm{FA}(\{\mathbb{P} : \mathbb{P}$ is proper and has the $\aleph_2$-c.c.$\})_\kappa \Rightarrow$
$\mathrm{MA}^{1.5}_\kappa$(stratified) $\Rightarrow \mathrm{MA}^{1.5}_\kappa \Longrightarrow \mathrm{MA}_\kappa$.

Theorem

(essentially A.–Mota) (GCH) Given any infinite cardinal κ, there is a cardinal–preserving forcing notion which forces $MA^{1.5}_{\kappa}$(stratified).

$\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) and $\square_{\omega_1,\omega_1}$

Recall: $(\mathcal{C}_\alpha : \alpha \in \mathrm{Lim}(\omega_2))$ is a $\square_{\omega_1,\omega_1}$-sequence iff for all $\alpha \in \mathrm{Lim}(\omega_2)$,

- $\mathcal{C}_\alpha$ is a set of clubs of α of order type at most ω_1,
- $|\mathcal{C}_\alpha| \leq \aleph_1$, and
- for every $C \in \mathcal{C}_\alpha$ and every limit point β of C, $C_\alpha \cap \beta \in \mathcal{C}_\beta$.

Theorem

(A.–Tananimit) $\mathsf{MA}^{1.5}_{\aleph_2}$(stratified) implies $\square_{\omega_1,\omega_1}$.

- Compare this result with the fact that strong enough forcing axioms at ω_1 (e.g. PFA) imply that $\square_\kappa$ fails for all κ.

- (Neeman) Weakenings of $\square_{\omega_1}$ (specifically $\square_{\omega_1,<\omega}$ and $\square^{ta}_{\omega_1,\omega}$) follow from other variants of $\mathrm{MA}^{1.5}_{\aleph_2}$ and from $\mathrm{FA}_{\omega_2}(\{\mathbb{P} : \mathbb{P}$ relaxed two-size proper forcing$\})$.

- (Neeman) $\mathrm{MA}^{1.5}_{\aleph_2}$ does not imply $\square_{\omega_1,\omega}$.

- I do not know of any consistent forcing axiom implying $\square_{\kappa,\kappa}$ for any $\kappa > \omega_1$.

Proof sketch of Theorem: Apply axiom to the forcing consisting of pairs $p = (h^p, \mathcal{N}^p)$, where:

(1) h^p is a finite approximation to a $\square_{\omega_1,\omega_1}$-sequence $\langle (C^\alpha_\nu)_{\nu<\omega_1} \,:\, \alpha \in \mathrm{Lim}(\omega_2)\rangle$ together with an index function i^p specifying, for $\alpha \in \mathrm{Lim}(\omega_2)$, $\nu < \omega_1$, and $\beta \in \mathrm{Lim}(C^\alpha_\nu)$, some $\bar\nu \in \omega_1$ such that $C^\alpha_\nu \cap \beta = C^\beta_{\bar\nu}$;

(2) $\mathcal{N}^p$ is a finite stratified family of countable $N \preccurlyeq (H(\omega_2); \in, \vec{e})$ for some fixed sequence $\vec{e} = (e_\alpha \,:\, \alpha < \omega_2)$, where $e_\alpha : |\alpha| \longrightarrow \alpha$ is a surjection for all α. (All countable $N \preccurlyeq H(\omega_2)$ being considered are sufficiently closed in this sense.)

(3) For every $N \in \mathcal{N}^p$ and every $\alpha \in \mathrm{dom}(h^p) \in [\mathrm{Lim}(\omega_2)]^{<\omega}$ such that $\alpha \in N$,
   (a) N is closed under the approximating functions h^p_α from h^p,
   (b) if $\mathrm{cf}(\alpha) = \alpha$, then $\delta_N \in \mathrm{dom}(h^p_\alpha)$ and $h^p_\alpha(\delta_N) = \sup(N \cap \alpha)$, and
   (c) N is closed under the index function at α.

$\square$

A consistent uniformization principle implying CH

Theorem

(Shelah) Suppose for every $F : S^{\omega_2}_{\omega_1} \longrightarrow 2$ there is a function $G : \omega_2 \longrightarrow 2$ and clubs $D_\alpha \subseteq \alpha$ (for $\alpha \in S^{\omega_2}_{\omega_1}$) such that for every $\alpha \in S^{\omega_2}_{\omega_1}$ and every $\xi \in D_\alpha$, $G(\xi) = F(\alpha)$. Then CH holds.

($S^{\omega_2}_{\omega_1} = \{\alpha < \omega_2 \,:\, \mathrm{cf}(\alpha) = \omega_1\}$.)

Proof: The hypothesis clearly implies the following stronger statement:

For every $F : S^{\omega_2}_{\omega_1} \longrightarrow {}^{\omega}2$ there is a function $G : \omega_2 \longrightarrow {}^{\omega}2$ and clubs $D_\alpha \subseteq \alpha$ (for $\alpha \in S^{\omega_2}_{\omega_1}$) such that for every $\alpha \in S^{\omega_2}_{\omega_1}$ and every $\xi \in D_\alpha$, $G(\xi) = F(\alpha)$.

Hence, if $2^{\aleph_0} \geq \aleph_2$, the following also holds:

For every $F : S^{\omega_2}_{\omega_1} \longrightarrow \omega_2$ there is a function $G : \omega_2 \longrightarrow \omega_2$ and clubs $D_\alpha \subseteq \alpha$ (for $\alpha \in S^{\omega_2}_{\omega_1}$) such that for every $\alpha \in S^{\omega_2}_{\omega_1}$ and every $\xi \in D_\alpha$, $G(\xi) = F(\alpha)$.

Now let F be the identity on $S^{\omega_2}_{\omega_1}$. Apply the hypothesis and get uniformizing function G. Let $\alpha \in S^{\omega_2}_{\omega_1}$ be such that $G``\alpha \subseteq \alpha$. But then there is no club $D_\alpha \subseteq \alpha$ such that $G(\xi) = \alpha$ for all $\xi \in D_\alpha$. Contradiction. $\square$

First forcing axiom failure

Given a regular $\kappa \geq \omega_1$, let SSP_κ denote the class of forcing notions preserving stationary subsets of μ for every uncountable regular $\mu \leq \kappa$.

Let $\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$ denote $\mathsf{FA}_{\aleph_2}(\mathsf{SSP}_{\omega_1})$.

Theorem

(A.–Tananimit) $\mathsf{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$ is inconsistent.

This theorem improves an earlier theorem of Shelah showing that there is no naive high analogue of MM:

For every regular cardinal $\kappa \geq \omega_2$, $\mathsf{FA}_\kappa(\mathsf{SSP}_\kappa)$ is false.

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This theorem improves an earlier theorem of Shelah showing that there is no naive high analogue of MM:

For every regular cardinal $\kappa \geq \omega_2$, $\mathsf{FA}_\kappa(\mathsf{SSP}_\kappa)$ is false.

Proof sketch of Theorem: Assume $\mathrm{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$. Then $2^{\aleph_0} \geq \aleph_3$ and so there is a function $F : S^{\omega_2}_{\omega_1} \longrightarrow 2$ which cannot be club-uniformized; i.e., there is no $G : \omega_2 \longrightarrow 2$, together with clubs $D_\alpha \subseteq \alpha$ (for $\alpha \in S^{\omega_2}_{\omega_1}$), such that for every $\alpha \in S^{\omega_2}_{\omega_1}$ and every $\xi \in D_\alpha$, $G(\xi) = F(\alpha)$. We will show that there is such a G after all, which is a contradiction.

By $\mathrm{MA}^{1.5}_{\aleph_2}(\text{stratified})$, there is a $\square_{\omega_1,\omega_1}$-sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \,:\, \alpha \in \mathrm{Lim}(\omega_2)\rangle$.

Let $\mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}}$ be the class of countable models N such that $N \cap \omega_2 = \bigcup_{\gamma \in C} e_\gamma``\delta_N$ for some $C \in \mathcal{C}_\alpha$, where $\alpha = \sup(N \cap \omega_2)$.

For every cardinal $\theta > \omega_1$, $\mathcal{K}^{\vec{e}}_{\vec{\mathcal{C}}} \cap H(\theta)$ is a projective stationary subset of $[H(\theta)]^{\aleph_0}$. This will guarantee that the forcing $\mathcal{Q}$ we will define is in SSP_{ω_1}.

A family $\mathcal{N}$ of countable models is $\vec{\mathcal{C}}$-*stratified* in case the following holds.

(1) $\mathcal{N} \subseteq \mathcal{K}_{\vec{\mathcal{C}}}^{\vec{e}}$

(2) For all $N_0, N_1 \in \mathcal{N}$, if $\delta_{N_0} = \delta_{N_1}$ but $N_0 \cap \omega_2 \neq N_1 \cap \omega_2$, then
   (a) $\alpha_i := \min((N_i \cap \omega_2) \setminus N_{1-i})$ exists for each $i \in 2$,
   (b) $\mathrm{cf}(\alpha_0) = \mathrm{cf}(\alpha_1) = \omega_1$, and
   (c) there is no ordinal α above $\sup(N_0 \cap N_1 \cap \omega_2)$ such that $\alpha \in \mathrm{cl}(N_0 \cap \omega_2) \cap \mathrm{cl}(N_1 \cap \omega_2)$.

(3) For all $N_0, N_1 \in \mathcal{N}$, if $\delta_{N_0} < \delta_{N_1}$, then

$$\alpha := \max(\mathrm{cl}(N_0 \cap \omega_2) \cap \mathrm{cl}(N_1 \cap \omega_2))$$

exists, $\alpha \in N_1$, and there is some $\nu < \delta_{N_1}$ such that

$$N_0 \cap \alpha = \bigcup_{\gamma \in C_{\alpha,\nu}} e_\gamma``\delta_{N_0}.$$

A $\vec{\mathcal{C}}$-stratified family $\mathcal{N}$ of models is *compatible with F* in case for all $N_0, N_1 \in \mathcal{N}$, if $\delta_{N_0} = \delta_{N_1}$, $N_0 \cap \omega_2 \neq N_1 \cap \omega_2$, and $\alpha_i = \min((N_i \cap \omega_2) \setminus N_{1-i})$ for each $i \in 2$, then $F(\alpha_0) = F(\alpha_1)$.

Now apply $\mathrm{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$ to the forcing $\mathcal{Q}$ consisting of pairs

$$q = ((\mathcal{I}^q_\alpha : \alpha \in X_q), \mathcal{N}_q)$$

such that:

(1) $X_q \in [S^{\omega_2}_{\omega_1}]^{<\omega}$

(2) For every $\alpha \in X_q$, $\mathcal{I}^q_\alpha$ is a finite collection of pairwise disjoint intervals of the form $[\gamma_0, \gamma_1)$ with $\gamma_0 < \gamma_1 < \alpha$.

(3) For all $\alpha_0, \alpha_1 \in X_q$, if $F(\alpha_0) \neq F(\alpha_1)$, then $\min(I) \neq \min(I')$ for all $I \in \mathcal{I}^q_{\alpha_0}$ and $I' \in \mathcal{I}^q_{\alpha_1}$.

(a) $\mathcal{N}_q$ is a finite family of countable $N \preccurlyeq (H(\omega_2); \in, \vec{e}, \vec{\mathcal{C}})$ which is $\vec{\mathcal{C}}$-stratified and compatible with F.

(b) For every $\alpha \in X_q$ and every $\beta < \alpha$, the following are equivalent:

  (a) $\beta = \min(I)$ for some $I \in \mathcal{I}^q_\alpha$;
  (b) $\beta = \sup(N \cap \alpha)$ for some $N \in \mathcal{N}_q$ such that $\alpha \in N$.

$\square$

Another proof of $\neg\,\mathrm{MM}_{\aleph_2}(\aleph_2\text{-c.c.})$

Theorem

(A.–Tananimit) $MA^{1.5}_{\aleph_2}(stratified)$ implies $\neg\, wCC$ (i.e., there is a function $f : \omega_1 \longrightarrow \omega_1$ such that $\{\nu < \omega_1 \;:\; g_\alpha(\nu) < f(\nu)\}$ contains a club for every $\alpha < \omega_2$ and every canonical function for α).

Using this theorem and assuming $MM_{\aleph_2}(\aleph_2\text{-c.c.})$, one can build a sequence $(f_n)_{n<\omega}$, $f_n : \omega_1 \longrightarrow \omega_1$, such that

- for all n, $\{\nu < \omega_1 \;:\; g_\alpha(\nu) < f_n(\nu)\}$ contains a club for every α and g_α;
- $\{\nu < \omega_1 \;:\; f_{n+1}(\nu) < f_n(\nu)\}$ contains a club C_n.

Then, if $\nu \in \bigcap_{n<\omega} C_n$, $(f_n(\nu))_{n<\omega}$ is infinite decreasing.

Question

Is $FA_{\aleph_2}(\{\mathcal{P} \;:\; \mathcal{P}$ proper and $\aleph_2$-c.c.$\})$ consistent?

$\mathrm{MA}^{1.5}_{\aleph_2}$(stratified) and LCS partial orders

Given infinite cardinals κ, λ, an *LCS*(κ, λ) *partial order* is a partial order $\leq^* \subseteq \lambda \times \kappa$ such that:

(1) $(\nu_0, \alpha_0) <^* (\nu_1, \alpha_1) \Rightarrow \alpha_0 < \alpha_1$;

(2) for every $(\nu, \alpha) \in \lambda \times \kappa$ and every $\bar{\alpha} < \alpha$ there are infinitely many ν' such that $(\nu', \bar{\alpha}) <^* (\nu, \alpha)$;

(3) for all (ν_0, α_0), $(\nu_0, \alpha_1) \in \lambda \times \kappa$ there is a finite set

$$b \subseteq \{(\nu, \alpha) \in \lambda \times \kappa \,:\, (\nu, \alpha) \leq^* (\nu_0, \alpha_0), (\nu_1, \alpha_1)\}$$

such that for every (ν, α), if $(\nu, \alpha) \leq^* (\nu_0, \alpha_0)$, (ν_1, α_1), then $(\nu, \alpha) \leq^* (\bar{\nu}, \bar{\alpha})$ for some $(\bar{\nu}, \bar{\alpha}) \in b$.

(Baumgartner-Shelah) The existence of an LCS(ω_2, ω) partial order can be forced.

Theorem

$MA^{1.5}_{\aleph_2}$(stratified) implies that there is a an $LCS(\omega_2, \omega)$ partial order.

Question

Is it consistent to have an $LCS(\omega_3, \omega)$ partial order?

On PFA(ω_1) and large continuum

Given a cardinal κ, let PFA$_\kappa(\omega_1)$ be FA$_\kappa(\{\mathbb{P} : \mathbb{P}$ proper, $|\mathbb{P}| = \aleph_1\})$.

Theorem

(A.–Golshani) (GCH) For every given κ, there is a cardinal-preserving poset forcing PFA$_\kappa(\omega_1)$.

The forcing does not use side conditions. It is a countable-support *memory iteration* $\langle \mathcal{P}_\alpha : \alpha \leq \kappa\rangle$: For a given sequence $\langle \mathcal{U}_\alpha : \alpha < \kappa\rangle$ with $\mathcal{U}_\alpha \in [\alpha]^{\leq\aleph_1}$ and $\mathcal{U}_\xi \subseteq \mathcal{U}_\alpha$ for all α and $\xi \in \mathcal{U}_\alpha$, each $\dot{\mathcal{Q}}_\alpha$ is a $\mathcal{P}_\alpha \restriction \mathcal{U}_\alpha$-name for a forcing on ω_1 and $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha$ is V-proper in $V[\dot{G}_\beta]$. Given $p \in \mathcal{P}_{\alpha+1}$, $p(\alpha)$ is a $\mathcal{P}_\alpha \restriction \mathcal{U}_\alpha$-name and $p \restriction \alpha \Vdash_\alpha p(\alpha) \in \dot{\mathcal{Q}}_\alpha$.

Thin high forcing axioms: Strong properness

(Mitchell) A partial order $\mathcal{P}$ is *strongly proper* iff for every large enough cardinal θ, there are club-many countable $N \preccurlyeq H(\theta)$ such that for every $p \in \mathcal{P} \cap N$ there is some $q \leq_{\mathcal{P}} p$ which is *strongly* $(N, \mathcal{P})$*-generic*, i.e., for every $q' \leq_{\mathcal{P}} q$ there is some $\pi_N(q') \in \mathcal{P} \cap N$ such that every $r \in \mathcal{P} \cap N$ with $r \leq_{\mathcal{P}} \pi_N(q')$ is compatible with q'.

Examples of strongly proper posets:

- Cohen forcing
- Baumgartner's forcing for adding a club of ω_1 with finite conditions.
- Given a cardinal $\lambda \geq \omega_2$, the forcing of finite $\in$-chains of countable $N \preccurlyeq H(\lambda)$.
- Given a cardinal $\lambda \geq \omega_2$, the forcing consisting of finite symmetric systems $\mathcal{N} \subseteq [H(\lambda)]^{\aleph_0}$.

Caution: ccc does not imply strongly proper. In fact, most ccc forcings are not strongly proper.

Some basic facts

Fact

If $\mathcal{P}$ is strongly proper, $N \preccurlyeq H(\theta)$ is countable, $\mathcal{P} \in N$, q is strongly $(N, \mathcal{P})$-generic, $G \subseteq \mathcal{P}$ is generic over V, and $q \in G$, then $G \cap N$ is $\mathcal{P} \cap N$-generic over V.

Fact

Every ω-sequence of ordinals added by a strongly proper forcing notion is in a generic extension of V by Cohen forcing.

Proof.

Let $\mathcal{P}$ be strongly proper, $\dot{r}$ a $\mathcal{P}$-name for an ω-sequence of ordinals, $p \in \mathcal{P}$, and $N \preccurlyeq H(\theta)$ countable and such that $\mathcal{P}$, p, $\dot{r} \in N$.

Let $q \leq_{\mathcal{P}} p$ be strongly $(N, \mathcal{P})$-generic. Then, if G is $\mathcal{P}$-generic over V and $q \in G$, $H = G \cap N$ is $\mathcal{P} \cap N$-generic over V.

But $\mathcal{P} \cap N$ is countable and non-atomic, and therefore forcing-equivalent to Cohen forcing.

And of course $\dot{r}_G = \dot{r}_H$. $\square$

Lemma

(Neeman) Suppose $\mathcal{P}$ *is strongly proper,* $\dot{f}$ *is a* $\mathcal{P}$*-name for a function with* $\mathrm{dom}(\dot{f}) = \alpha \in Ord$*. Let* $N \prec H(\theta)$ *countable and such that* $\mathcal{P}, \dot{f} \in N$*. Let* q *be strongly* $(N, \mathcal{P})$*-generic, let* G *be* $\mathcal{P}$*-generic over* V *such that* $q \in G$*, and suppose* $\dot{f}_G \restriction N \in V$*. Then* $\dot{f}_G \in V$*.*

Corollary

(Neeman) Suppose $\mathcal{P}$ *is strongly proper. Then* $\mathcal{P}$ *does not add new branches through trees* T *such that* $\mathrm{cf}(ht(T)) \geq \omega_1$*.*

Lemma

(Neeman) Suppose $\mathcal{P}$, $\mathcal{Q}$ *are forcing notions,* $N \preccurlyeq H(\theta)$ *is countable and such that* $\mathcal{P}$, $\mathcal{Q} \in N$, p *is strongly* $(N, \mathcal{P})$*-generic, and* q *is* $(N, \mathcal{Q})$*-generic. Then* (p, q) *is* $(N, \mathcal{P} \times \mathcal{Q})$*-generic.*

Extending to $\kappa > \omega$

The notion of strong properness can be naturally extended to higher cardinals:

Suppose κ is an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$. A partial order $\mathcal{P}$ is *κ-strongly proper* iff for every $N \preccurlyeq H(\theta)$ such that $\mathcal{P} \in N$ and such that

- $|N| = \kappa$, and
- ${}^{<\kappa}N \subseteq N$,

every $\mathcal{P}$-condition in N can be extended to a strongly $(N, \mathcal{P})$-generic condition.

We will need the following closure property:

Given an infinite regular cardinal κ, a partial order $\mathcal{P}$ is $<\kappa$*-directed closed with greatest lower bounds* in case every directed subset X of $\mathcal{P}$ (i.e., every finite subset of X has a lower bound in $\mathcal{P}$) such that $|X| < \kappa$ has a greatest lower bound in $\mathcal{P}$.

We will also say that $\mathcal{P}$ *is* κ*-lattice*.

All facts about strongly proper (i.e., ω-strongly proper) forcing we have seen extend naturally to κ-strongly proper forcing notion which are κ-lattice (assuming $\kappa^{<\kappa} = \kappa$).

For example, every κ-sequence of ordinals added by a forcing in this class belongs to a generic extension by adding a Cohen subset of κ.

Lemma

(Reflection Lemma) Let κ be an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$. Suppose $\mathcal{P}$ is a κ-lattice and κ-strongly proper forcing. If θ is large enough and $(Q_i)_{i<\kappa^+}$ is a $\subseteq$-continuous $\in$-chain of elementary submodels of $H(\theta)$ such that $\mathcal{P} \in Q_i$, $|Q_i| = \kappa$, and ${}^{<\kappa}Q_i \subseteq Q_i$ for all $i \in S^{\kappa^+}_{\kappa}$, then $\mathcal{P} \cap Q$ is κ-lattice and κ-strongly proper, for $Q = \bigcup_{i<\kappa^+} Q_i$.

Proof.

Given large enough cardinal χ and $N \preccurlyeq H(\chi)$ such that $\mathcal{P}$, $(Q_i)_{i<\kappa^+} \in N$, $|N| = \kappa$ and ${}^{<\kappa}N \subseteq N$, $N \cap Q = Q_\delta \in Q$ for $\delta = N \cap \kappa^+$. But any strongly $(Q_\delta, \mathcal{P})$-generic $q \in Q$ is strongly $(N, \mathcal{P} \cap Q)$-generic. □

Compare the above reflection property with the reflection of κ-c.c. forcing to substructures Q such that ${}^{<\kappa}Q \subseteq Q$.

Theorem

(A.–Cox–Karagila–Weiss) Assume GCH, and let κ be infinite regular cardinal. Then there is a κ-lattice and κ-strongly proper forcing $\mathcal{P}$ which forces $2^\kappa = \kappa^{++}$ together with the κ-Str PFA ($= FA_{\kappa^+}(\kappa$-lattice + κ-strongly proper$)$).

Proof sketch: Let $\theta = \kappa^{++}$. By first forcing with $\mathsf{Coll}(\kappa^+, {<}\theta)$, we may assume that $\diamondsuit(S^\theta_{\kappa^+})$ holds. Hence there is a 'diamond sequence' $\vec{A} = (A_\alpha)_{\alpha \in S^\theta_{\kappa^+}}$, where $A_\alpha \subseteq H(\theta)$ for all α.

Our forcing $\mathcal{P}$ is $\mathcal{P}_\theta$, where $(\mathcal{P}_\alpha : \alpha \in E \cup \{\theta\})$ is a ${<}\kappa$-support iteration with side conditions. At stage $\alpha + 1 < \kappa$, we force with A_α if A_α happens to be a $\mathcal{P}_\alpha$-name for a κ-lattice κ-strongly proper forcing.

The Reflection Property is used to show that our construction captures κ-strongly proper forcings of arbitrary size.

The proof uses the fact that every κ-sequence of ordinals is in a κ-Cohen extension since each $\mathcal{P}_\alpha$ is κ-lattice and κ-strongly proper, which enables a typical element of a filtration of $H(\theta)$ to have access to sufficiently many $\mathcal{P}_\alpha$-names for κ-sized elementary submodels N (so the relevant A_α's are in fact such that $\Vdash_{\mathcal{P}_\alpha} A_\alpha$ is κ-strongly proper).

Also: The proof crucially uses the fact that our forcings are κ-lattice (it would not work if we just assumed $<\kappa$-directed closedness). $\square$

κ-Str PFA does not decide 2^κ. In fact:

Theorem

Assume GCH, and let $\kappa < \kappa^+ < \kappa^{++} \leq \theta$ be infinite regular cardinals. Suppose $\diamondsuit(S^{\kappa^{++}}_{\kappa^+})$ holds. Then there is a κ-lattice and κ-strongly proper forcing $\mathcal{P}$ which forces $2^\kappa = \theta$ together with κ-Str PFA.

Proof sketch: We fix diamond sequence $\vec{A} = \langle A_\alpha \,:\, \alpha \in S^{\kappa^{++}}_{\kappa^+}\rangle$ and build an iteration $(\mathcal{P}_\alpha \,:\, \alpha \in E \cup \{\kappa^{++}\})$ as before, except that at each stage $\alpha \in E$ now we look at whether A_α is a $\mathcal{P}_\alpha \times \mathrm{Add}(\kappa, \kappa^+)$-name for a κ-lattice and κ-strongly proper poset (and if so we force with $\mathrm{Add}(\kappa, \kappa^+) * A_\alpha$).

The forcing witnessing the theorem is

$$\mathcal{P} = \mathcal{P}_{\kappa^{++}} \times \mathrm{Add}(\kappa, \theta)$$

To see this, take a κ-lattice κ-strongly proper forcing in the

extension via $\mathcal{P}$. By the Reflection Property it reflects to a forcing of size κ^+. Let $\dot{Q}$ be a $\mathcal{P}$-name for the corresponding forcing.

By κ^{++}-c.c. of $\mathcal{P}$ we may identify $\dot{Q}$ with a $\mathcal{P}_{\kappa^{++}} \times \mathrm{Add}(\kappa, \kappa^+)$-name, which of course we may assume is a subset of κ^{++}. Now we use our diamond $\vec{A}$ to capture $\dot{Q}$ by some A_α as in the proof of the previous theorem.

Again, we use the fact that every κ-sequence of ordinals is in a κ-Cohen extension since $\mathcal{P}_\alpha \times \mathrm{Add}(\kappa, \kappa^+)$ is κ-lattice and κ-strongly proper.

$\square$

As far as I know this is the first example of a forcing axiom $\mathsf{FA}_{\kappa^+}(\Gamma)$ such that $\mathsf{FA}_{\kappa^{++}}(\Gamma)$ is false but nevertheless $\mathsf{FA}_{\kappa^+}(\Gamma)$ is compatible with 2^κ arbitrarily large:

To see that $\mathsf{FA}_{\kappa^{++}}$($\kappa$-lattice + κ-strongly proper) is false, look at the forcing $\mathcal{P}$ of $<\kappa$-length $\in$-chains of suitable models $N \preccurlyeq H(\kappa^{++})$ of size κ. An application of $\mathsf{FA}_{\kappa^{++}}(\{\mathcal{P}\})$ would cover κ^{++} with a κ^+-chain of models of size κ.

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Applications of κ-$\mathsf{Str}\,\mathrm{PFA}$

Not many.

- $\mathfrak{d}(\kappa) > \kappa^+$
- The covering number of natural meagre ideals is $> \kappa^+$.
- Weak failures of Club-Guessing at κ.

Relaxing strongness?

Let us say that a forcing $\mathcal{P}$ is κ-MRP*-strongly proper* if for every large enough θ, every $N \preccurlyeq H(\theta)$ of size κ such that ${}^{<\kappa}N \subseteq N$ and $\mathcal{P} \in N$, and every $p \in N \cap \mathcal{P}$ there is $q \leq_{\mathcal{P}} p$ such that for every $q' \leq_{\mathcal{P}} q$,

$$\mathcal{X}_{q'} = \{X \in [N]^{\kappa} \,:\, \exists \pi_X(q') \in \mathcal{P} \cap X \, \forall r \leq_{\mathcal{P}} \pi_X(q'), r \in X \longrightarrow r ||_{\mathcal{P}} q'\}$$

is N-stationary (i.e., for every club $E \in N$ there is some $X \in E \cap \mathcal{X}_{q'} \cap N$).

$\mathrm{FA}_{\kappa^+}(\{\mathcal{P} \,:\, \mathcal{P}\ \kappa\text{-lattice and } \kappa\text{-MRP-strongly proper}\})$ implies a natural high analogue of MRP which in turn implies $2^{\kappa^+} = \kappa^{++}$.

Theorem

Suppose $\kappa \geq \omega_1$ is a regular cardinal and $\kappa^{<\kappa} = \kappa$. Then

$$\mathrm{FA}_{\kappa^+}(\{\mathcal{P} \,:\, \mathcal{P}\ \kappa\text{-lattice, } \kappa^+\text{-c.c., and } \kappa\text{-MRP-strongly proper}\})$$

is false.

Proof sketch: For the proof we use...

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is false.

Proof sketch: For the proof we use...

An inconsistent uniformization principle

Theorem

(Shelah) Let $\kappa \geq \omega_1$ be a regular cardinal and let $\langle C_\alpha : \alpha \in S^{\kappa^+}_\kappa \rangle$ be a club-sequence. Then there is a sequence $\langle f_\alpha : \alpha \in S^{\kappa^+}_\kappa \rangle$ of colourings, with $f_\alpha : C_\alpha \longrightarrow \{0, 1\}$ for all α, for which there is no function $G : \kappa^+ \longrightarrow 2$ such that for all $\alpha \in S^{\kappa^+}_\kappa$, $G(\xi) = f_\alpha(\xi)$ for club-many $\xi \in C_\alpha$.

Now let $\langle C_\alpha : \alpha \in S^{\kappa^+}_\kappa \rangle$ be a club-sequence and $\langle f_\alpha : \alpha \in S^{\kappa^+}_\kappa \rangle$ be a sequence of colourings which cannot be club-uniformized.

Let $\mathcal{P}$ be the forcing consisting of $<\kappa$-sized functions p with $\mathrm{dom}(p) \subseteq S^{\kappa^+}_\kappa$ such that

(1) for all $\alpha \in \mathrm{dom}(p)$, $p(\alpha) < \alpha$, and

(2) for all $\alpha_0 < \alpha_1$ in $\mathrm{dom}(p)$, if $\xi \in (C_{\alpha_0} \setminus p(\alpha_0)) \cap (C_{\alpha_1} \setminus p(\alpha_1))$, then $f_{\alpha_0}(\xi) = f_{\alpha_1}(\xi)$.

Then $\mathcal{P}$ is κ^+-c.c., κ-lattice, and κ-MRP-strongly proper, so an application of $\mathrm{FA}_{\kappa^+}(\{\mathcal{P}\})$ gives a function $G : \kappa^+ \longrightarrow \{0, 1\}$ which in fact uniformizes $\langle f_\alpha : \alpha \in S^{\kappa^+}_\kappa \rangle$ modulo co-bounded sets — for each $\alpha \in S^{\kappa^+}_\kappa$ there is $p(\alpha) < \alpha$ such that $G(\xi) = f_\alpha(\xi)$ for all $\xi \in C_\alpha \setminus p(\alpha)$. $\square$

Getting rid of g.l.b.'s?

No:

Theorem

(Shelah) Suppose $\kappa \geq \omega_1$ is a regular cardinal and $\kappa^{<\kappa} = \kappa$. Then

$$FA_{\kappa^+}(\{\mathcal{P} : \mathcal{P} \text{ } {<}\kappa\text{-directed closed, } \kappa^+\text{-c.c., and } \kappa\text{-strongly proper}\})$$

is false.

Proof.

Similar as previous proof, with a natural forcing for adding $G : \kappa^+ \longrightarrow \{0, 1\}$ and clubs $D_\alpha \subseteq C_\alpha$ (for $\alpha \in S^{\kappa^+}_\kappa$) such that $G(\xi) = f_\alpha(\xi)$ for all α and all $\xi \in D_\alpha$. $\square$

κ-strong semiproperness

Let κ be an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$. Let us say that a forcing notion $\mathcal{P}$ is *κ-strongly semiproper* if and only if for every large enough θ and every $N \preccurlyeq H(\theta)$ such that $\mathcal{P} \in N$, $|N| = \kappa$, and ${}^{<\kappa}N \subseteq N$, every $p \in \mathcal{P} \cap N$ can be extended to some $q \in \mathcal{P}$ which is *κ-strongly semiproper*, i.e., there is some $\sigma \in [H(\theta)]^{\leq\kappa}$ such that

(1) $\mathrm{Sk}(N, \sigma) \cap \kappa^+ = N \cap \kappa^+$, and

(2) q is strongly $(\mathrm{Sk}(N, \sigma), \mathcal{P})$-generic.

Given infinite regular κ, let the *κ-Strongly Semiproper Forcing Axiom* be

$$\mathrm{FA}_{\kappa^+}(\{\mathcal{P} : \mathcal{P} \ \kappa\text{-lattice and } \kappa\text{-strongly semiproper}\})$$

A family of reflection principles

Given an infinite regular κ such that $\kappa^{<\kappa} = \kappa$ and a cardinal $\mu \leq \kappa$, let $\mathrm{SRP}(\kappa^+, \mu)$ be the following reflection principle: Suppose X is a set and $\mathcal{S} \subseteq [X]^\kappa$. If λ is such that $X \in H(\lambda)$, there is a $\subseteq$-continuous $\in$-chain $(N_i)_{i<\kappa^+}$ such that for each $i < \kappa^+$ such that $\mathrm{cf}(i) = \kappa$:

(1) $N_i \preccurlyeq H(\lambda)$ and $|N_i| = \kappa$.

(2) $N_i \cap X \notin \mathcal{S}$ if and only if there is no $\sigma \in [X]^{\leq\mu}$ such that

  (a) $\mathrm{Sk}_\lambda(N \cup \sigma)$ is a κ^+-end-extension of N (i.e., $\mathrm{Sk}_\lambda(N \cup \sigma) \cap \kappa^+ = N \cap \kappa^+$), and

  (b) $\mathrm{Sk}_\lambda(N \cup \sigma) \cap X \in \mathcal{S}$.

Obvious: If $\mu_0 < \mu_1$, then $\mathrm{SRP}(\kappa^+, \mu_1) \Rightarrow \mathrm{SRP}(\kappa^+, \mu_0)$.

Easy: The κ-Strongly Semiproper Forcing Axiom implies $\mathrm{SRP}(\kappa^+, \kappa)$.

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Theorem

For every $\kappa \geq \omega_1$, SRP(κ^+, ω) is false. In particular, the κ-Strongly Semiproper Forcing Axiom is false.

Proof: Let $\mathcal{S}$ be the collection of $X \in [\kappa^{++}]^\kappa$ such that $\mathrm{cf}(X) = \omega$.

By an application of $\mathrm{SRP}^+(\kappa^+, \omega)$ to $\mathcal{S}$ there is a $\subseteq$-continuous $\in$-chain $(N_i)_{i<\kappa^+}$ of models of size κ such that for each $i < \kappa^+$ such that $\mathrm{cf}(i) = \kappa$, if

$$\mathrm{cf}(N_i \cap \kappa^{++}) \neq \omega,$$

then there is no countable $\sigma \subseteq \kappa^{++}$ such that

- $\mathrm{Sk}(N_i \cup \sigma) \cap \kappa^+ = N_i \cap \kappa^+$ and
- $\mathrm{cf}(\mathrm{Sk}(N_i \cup \sigma)) = \omega$.

Claim:

$$S = \{i \in S^{\kappa^+}_\kappa \ : \ \text{there is no countable } \sigma \subseteq \kappa^{++} \text{ as above for } N_i\}$$

cannot be stationary: Suppose S is stationary. Let $\alpha \in \kappa^{++}$, $\mathrm{cf}(\alpha) = \omega$, such that $F``[\alpha]^{<\omega} \cap \kappa^{++} \subseteq \alpha$ for some $F : [H(\lambda)]^{<\omega} \longrightarrow H(\lambda)$ generating club of elementary submodels N such that $(N_i)_{i<\kappa^+} \in N$.

Now we can easily find $X \subseteq \alpha$ cofinal in α, such that $N = F``[X]^{<\omega}$ is such that $|N| = \kappa$ and $N \cap \kappa^+ \in S$. Let $\sigma \subseteq X$ be countable and cofinal in X. But then N is a κ^+-end-extension of N_i and $\mathrm{cf}(N \cap \kappa^{++}) = \omega$, and so σ witnesses that $N_i \notin S$. Contradiction. $\square$

Now we get club-many i such that if $\mathrm{cf}(i) = \kappa$, then $\mathrm{cf}(N_i \cap \kappa^{++}) = \omega$. But this is impossible since $(\sup(N_i \cap \kappa^{++}))_{i<\kappa^+}$ is strictly increasing and continuous and therefore $\mathrm{cf}(N_i \cap \kappa^{++}) = \kappa > \omega$ if $\mathrm{cf}(i) = \kappa$. $\square$

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Saturation

Given an infinite regular κ and a stationary $S \subseteq \kappa^+$, $\mathrm{NS}_{\kappa^+} \restriction S$ *is saturated* iff every collection $\mathcal{A}$ of stationary subsets of S such that $S_0 \cap S_1$ is nonstationary for all $S_0 \neq S_1$ in $\mathcal{A}$ is such that $|\mathcal{A}| \leq \kappa^+$.

Fact

If κ is an infinite regular cardinal, $SRP(\kappa^+, 1)$ implies that $\mathrm{NS}_{\kappa^+} \restriction S^{\kappa^+}_{\kappa}$ is saturated.

Proof: Let $\mathcal{A}$ be a collection of stationary subsets of $S^{\kappa^+}_{\kappa}$ with pairwise nonstationary intersection. We want to show $|\mathcal{A}| \leq \kappa^+$. Let $X = \mathcal{A} \cup \kappa^+$ and let $\mathcal{S}$ be the collection of $Z \in [X]^{\kappa}$ such that

- $\delta_Z := Z \cap \kappa^+ \in \kappa^+$ and
- $\delta_Z \in S$ for some $S \in \mathcal{A} \cap Z$.

Let $(N_i)_{i<\kappa^+}$ be a reflecting sequence for $\mathcal{S}$ as given by $\mathrm{SRP}(\kappa^+, 1)$, and suppose $S \in \mathcal{A} \setminus \bigcup_{i<\kappa^+} N_i$. Let $N_i' = \mathrm{Sk}_\lambda(N_i \cup \{S\})$ for all i and note that

$$\{i < \kappa^+ \, : \, \mathrm{cf}(i) = \kappa \Rightarrow N_i' \cap \kappa^+ = N_i \cap \kappa^+\}$$

contains a club $C \subseteq \kappa^+$.

Hence, for every $i \in C \cap S$ there is some $S(i) \in N_i$ such that $N_i \cap \kappa^+ \in S(i)$. By Fodor's lemma there is some S_0 such that

$$T = \{i \in S \cap C \,:\, S(i) = S_0\}$$

is stationary. But that is a contradiction since $N_i \cap \kappa^+ \in S \cap S_0$ for every $i \in T$ and therefore $S \cap S_0$ is stationary. $\square$

Let us say that a forcing $\mathcal{P}$ is *κ-strongly 1-semiproper* iff it satisfies the definition of 'κ-strongly semiproper' replacing $\mathrm{Sk}(N, \sigma)$, for $|\sigma| \leq \kappa$, with $\mathrm{Sk}(N, \sigma)$, for $|\sigma| \leq 1$.

κ-strong 1-semiproperness is the weakest extension of κ-strong properness into the realm of semiproperness. $\mathsf{FA}_{\kappa^+}(\{\mathcal{P} : \mathcal{P}\ \kappa\text{-lattice},\ \kappa\text{-strongly 1-semiproper}\})$ implies $\mathsf{SRP}(\kappa^+, 1)$.

Question: Is $\mathsf{FA}_{\kappa^+}(\{\mathcal{P} : \mathcal{P}\ \kappa\text{-lattice},\ \kappa\text{-strongly 1-semiproper}\})$ consistent for any $\kappa \geq \omega_1$? Is $\mathsf{SRP}(\kappa^+, 1)$ consistent for any $\kappa \geq \omega_1$?

Question: Suppose $\kappa \geq \omega_1$ is regular and $\mathsf{NS}_{\kappa^+} \restriction S^{\kappa^+}_{\kappa}$ is saturated. Does it follow that GCH cannot hold below κ?

On high properness when adding reals

Neeman considers side conditions consisting of *nodes* of either of the following types.

(1) (Type ω_1) These are models $N \preccurlyeq H(\theta)$ such that $|N| = \aleph_1$ and N is internally club.

(2) (Countable type elementary) These are models $M \preccurlyeq H(\theta)$ such that $|M| = \aleph_0$.

(3) (Countable type tower.) These are countable $\in$-chains $\mathcal{T}$ of nodes of type ω_1 such that $\mathcal{T} \cap N \in N$ for all $N \in \mathcal{T}$.

Definition

(Neeman) A *two-size side condition* is a finite set $\mathcal{N}$ of nodes of the above types which is $\in$-increasing (i.e., every node belongs to the next), and closed under intersection in the sense that:

- If $N, M \in \mathcal{N}$, $N \in M$, N of type ω_1 and M countable elementary, then $M \cap N \in \mathcal{N}$.
- If $N, \mathcal{T} \in \mathcal{N}$, $N \in \mathcal{T}$, N of type ω_1 and $\mathcal{T}$ of type tower, and $\mathcal{T} \cap N \neq \emptyset$, then there is a tower $\mathcal{T}' \supseteq \mathcal{T} \cap N$ occurring in $\mathcal{N}$ before N.

Definition

(Neeman) A partial order $\mathcal{P}$ is *two-size proper* if for every large enough θ there is a function $f : [H(\theta)]^{<\omega} \longrightarrow H(\theta)$ such that for every two-size side condition $\mathcal{N}$ with all models involved closed under f, every $Q \in \mathcal{N}$, and every $p \in \mathcal{P} \cap Q$, if p is $(R, \mathcal{P})$-generic for every $R \in \mathcal{N} \cap Q$, then there is $q \leq_{\mathcal{P}} p$ which is $(R, \mathcal{P})$-generic for all $R \in \mathcal{N}$. (If $\mathcal{T}$ is a tower, a condition is $(\mathcal{T}, \mathcal{P})$-generic iff it is $(N, \mathcal{P})$-generic for all $N \in \mathcal{T}$.)

Theorem

(Neeman) If κ is a supercompact cardinal, then there is a partial order $\mathcal{P} \subseteq V_\kappa$ forcing $FA_{\aleph_2}(\{\mathcal{P} : \mathcal{P} \text{ two-size proper}\})$.

A partial order $\mathcal{P}$ is *two-size strongly semiproper* if for every large enough θ there is a function $f : [H(\theta)]^{<\omega} \longrightarrow H(\theta)$ such that for every two-size side condition $\mathcal{N}$ with all models involved closed under f, every $Q \in \mathcal{N}$, and every $p \in \mathcal{P} \cap Q$, if p is $(R, \mathcal{P})$-strongly semigeneric for every $R \in \mathcal{N} \cap Q$, then there is $q \leq_{\mathcal{P}} p$ which is $(R, \mathcal{P})$-strongly semigeneric for all $R \in \mathcal{N}$.

Theorem

$FA_{\aleph_2}(\{\mathcal{P} : \mathcal{P}$ two-size strongly semiproper$\})$ implies $SRP(\omega_2, \omega)$.

Corollary

$FA_{\aleph_2}(\{\mathcal{P} : \mathcal{P}$ two-size strongly semiproper$\})$ is inconsistent.

Two-size strong 1-semiproperness is the weakest extension of two-size properness into the realm of semiproperness.

$\mathrm{FA}_{\aleph_2}(\{\mathcal{P} : \mathcal{P}$ two-size strongly 1-semiproper$\})$ implies $\mathrm{SRP}(\omega_2, 1)$.

Question: Is $\mathrm{FA}_{\aleph_2}(\{\mathcal{P} : \mathcal{P}$ two-size strongly 1-semiproper$\})$ consistent?

Club Bounding on $S^{\omega_2}_{\omega_1}$: For every function $f : \omega_2 \longrightarrow \omega_2$ there is some $\alpha < \omega_3$ such that

$$\{\nu < \omega_2 \,:\, \mathrm{cf}(\nu) = \omega_1 \longrightarrow f(\nu) < \mathrm{ot}(\pi``\nu)\}$$

contains a club of ω_2 for every surjection $\pi : \omega_2 \longrightarrow \alpha$.

(A.–Veličković, work in progress): Forcing Club Bounding on $S^{\omega_2}_{\omega_1}$ (using virtual models of two types).

The construction does **not** work for $\psi^{S^{\omega_2}_{\omega_1}}_{\mathrm{AC}}$.

Question: Is there any consistent high analogue R^* of any reflection principle R following from MM^{++} such that R^* implies $2^{\aleph_0} = \aleph_3$?