Iteration, reflection, and Prikry type forcing

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Looking at SCH in the context of stationary reflection addresses both.

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Question: can we get the failure of SCH at κ together with stationary reflection at κ^+?

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Warm up: stationary reflection in a vanilla Prikry extension.

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For $p = \langle s, A\rangle \in \mathbb{P}$, set $\mathrm{lh}(p) = |s|$.

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   The proof uses that $\kappa^+ \cap \mathrm{cof}^{\mathsf{V}}(< \kappa)$ reflects in V.

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Thm: If κ is κ^+-supercompact in V, then in V[G] every stationary subset of $\kappa^+ \cap \mathrm{cof}^{\mathsf{V}}(<\kappa)$ reflects.

Proof: Let $\mathsf{T} \subset \kappa^+ \cap \mathrm{cof}^{\mathsf{V}}(<\kappa)$ be stationary.

For each stem s, look at

$$\mathsf{T}_{\mathsf{s}} := \{\alpha \mid \exists \text{ p with stem s, p} \Vdash \alpha \in \dot{\mathsf{T}}\}.$$

Densely often these sets are stationary in V.

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This is true in any Prikry that singularizes a regular cardinal κ.

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Suppose that $\mathbb{P}$ is a nice enough Prikry type forcing, and $\dot{T}$ satisfies $(\dagger)$. Then we can kill its stationarity in a Prikry type way.

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Weakly Σ Prikry forcing: as above but, with a relaxed first clause.

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Next: we want to iterate this.

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EBF and with collapses

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EBFC is weakly Σ-Prikry.

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THANK YOU