Few New Reals+

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Introduction

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If $q \in P$ and $N \prec H(\theta)$ with $|N| = \aleph_0$, then

1. q is said to be (N, P)-generic iff for every dense subset D of P belonging to N, $D \cap N$ is predense below q.
2. q is said to be strongly (N, P)-generic iff for every dense subset D of $P \cap N$, D is predense below q.

R1 By elementarity, if D is a dense subset of P and $D, P \in N$, then $D \cap N$ is a dense subset of $P \cap N$. So, if $P \in N$, then $2 \Rightarrow 1$.
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A typical condition of a forcing P equipped with side cond. is a pair (x, Δ) where x is an approximation to the desired generic object and Δ is a finite set of ctble. elementary substructures such that if $N \in \Delta$, then (x, Δ) is (N, P)-generic.

This an easy way to guarantee that ω_1 is not collapsed.

If additionally, we want to have the $\aleph_2$-chain condition, it is often necessary to start from a model of CH and require that the models living in Δ satisfy suitable symmetry properties. We call (finite) sets of models having these properties T-symmetric systems (for a fixed $T \subseteq H(\kappa)$).

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Definition

Let $T \subseteq H(\theta)$ and let $\mathcal{N}$ be a finite set of countable subsets of $H(\theta)$. We will say that $\mathcal{N}$ *is a T–symmetric system* iff

(A) For every $N \in \mathcal{N}$, $(N, \in, T) \prec (H(\theta), \in, T)$.

(B) Given distinct N, N' in $\mathcal{N}$, if $\delta_N = \delta_{N'}$, then there is a unique isomorphism

$$\Psi_{N,N'} : (N, \in, T) \longrightarrow (N', \in, T)$$

Furthermore, $\Psi_{N,N'}$ is the identity on $N \cap N'$.

(C) $\mathcal{N}$ is closed under isomorphisms. That is, for all N, N', M in $\mathcal{N}$, if $M \in N$ and $\delta_N = \delta_{N'}$, then $\Psi_{N,N'}(M) \in \mathcal{N}$.

(D) For all N, M in $\mathcal{N}$, if $\delta_M < \delta_N$, then there is some $N' \in \mathcal{N}$ such that $\delta_{N'} = \delta_N$ and $M \in N'$.

Lemma

Let $T \subseteq H(\theta)$, let $\mathcal{N}$ be a T–symmetric system and let $N \in \mathcal{N}$. Then the following holds.

1. *$\mathcal{N} \cap N$ is a T–symmetric system.*
2. *Suppose $\mathcal{N}^* \in N$ is a T–symmetric system such that $\mathcal{N} \cap N \subseteq \mathcal{N}^*$. Let*

$$\mathcal{M} = \mathcal{N} \cup \bigcup \{\Psi_{N,N'} \text{``} \mathcal{N}^* \ : \ N' \in \mathcal{N},\, \delta_{N'} = \delta_N\}$$

   Then $\mathcal{M}$ is the $\subseteq$–minimal T–symmetric system $\mathcal{W}$ such that $\mathcal{N} \cup \mathcal{N}^* \subseteq \mathcal{W}$.

Given $T \subseteq H(\theta)$ and T–symmetric systems $\mathcal{N}_0$, $\mathcal{N}_1$, let us write $\mathcal{N}_0 \cong \mathcal{N}_1$ iff

- $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1) = R$ and
- for some $m < \omega$, there are enumerations $(N_i^0)_{i<m}$ and $(N_i^1)_{i<m}$ of $\mathcal{N}_0$ and $\mathcal{N}_1$, respectively, together with an isomorphism between

  $$\langle \bigcup \mathcal{N}_0, \in, T, R, N_i^0 \rangle_{i<m}$$

  and

  $$\langle \bigcup \mathcal{N}_1, \in, T, R, N_i^1 \rangle_{i<m}$$

  which is the identity on R.

Lemma

Let $T \subseteq H(\theta)$ and let $\mathcal{N}_0$ and $\mathcal{N}_1$ be T–symmetric systems. Suppose $\mathcal{N}_0 \cong \mathcal{N}_1$. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is a T–symmetric system.

Definition

The poset $\mathcal{P}_0$ is the set of all the T–symmetric systems. Given q_1 and q_0 in $\mathcal{P}_0$, $q_1 \leq_{\mathcal{P}_0} q_0$ iff $q_0 \subseteq q_1$.

Corollary

1. $\mathcal{P}_0$ *is (strongly) proper.*
2. *(CH) If there is a bijection between* θ *and* $H(\theta)$ *which is definable in* $(H(\theta), \in, T)$*, then* $\mathcal{P}_0$ *is* $\aleph_2$*–Knaster.*
3. *(CH) If there is a bijection between* θ *and* $H(\theta)$ *which is definable in* $(H(\theta), \in, T)$*, then* $\mathcal{P}_0$ *preserves CH.*

Proof of (1). Suppose that κ is regular and N^* is a ctble. elem. substr. of $H(\kappa)$ s. t. $\mathcal{P}_0$ and the cond. s are in N^*. Then, letting $N = N^* \cap H(\theta)$ and $s' = s \cup \{N\}$, s' is $(N^*, \mathcal{P}_0)$–generic.

Let E be a dense subset of $\mathcal{P}_0$ in N^*. It suffices to show that there is some condition in $E \cap N^*$ compatible with s'. Notice first that $s' \cap N \in \mathcal{P}_0$. Hence, we may find a condition $s^\circ \in E \cap N$ extending $s' \cap N$. Now let

$$s^* = s' \cup \{\Psi_{N,\overline{N}}(M) : M \in q^\circ, \overline{N} \in s', \delta_{\overline{N}} = \delta_N\}.$$

So, s^* is a condition in $\mathcal{P}_0$ extending both s' and s°.

Proof of (2). Suppose that $s_\xi = \{N_i^\xi : i < m\}$ is a $\mathcal{P}_0$–condition for each $\xi < \omega_2$. By CH we may assume that $\{\bigcup_{i<m} N_i^\xi : \xi < \omega_2\}$ forms a Δ–system with root X. Moreover, by CH we may assume, for all $\xi, \xi' < \omega_2$, that the structures $\langle \bigcup_{i<m} N_i^\xi, \in, P, X, N_i^\xi \rangle_{i<m}$ and $\langle \bigcup_{i<m} N_i^{\xi'}, \in, P, X, N_i^{\xi'} \rangle_{i<m}$ are isomorphic and that the isomorphism fixes X.

The first assertion follows from the fact that there are only $\aleph_1$–many iso. types for such structures. For the second assertion note that, if Ψ is the unique isomorphism between $\langle \bigcup_{i<m} N_i^{\xi}, \in, T, X, N_i^{\xi}\rangle_{i<m}$ and $\langle \bigcup_{i<m} N_i^{\xi'}, \in, T, X, N_i^{\xi'}\rangle_{i<m}$, then the restriction of Ψ to $X \cap \theta$ has to be the identity on $X \cap \theta$. Since there is a bijection between θ and $H(\theta)$ which is definable in $(H(\theta), \in, T)$, we have that Ψ fixes X if and only if it fixes $X \cap \theta$. It follows that Ψ fixes X. Hence, for all ξ, $\xi' < \omega_2$, $s_\xi \cup s_{\xi'}$ extends both s_ξ and $s_{\xi'}$.

Proof of (3). Suppose $s \in \mathcal{S}_P$, $\dot{r}_\alpha$ (for $\alpha < \omega_2$) are $\mathcal{P}_0$–names , and s forces that $\dot{r}_\alpha$, for $\alpha < \omega_2$, are pairwise distinct reals . By the $\aleph_2$–chain condition of $\mathcal{P}_0$ we may assume that each $\dot{r}_\alpha$ is in $H(\theta)$. Let κ be a regular cardinal such that $\mathcal{P}_0 \in H(\kappa)$. For each α let N_α be such that $\{q, \dot{r}_\alpha\} \in N_\alpha$ and N_α is a countable elementary substructure of $(H(\theta), \in, P, \mathcal{S}_P)$. We can also assume that for each α, there is a countable elementary substructure $N^*_\alpha \prec H(\kappa)$ such that $N_\alpha = H(\theta) \cap N^*_\alpha$. By CH, there are distinct α, α' such that $(N_\alpha, \in, P, \mathcal{P}_0, s, \dot{r}_\alpha)$ and $(N_\alpha, \in, P, \mathcal{P}_0, s, \dot{r}_\alpha)$ are isomorphic.

By the above lemmas we may also assume that $s \cup \{N_\alpha, N_{\alpha'}\}$ is a $\mathcal{P}_0$–condition. So, $s \cup \{N_\alpha, N_{\alpha'}\}$ is $(N^*_\alpha, \mathcal{P}_0)$–generic and $(N^*_{\alpha'}, \mathcal{P}_0)$–generic. Let Ψ be the unique isomorphism between N_α and $N_{\alpha'}$ and note that for every natural number n and for every condition s' $\mathcal{P}_0$–extending $s \cup \{N_\alpha, N_{\alpha'}\}$, there are conditions s'' and r such that $r \in N_\alpha$, r decides the nth value of $\dot{r}_\alpha$ and s'' is a common $\mathcal{P}_0$–extension of r and s'. Since symmetric systems are closed under isomorphism, s'' also $\mathcal{P}_0$–extends $\Psi(r) \in N_{\alpha'}$. By correctness of Ψ, $\Psi(r)$ forces that the nth value of $\Psi(\dot{r}_\alpha) = \dot{r}_{\alpha'}$ is equal to the nth value of $\dot{r}_\alpha$. So, $s \cup \{N_\alpha, N_{\alpha'}\}$ forces that $\dot{r}_\alpha = \dot{r}_{\alpha'}$. This is a contradiction.

Applications in the context of iterated forcing

Something one may naturally envision at this point is the possibility to build a suitable forcing iteration with systems of models as side conditions while strengthening the symmetry constraints, so as to make them apply not only to the side condition part of the forcing but also to the working parts; one would hope to exploit the above idea in order to show that the iteration thus constructed preserves *CH*, and would of course like to be able to do that while at the same time forcing some interesting statement.

Indeed, starting with a model of *GCH* and doing such an iteration in length ω_2, Asperó and I proved the consistency of Measuring together with CH.

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Definition

Measuring holds if and only if for every sequence $\vec{C} = (C_\delta : \delta \in \omega_1)$, if each C_δ is a closed subset of δ in the order topology, then there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta) \setminus \alpha \subseteq C_\delta$, or
- $(C \setminus \alpha) \cap C_\delta = \emptyset$.

That is, a tail of $(C \cap \delta)$ is either contained in or disjoint from C_δ.

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Theorem

(CH) Let $\kappa \geq \omega_2$ be a regular cardinal such that $2^{<\kappa} = \kappa$. Then there is a partial order $\mathcal{P} \subseteq H(\kappa)$ with the following properties.

1. *$\mathcal{P}$ is proper.*
2. *$\mathcal{P}$ is $\aleph_2$-Knaster.*
3. *$\mathcal{P}$ forces the following statements.*
    1. *Measuring*
    2. *CH*

More surprisingly, the proof of the above theorem can be slightly modified in such a way we also get the following result.

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    1. *Measuring*
    2. *$2^\lambda = \kappa$ for every infinite cardinal $\lambda < \kappa$.*

Some ingredients of the CH construction

Our construction can be roughly described as a finitely supported forcing iteration $\langle \mathcal{P}_\beta : \beta \leq \kappa \rangle$ in which conditions come together with a side condition consisting of a graph of edges $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle$, where each (N_i, ρ_i) is a model with markers, with suitable structural properties. Given any such edge, all information carried by the condition—including both its working part and its side condition—contained in N_0 is to be copied over into N_1 in an appropriate way. The working part consists of conditions for natural forcing notions adding instances of Measuring.

Let us assume CH. We will recursively build a sequence $(\mathcal{P}_\beta : \beta \leq \kappa)$ of forcing notions, together with a sequence of predicates $(\Phi_\alpha : \alpha < \kappa)$.

To start with, let us fix a (bookkeeping) function $\Phi : \kappa \longrightarrow H(\kappa)$ with the property that $\{\alpha < \kappa : \Phi(\alpha) = x\}$ is unbounded in κ for each $x \in H(\kappa)$ (which exists by $\kappa^{<\kappa} = \kappa$), and let Φ_0 be the satisfaction predicate for the structure $(H(\kappa); \in, \Phi)$.

A model with marker is an ordered pair (N, ρ) satisfying

- N is a countable elementary submodel of $(H(\kappa); \in, \Phi_0)$,
- $\rho < \kappa$, and
- for every $\alpha \in N \cap \rho$, N is an elementary submodel of $(H(\kappa); \in, \Phi_{\alpha+1})$.

The presence of the marker ρ will tell us that N is to be seen as 'active' for all stages in $N \cap \rho$.

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Some graph theory

Given an ordered pair

$$e = \langle (N_0, \rho_0), (N_1, \rho_1) \rangle$$

of models with markers, we will call e an **edge** in case

1. $N_0 \cong N_1$;
2. for every $\alpha \in N_0 \cap \rho_0$,
   1. $\bar{\alpha} = \Psi_{N_0,N_1}(\alpha) < \rho_1$ and
   2. Ψ_{N_0,N_1} is an isomorphism between

      $$(N_0; \in, \Phi_{\alpha+1})$$

      and

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If $e = \langle (N_0, \rho_0), (N_1, \rho_1) \rangle$ is an edge, we write Ψ_e for Ψ_{N_0,N_1}.

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We say that a set $\mathcal{G}$ of edges is **reflexive** if $\langle (N, \rho), (N, \rho) \rangle \in \mathcal{G}$ for every (N, ρ) in the field of $\mathcal{G}$, and that it is **symmetric** if $\langle (N_1, \rho_1), (N_0, \rho_0) \rangle \in \mathcal{G}$ for every $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \mathcal{G}$.

We note that if $\mathcal{G}$ is a symmetric set of edges and $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \mathcal{G}$, then for every $\alpha \in N_0$, $\alpha < \rho_0$ if and only if $\Psi_{N_0,N_1}(\alpha) < \rho_1$.

We say that a set $\mathcal{G}$ of edges is **closed under copying** in case for all edges $e = \langle (N_0, \rho_0), (N_1, \rho_1) \rangle$ and $e' = \langle (N'_0, \rho'_0), (N'_1, \rho'_1) \rangle$ in $\mathcal{G}$, if $e' \in N_0$ and $\rho'_0, \rho'_1 \leq \rho_0$, then $\Psi_e(e') \in \mathcal{G}$.

We say that a set $\mathcal{G}$ of edges is **reflexive** if $\langle (N,\rho),(N,\rho)\rangle \in \mathcal{G}$ for every (N,ρ) in the field of $\mathcal{G}$, and that it is **symmetric** if $\langle (N_1,\rho_1),(N_0,\rho_0)\rangle \in \mathcal{G}$ for every $\langle (N_0,\rho_0),(N_1,\rho_1)\rangle \in \mathcal{G}$.

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If Δ is a set of models with markers and $\beta < \kappa$, we let

$$\mathcal{N}_\beta^\Delta = \{N : (N, \beta) \in \Delta, \beta \in N\}$$

Also, let $\Delta(\mathcal{G})$ denote the field of the graph $\mathcal{G}$.
A set $\mathcal{G}$ of edges is **sticky** in case for every ordinal α and for all $N_0, N_1 \in \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G})}$, if $\delta_{N_0} = \delta_{N_1}$, then $\langle (N_0, \alpha+1), (N_1, \alpha+1) \rangle \in \mathcal{G}$.

Our sequence $(\mathcal{P}_\beta : \beta \leq \kappa)$ will turn to be a forcing iteration, in the sense that $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$ for all $\alpha < \beta \leq \kappa$, but we have to be careful since copying things from the past could interfere with the future. For that reason, it is natural to consider the following notions.

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Also, let $\Delta(\mathcal{G})$ denote the field of the graph $\mathcal{G}$.
A set $\mathcal{G}$ of edges is **sticky** in case for every ordinal α and for all N_0, $N_1 \in \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G})}$, if $\delta_{N_0} = \delta_{N_1}$, then $\langle (N_0, \alpha+1), (N_1, \alpha+1) \rangle \in \mathcal{G}$.

Our sequence $(\mathcal{P}_\beta : \beta \leq \kappa)$ will turn to be a forcing iteration, in the sense that $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$ for all $\alpha < \beta \leq \kappa$, but we have to be careful since copying things from the past could interfere with the future. For that reason, it is natural to consider the following notions.

Given functions $f_0, \ldots, f_n$, for some $n < \omega$, we let

$$f_n \circ \ldots \circ f_0$$

denote the function f with domain the set of $x \in \operatorname{dom}(f_0)$ such that for every $i < n$, $(f_i \circ \ldots \circ f_0)(x) \in \operatorname{dom}(f_{i+1})$, and such that for every $x \in \operatorname{dom}(f)$, $f(x) = f_n((f_{n-1} \circ \ldots \circ f_0)(x))$.

If $\vec{\mathcal{E}} = (\langle (N_0^i, \rho_0^i), (N_1^i, \rho_1^i) \rangle \ : \ i < n)$, for some $n < \omega$, is a sequence of pairs of models with markers such that $N_0^i \cong N_1^i$ for all $i < n$, we denote $\Psi_{N_0^{n-1}, N_1^{n-1}} \circ \ldots \circ \Psi_{N_0^0, N_1^0}$ by $\Psi_{\vec{\mathcal{E}}}$.

If $\mathcal{G}$ is a set of edges and $\alpha < \kappa$, we call $\langle \alpha, \vec{\mathcal{E}} \rangle$ a $\mathcal{G}$*-thread* if $\vec{\mathcal{E}}$ is a finite sequence of edges in $\mathcal{G}$ and $\alpha \in \operatorname{dom}(\Psi_{\vec{\mathcal{E}}})$.

If $\mathcal{G}$ is a set of edges and $\alpha < \kappa$, we denote by $\mathcal{G}|_\alpha$ the set of edges $\langle (N_0, \bar{\rho}_0), (N_1, \bar{\rho}_1) \rangle$, with the following properties.

1. $\bar{\rho}_0$ is the supremum of the set of ordinals $\xi + 1$, with $\xi \in N_0$ being such that
   1. $\xi < \rho_0$ for some edge $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \mathcal{G}$;
   2. $\Psi_{\vec{\mathcal{E}}}(\xi) < \alpha$ and $\Psi_{\vec{\mathcal{E}}}(\xi) < \rho_1^n$ for every $\mathcal{G}$-thread

      $$\langle \xi, (\langle (N_0^i, \rho_0^i), (N_1^i, \rho_1^i) \rangle \ : \ i \leq n) \rangle$$

      such that $N_0 \in N_0^0 \cup \{N_0^0\}$.
2. $\bar{\rho}_1 = sup\{\Psi_{N_0,N_1}(\xi) + 1 \ : \ \xi \in N_0 \cap \bar{\rho}_0\}$.

We will call a finite function F *pertinent* if $\mathrm{dom}\,(F) \in [\kappa]^{<\omega}$ and for every $\alpha \in \mathrm{dom}\,(F)$, $F(\alpha) = (b_\alpha, d_\alpha)$, where

- $b_\alpha \in [Lim(\omega_1) \times \omega_1]^{<\omega}$ is such that $\delta_1 < \delta_0$ for every $(\delta_0, \delta_1) \in b_\alpha$;
- $d_\alpha \in [\omega_1 \times H(\kappa)]^{<\omega}$.

If $\mathcal{G}$ is a set of edges and F is a pertinent function, we denote by $F|_\mathcal{G}$ the function F' with domain $\{\alpha \in \mathrm{dom}\,(F) \,:\, \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G})} \neq \emptyset\}$ defined by letting

$$F'(\alpha) = (\{(\delta_N, \delta) \in b_\alpha^F \,:\, N \in \mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G})}\}, d_\alpha^F)$$

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$$F'(\alpha) = (\{(\delta_N, \delta) \in b^F_\alpha : N \in \mathcal{N}^{\Delta(\mathcal{G})}_{\alpha+1}\}, d^F_\alpha)$$

Given any $\alpha < \kappa$, and assuming $\mathcal{P}_\alpha$ has been defined, we let $\dot{C}^\alpha$ be some canonically chosen (using Φ) $\mathcal{P}_\alpha$-name for a club-sequence on ω_1^V such that $\mathcal{P}_\alpha$ forces that

- $\dot{C}^\alpha = \Phi(\alpha)$ in case $\Phi(\alpha)$ is a $\mathcal{P}_\alpha$-name for a club-sequence on ω_1, and that
- $\dot{C}^\alpha$ is some fixed club-sequence on ω_1 in the other case.

Given $\delta \in Lim(\omega_1)$, we let $\dot{C}^\alpha_\delta$ be a name for $\dot{C}^\alpha(\delta)$.

Let $\beta < \kappa$. An ordered pair $q = (F_q, \mathcal{G}_q)$ is a $\mathcal{P}_\beta$-condition if and only if it has the following properties.

1. $\mathcal{G}_q$ is a sticky, reflexive and symmetric set of edges closed under copying, and such that
   1. $\mathcal{N}_0^{\Delta(\mathcal{G}_q)}$ is a Φ_0-symmetric system;
   2. for every $\alpha < \beta$, $\mathcal{N}_{\alpha+1}^{\Delta(\mathcal{G}_q)}$ is a $\Phi_{\alpha+1}$-symmetric system;
   3. $\rho_0, \rho_1 \leq \beta$ for every $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \mathcal{G}_q$.
2. F_q is a pertinent function with $\mathrm{dom}\,(F_q) \subseteq \beta$.
3. For every $\alpha < \beta$, *the restriction of q to α*, $q|_\alpha$, is a condition in $\mathcal{P}_\alpha$, where
$$q|_\alpha := (F_q|_{\mathcal{G}_q|_\alpha}, \mathcal{G}_q|_\alpha)$$
4. $F_q = F_q|_{\mathcal{G}_q}$ and $\mathcal{G}_q = \mathcal{G}_q|_\beta$.

If $\alpha \in \mathrm{dom}\,(F_q)$, then $F_q(\alpha) = (b^q_\alpha, d^q_\alpha)$ has the following properties.

1. $\mathrm{dom}\,(b^q_\alpha) \subseteq \{\delta_N : N \in \mathcal{N}^q_{\alpha+1}\}$
2. For every $N \in \mathcal{N}^q_{\alpha+1}$ and $(\delta_0, \delta_1) \in b^q_\alpha$, if $\delta_1 < \delta_N < \delta_0$, then $q|_\alpha \Vdash_\alpha \delta_N \notin \dot{C}^\alpha_{\delta_0}$.
3. For every $N \in \mathcal{N}^q_{\alpha+1}$, $(\delta, a) \in d^q_\alpha \cap N$ and $N' \in \mathcal{N}^q_{\alpha+1}$, if $\delta_{N'} = \delta_N$, then $(\delta, \Psi_{N,N'}(a)) \in d^q_{\alpha+1}$.
4. For every $(\delta, a) \in d^q_\alpha$ and $N \in \mathcal{N}^q_{\alpha+1}$, if $\delta < \delta_N$, then there is some $N' \in \mathcal{N}^q_{\alpha+1}$ such that $\delta_{N'} = \delta_N$ and $a \in N'$.

The idea here is that $\mathcal{P}_\alpha$ will force that the set $D^G_\alpha = \{\delta_N : N \in \mathcal{N}^{\Delta(\mathcal{G}_q)}_{\alpha+1}, q \in \dot{G}\}$ is a club witnessing measuring for the instance $\dot{C}^\alpha = \Phi(\alpha)$.

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Suppose $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \mathcal{G}_q$, $\alpha \in \mathrm{dom}\,(F_q) \cap N_0 \cap \rho_0$ and $\Psi_{N_0,N_1}(\alpha) = \bar{\alpha}$. Then:

1. $\bar{\alpha} \in \mathrm{dom}\,(F_q)$;
2. $b^q_\alpha \cap N_0 = b^q_{\bar{\alpha}}$;
3. Ψ_{N_0,N_1} “$d^q_\alpha = d^q_{\bar{\alpha}} \cap N_1$.

Finally, for every $\alpha < \beta$ and $N \in \mathcal{N}^q_{\alpha+1}$, if (α, δ_N) is copied to some $(\bar{\alpha}, \delta_N)$ where $b^q_{\bar{\alpha}}(\delta_N)$ is defined, then $q|_\alpha$ forces that for every $a \in N$ there is some $M \in \mathcal{N}^{\dot{G}_\alpha}_\alpha \cap \mathcal{T}_{\alpha+1} \cap N$ such that δ_M does not touch anything bad in the process of doing copies of M

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1. $\bar{\alpha} \in \mathrm{dom}\,(F_q)$;
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Given $\mathcal{P}_\beta$-conditions q_i, for $i = 0, 1$, $q_1 \leq_\beta q_0$ if and only if the following holds.

1. dom $(F_{q_0}) \subseteq$ dom (F_{q_1}) and the following holds for every $\alpha \in$ dom (F_{q_0}).
   1. $b_\alpha^{q_0} \subseteq b_\alpha^{q_1}$
   2. $d_\alpha^{q_0} \subseteq d_\alpha^{q_1}$
2. For every $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \mathcal{G}_{q_0}$ there are $\rho_0' \geq \rho_0$ and $\rho_1' \geq \rho_1$ such that $\langle (N_0, \rho_0'), (N_1, \rho_1') \rangle \in \mathcal{G}_{q_1}$.