A tail of a generic real
Classifying invariants for E_1

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An intersection model

Let $x \in \mathbb{R}^\omega$ be Cohen generic. Define the tail intersection model

$$M = \bigcap_{n<\omega} V[\langle x_n, x_{n+1}, \ldots\rangle].$$

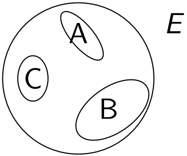
This model was considered by Kanovei-Sabok-Zapletal (2013) and Larson-Zapletal (2020), while studying E_1.
E_1 is the equivalence relation on $\mathbb{R}^\omega$:
$x \; E_1 \; y \iff (\exists n)(\forall m > n) x(m) = y(m)$.
What this model looks like was left open. In particular: does it satisfy choice?

We will see some structural results about this model.
The main topic of this talk is: what do the properties of this model tell us about E_1?
Specifically: we will define and study a *weak* notion of "classifying invariants".

Complete classifications

Let E be an equivalence relation on X.
A **complete classification** of E is a map $c\colon X \to I$

$$x \mathrel{E} y \iff c(x) = c(y).$$

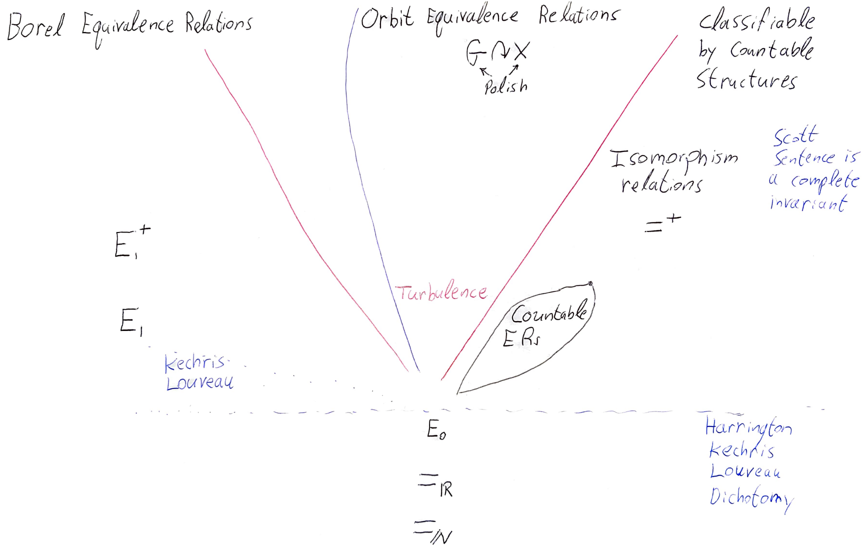


Some "bad" examples:
- $c\colon X/E \to X$ choice function $c([x]_E) \in [x]_E$. (Not definable)
- $x \mapsto [x]_E$. (Hard to *describe* $c(x)$ from x)

Say that c is **absolute** if: • c is definable.
• c remains a complete classification in generic extensions.
• $c(x)^V = c(x)^{V[G]}$ for $x \in V$. ("local computation")

E,F E.R.s on Polish spaces X, Y. $f\colon X \to Y$ is a **reduction** if
$$x \mathrel{E} y \iff f(x) \mathrel{F} f(y).$$
Classifying invariants for F can be used to classify E.
E is Borel reducible to F is there is a Borel reduction.

An extremely partial picture of Borel ERs



Generically absolute classifications

Definition: $c\colon X \to I$ a definable complete classification of E.
Say that c is **generically absolute** if

- it remains a complete classification in a Cohen-real extension.
- $c(x)^V = c(x)^{V[G]}$ for $x \in V$.

Main point: allow some non-orbit relations to "be classifiable" too, maintaining turbulence as the anti-classification phenomenon.

Theorem

1. E_1 is generically classifiable. (Using $\mathfrak{b}$ many of E_0-classes.)

A. Choice fails in M. (for $\mathfrak{b}$-sequences of E_0-classes)

2. E_1 does not admit an absolute classification.

B. $M = V(A)$ for a set (of reals) A.

3. E_1 is *not* gen. class. using $< \mathbf{add}(\mathcal{B})$ many E_0-classes.

C. An analysis of reals in M. (Question: Does $M \models \mathrm{DC}_{<\mathbf{add}(\mathcal{B})}$?)

Question: is (1) optimal? Chichon-Pawlikowsky: $\mathfrak{b}^{V[x]} = \mathbf{add}(\mathcal{B})^V$

Parts (1) and (A): Classifying invariants for E_1

Fix $x \in (2^\omega)^\omega$. Given $f \in \omega^\omega$, Let $[x \restriction f]$ be the set of all finite changes of $x \restriction f$.
This is E_1-invariant. ($[x \restriction f]$ is an E_0-class.)

$$\begin{array}{ll} x & \\ \begin{matrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{matrix} & x \restriction f \end{array}$$

Fix $\langle f_\alpha \mid \alpha < \mathfrak{b} \rangle$, $<^*$-unbdd, f_α increasing.
Claim: $x \mapsto \langle [x \restriction f_\alpha] \mid \alpha < \mathfrak{b} \rangle$ is a complete classification of E_1.
Moreover, this is true in any model in which $\langle f_\alpha \mid \alpha < \mathfrak{b} \rangle$ is unbounded.
(In particular, in a Cohen-real extension.)
Note: $\langle [x \restriction f_\alpha] \mid \alpha < \mathfrak{b} \rangle \in M$.
(Any E_1-invariant is in M.)
Claim: $\langle [x \restriction f_\alpha] \mid \alpha < \mathfrak{b} \rangle$ has no choice function in M.

Thanks for listening!

Lower bounds for possible generic invariants

Assume for contradiction that there is a generically absolute classification of E_1 using $< \mathbf{add}(\mathcal{B})$-sequences of E_0-classes.
$x \in (2^\omega)^\omega$ Cohen-generic. $A = A_x = \langle A_\alpha \mid \alpha < \kappa \rangle$ its invariant.
Then $A \in M$ (the intersection model). $\kappa < \mathbf{add}(\mathcal{B})^V = \mathfrak{b}^{V[x]}$.

Lemma

For any real $z \in M$ there is a function $f \in M$ so that

$$n \in z \iff x \restriction f \Vdash \check{n} \in \dot{z}.$$

Sketch of lower bound proof (part (3) of thm) using the lemma:
Working in $V[x]$, associate to each real $z \in \bigcup_\alpha A_\alpha$ a function f_z, and find some f dominating all of them.
Change x (generically) above f, get a new generic x', agreeing with x below f. So x, x' are **not** E_1-related.
It follows however that the invariant $A_{x'}$ is also equal to A, so these are not classifying invariants.

Thanks for listening!