On P-measures in the random model

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Joint work with Piotr Borodulin-Nadzieja.

P-points

Definition

A free ultrafilter $\mathcal{U}$ on ω is a *P-point* if for every sequence $\langle A_n \colon\ n \in \omega \rangle$ in $\mathcal{U}$ there is $A \in \mathcal{U}$ such that $A \subseteq^* A_n$ for every $n \in \omega$.

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P-points exist under any of the following assumptions:

- CH (W. Rudin)
- MA+$\neg$CH
- $\mathfrak{d} = \mathfrak{c}$ (J. Ketonen)
- $\Diamond(\mathfrak{r})$ (J.T. Moore, M. Hrušák, M. Džamonja)

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P-points may not exist:

- S. Shelah first found a model without P-points
- the Silver model (D. Chodounský, O. Guzmán)

Random forcing

$\kappa \geqslant \omega$

λ_κ — the σ-additive standard product measure on 2^κ

$\mathcal{N}_\kappa = \{A \in \mathrm{Bor}(2^\kappa) \colon \ \lambda_\kappa(A) = 0\}$

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Cardinal characteristics

(CH) If $\kappa = \kappa^\omega$, then $V^{\mathbb{M}_\kappa} \models 2^\omega = \kappa \ \wedge\ \mathfrak{d} = \omega_1$.

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(Again) Open question

Are there P-points in the random model?

Some positive results

Theorem (K. Kunen)

Adding ω_1 Cohen reals and then ω_2 random reals to the ground model produces a P-point.

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Theorem (A. Dow)

Adding ω_2 random reals to a model of CH+$\square_{\omega_1}$ produces a P-point.

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A (finitely additive probability) measure μ on ω is a *P-measure* if for every sequence (A_n) of pairwise disjoint subsets of ω, there is $A \subseteq \omega$ such that $A_n \subseteq^* A$ for each $n \in \omega$ and $\mu(A) = \sum_{n \in \omega} \mu(A_n)$.

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Other names: μ has the Additive Property, (AP), AP(*)...

P-measures vs. P-points

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Fact

Let $\mathcal{U}$ be a free ultrafilter on ω. Then, $\delta_{\mathcal{U}}$ is a P-measure if and only if $\mathcal{U}$ is a P-point.

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Trivial fact

If there is a P-point, then there is a P-measure.

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Theorem (R. Frankiewicz, S. Shelah, P. Zbierski)

There is a model of ZFC with no ccc P-sets in ω^*. In particular, there are no P-measures in this model.

Density measures

The asymptotic density

For every $A \in \wp(\omega)$ we set:

$$d(A) = \lim_{n\to\infty} \frac{|A \cap \{0, \ldots, n\}|}{n+1}$$

(if the limit exists).

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A measure μ on ω is a *density measure* if $\mu \restriction \mathcal{D} = d$.

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An ultrafilter density

Let $\mathcal{U}$ be a free ultrafilter on ω. For every $A \in \wp(\omega)$ we set:

$$d_{\mathcal{U}}(A) = \lim_{n\to\mathcal{U}} \frac{|A \cap \{0,\dots,n\}|}{n+1}.$$

Density measures vs. P-measures

Theorem (J. Grebík)

TFAE:

1. There is a P-point.
2. There is an ultrafilter density which is a P-measure.

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Question

Assume there is a density P-measure. Does there exist an ultrafilter density which is a P-measure?

P-measures in the random model, prelim.

$$\alpha < \kappa,\ i \in \{0,1\} \quad \longmapsto \quad C^i_\alpha = \{x \in 2^\kappa\colon\ x(\alpha) = i\}$$

$\dot r = \{\langle\langle\alpha, i\rangle, C^i_\alpha\rangle\colon\ \alpha < \kappa, i \in \{0,1\}\}$ — $\mathbb{M}_\kappa$-name for the generic element of 2^κ

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$$M \in \mathbb{M}^\omega_\kappa \quad \longmapsto \quad \dot{M} = \big\{\langle k, M(k)\rangle\colon\ k \in \omega\big\}$$ — $\mathbb{M}_\kappa$-name

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$\dot{\mu}_\mathcal{U}$ — *Solovay measure associated with* $\mathcal{U}$

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Theorem (DS.–P. Borodulin-Nadzieja)

Let $\mathcal{U}$ be an ultrafilter on ω. Then, TFAE:

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2. $\Vdash_{\mathbb{M}_\kappa} \dot{\mu}_{\mathcal{U}}$ is non-atomic.

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There is a ccc P-set in ω^* in the random model.

Application of Solovay measures

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An ultrafilter $\mathcal{U}$ on ω is *semi-selective* if whenever $(a_n) \subseteq \mathbb{R}$ is such that $\lim_{n\to\mathcal{U}} a_n = 0$, then there is $X \in \mathcal{U}$ such that $\sum_{n\in X} a_n < \infty$.

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$$\bigwedge_{k \in X} \bigvee_{\substack{i \in X \\ i > k}} P_i = \emptyset;$$
3. $\Vdash_{\mathbb{M}_\kappa}$ Radon extension of $\dot{\mu}_{\mathcal{U}}$ is strictly positive on $\bigcap_{X \in \mathcal{U}} [X]$.

The end

Thank you for your attention!